IMO Shortlist

From 2003 To 2013

Problems with Solutions

International Mathematics Olympiad 2015

Olympiad Training Materials

For IMO 2015

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44th International Mathematical Olympiad

Short-listed Problems and Solutions

Tokyo Japan

July 2003

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The Problem Selection Committee and the Organising Committee of IMO 2003 thank the following thirty-eight countries for contributing problem proposals.

Armenia	Greece	New Zealand
Australia	Hong Kong	Poland
Austria	India	Puerto Rico
Brazil	Iran	Romania
Bulgaria	Ireland	Russia
Canada	Israel	South Africa
Colombia	Korea	Sweden
Croatia	Lithuania	Taiwan
Czech Republic	Luxembourg	Thailand
Estonia	Mexico	Ukraine
Finland	Mongolia	United Kingdom
France	Morocco	United States
Georgia	Netherlands	

The problems are grouped into four categories: algebra (A), combinatorics (C), geometry (G), and number theory (N). Within each category, the problems are arranged in ascending order of estimated difficulty, although of course it is very hard to judge this accurately.

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Part I

Problems

Algebra

A1. Let a_{ij}, $i = 1, 2, 3$; $j = 1, 2, 3$ be real numbers such that a_{ij} is positive for $i = j$ and negative for $i \neq j$.

Prove that there exist positive real numbers c_1, c_2, c_3 such that the numbers

$$a_{11}c_1 + a_{12}c_2 + a_{13}c_3, \qquad a_{21}c_1 + a_{22}c_2 + a_{23}c_3, \qquad a_{31}c_1 + a_{32}c_2 + a_{33}c_3$$

are all negative, all positive, or all zero.

A2. Find all nondecreasing functions $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ such that

(i) $f(0) = 0$, $f(1) = 1$;

(ii) $f(a) + f(b) = f(a)f(b) + f(a + b - ab)$ for all real numbers a, b such that $a < 1 < b$.

A3. Consider pairs of sequences of positive real numbers

$$a_1 \geq a_2 \geq a_3 \geq \cdots, \qquad b_1 \geq b_2 \geq b_3 \geq \cdots$$

and the sums

$$A_n = a_1 + \cdots + a_n, \quad B_n = b_1 + \cdots + b_n; \qquad n = 1, 2, \ldots.$$

For any pair define $c_i = \min\{a_i, b_i\}$ and $C_n = c_1 + \cdots + c_n$, $n = 1, 2, \ldots$.

(1) Does there exist a pair $(a_i)_{i\geq 1}$, $(b_i)_{i\geq 1}$ such that the sequences $(A_n)_{n\geq 1}$ and $(B_n)_{n\geq 1}$ are unbounded while the sequence $(C_n)_{n\geq 1}$ is bounded?

(2) Does the answer to question (1) change by assuming additionally that $b_i = 1/i$, $i = 1, 2, \ldots$?

Justify your answer.

A4. Let n be a positive integer and let $x_1 \leq x_2 \leq \cdots \leq x_n$ be real numbers.

(1) Prove that

$$\left(\sum_{i,j=1}^{n} |x_i - x_j|\right)^2 \leq \frac{2(n^2-1)}{3} \sum_{i,j=1}^{n} (x_i - x_j)^2.$$

(2) Show that the equality holds if and only if $x_1, \ldots, x_n$ is an arithmetic sequence.

A5. Let $\mathbb{R}^+$ be the set of all positive real numbers. Find all functions $f\colon \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ that satisfy the following conditions:

(i) $f(xyz) + f(x) + f(y) + f(z) = f(\sqrt{xy})f(\sqrt{yz})f(\sqrt{zx})$ for all $x, y, z \in \mathbb{R}^+$;

(ii) $f(x) < f(y)$ for all $1 \leq x < y$.

A6. Let n be a positive integer and let $(x_1, \ldots, x_n)$, $(y_1, \ldots, y_n)$ be two sequences of positive real numbers. Suppose $(z_2, \ldots, z_{2n})$ is a sequence of positive real numbers such that

$$z_{i+j}^2 \geq x_i y_j \qquad \text{for all } 1 \leq i, j \leq n.$$

Let $M = \max\{z_2, \ldots, z_{2n}\}$. Prove that

$$\left(\frac{M + z_2 + \cdots + z_{2n}}{2n}\right)^2 \geq \left(\frac{x_1 + \cdots + x_n}{n}\right)\left(\frac{y_1 + \cdots + y_n}{n}\right).$$

Combinatorics

C1. Let A be a 101-element subset of the set $S = \{1, 2, \ldots, 1000000\}$. Prove that there exist numbers t_1, t_2, …, t_{100} in S such that the sets

$$A_j = \{x + t_j \mid x \in A\}, \qquad j = 1, 2, \ldots, 100$$

are pairwise disjoint.

C2. Let D_1, …, D_n be closed discs in the plane. (A closed disc is the region limited by a circle, taken jointly with this circle.) Suppose that every point in the plane is contained in at most 2003 discs D_i. Prove that there exists a disc D_k which intersects at most $7 \cdot 2003 - 1$ other discs D_i.

C3. Let $n \geq 5$ be a given integer. Determine the greatest integer k for which there exists a polygon with n vertices (convex or not, with non-selfintersecting boundary) having k internal right angles.

C4. Let x_1, …, x_n and y_1, …, y_n be real numbers. Let $A = (a_{ij})_{1 \leq i,j \leq n}$ be the matrix with entries

$$a_{ij} = \begin{cases} 1, & \text{if } x_i + y_j \geq 0; \\ 0, & \text{if } x_i + y_j < 0. \end{cases}$$

Suppose that B is an $n \times n$ matrix with entries 0, 1 such that the sum of the elements in each row and each column of B is equal to the corresponding sum for the matrix A. Prove that $A = B$.

C5. Every point with integer coordinates in the plane is the centre of a disc with radius 1/1000.

(1) Prove that there exists an equilateral triangle whose vertices lie in different discs.

(2) Prove that every equilateral triangle with vertices in different discs has side-length greater than 96.

C6. Let $f(k)$ be the number of integers n that satisfy the following conditions:

(i) $0 \leq n < 10^k$, so n has exactly k digits (in decimal notation), with leading zeroes allowed;

(ii) the digits of n can be permuted in such a way that they yield an integer divisible by 11.

Prove that $f(2m) = 10f(2m-1)$ for every positive integer m.

Geometry

G1. Let $ABCD$ be a cyclic quadrilateral. Let P, Q, R be the feet of the perpendiculars from D to the lines BC, CA, AB, respectively. Show that $PQ = QR$ if and only if the bisectors of $\angle ABC$ and $\angle ADC$ are concurrent with AC.

G2. Three distinct points A, B, C are fixed on a line in this order. Let Γ be a circle passing through A and C whose centre does not lie on the line AC. Denote by P the intersection of the tangents to Γ at A and C. Suppose Γ meets the segment PB at Q. Prove that the intersection of the bisector of $\angle AQC$ and the line AC does not depend on the choice of Γ.

G3. Let ABC be a triangle and let P be a point in its interior. Denote by D, E, F the feet of the perpendiculars from P to the lines BC, CA, AB, respectively. Suppose that

$$AP^2 + PD^2 = BP^2 + PE^2 = CP^2 + PF^2.$$

Denote by I_A, I_B, I_C the excentres of the triangle ABC. Prove that P is the circumcentre of the triangle $I_AI_BI_C$.

G4. Let Γ_1, Γ_2, Γ_3, Γ_4 be distinct circles such that Γ_1, Γ_3 are externally tangent at P, and Γ_2, Γ_4 are externally tangent at the same point P. Suppose that Γ_1 and Γ_2; Γ_2 and Γ_3; Γ_3 and Γ_4; Γ_4 and Γ_1 meet at A, B, C, D, respectively, and that all these points are different from P.

Prove that

$$\frac{AB \cdot BC}{AD \cdot DC} = \frac{PB^2}{PD^2}.$$

G5. Let ABC be an isosceles triangle with $AC = BC$, whose incentre is I. Let P be a point on the circumcircle of the triangle AIB lying inside the triangle ABC. The lines through P parallel to CA and CB meet AB at D and E, respectively. The line through P parallel to AB meets CA and CB at F and G, respectively. Prove that the lines DF and EG intersect on the circumcircle of the triangle ABC.

G6. Each pair of opposite sides of a convex hexagon has the following property:

> the distance between their midpoints is equal to $\sqrt{3}/2$ times the sum of their lengths.

Prove that all the angles of the hexagon are equal.

G7. Let ABC be a triangle with semiperimeter s and inradius r. The semicircles with diameters BC, CA, AB are drawn on the outside of the triangle ABC. The circle tangent to all three semicircles has radius t. Prove that

$$\frac{s}{2} < t \le \frac{s}{2} + \left(1 - \frac{\sqrt{3}}{2}\right) r.$$

Number Theory

N1. Let m be a fixed integer greater than 1. The sequence x_0, x_1, x_2, $\ldots$ is defined as follows:

$$x_i = \begin{cases} 2^i, & \text{if } 0 \leq i \leq m-1; \\ \sum_{j=1}^{m} x_{i-j}, & \text{if } i \geq m. \end{cases}$$

Find the greatest k for which the sequence contains k consecutive terms divisible by m.

N2. Each positive integer a undergoes the following procedure in order to obtain the number $d = d(a)$:

(i) move the last digit of a to the first position to obtain the number b;

(ii) square b to obtain the number c;

(iii) move the first digit of c to the end to obtain the number d.

(All the numbers in the problem are considered to be represented in base 10.) For example, for $a = 2003$, we get $b = 3200$, $c = 10240000$, and $d = 02400001 = 2400001 = d(2003)$.

Find all numbers a for which $d(a) = a^2$.

N3. Determine all pairs of positive integers (a, b) such that

$$\frac{a^2}{2ab^2 - b^3 + 1}$$

is a positive integer.

N4. Let b be an integer greater than 5. For each positive integer n, consider the number

$$x_n = \underbrace{11\cdots1}_{n-1}\underbrace{22\cdots2}_{n}5,$$

written in base b.

Prove that the following condition holds if and only if $b = 10$:

> there exists a positive integer M such that for any integer n greater than M, the number x_n is a perfect square.

N5. An integer n is said to be *good* if $|n|$ is not the square of an integer. Determine all integers m with the following property:

> m can be represented, in infinitely many ways, as a sum of three distinct good integers whose product is the square of an odd integer.

N6. Let p be a prime number. Prove that there exists a prime number q such that for every integer n, the number $n^p - p$ is not divisible by q.

N7. The sequence $a_0, a_1, a_2, \ldots$ is defined as follows:

$$a_0 = 2, \qquad a_{k+1} = 2a_k^2 - 1 \quad \text{for } k \geq 0.$$

Prove that if an odd prime p divides a_n, then 2^{n+3} divides $p^2 - 1$.

N8. Let p be a prime number and let A be a set of positive integers that satisfies the following conditions:

(i) the set of prime divisors of the elements in A consists of $p-1$ elements;

(ii) for any nonempty subset of A, the product of its elements is not a perfect p-th power.

What is the largest possible number of elements in A?

Part II

Solutions

Algebra

A1. Let a_{ij}, $i = 1, 2, 3$; $j = 1, 2, 3$ be real numbers such that a_{ij} is positive for $i = j$ and negative for $i \neq j$.

Prove that there exist positive real numbers c_1, c_2, c_3 such that the numbers

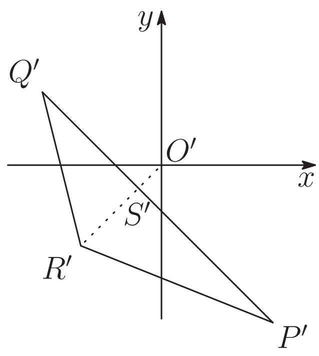
$$a_{11}c_1 + a_{12}c_2 + a_{13}c_3, \qquad a_{21}c_1 + a_{22}c_2 + a_{23}c_3, \qquad a_{31}c_1 + a_{32}c_2 + a_{33}c_3$$

are all negative, all positive, or all zero.

Solution. Set $O(0,0,0)$, $P(a_{11}, a_{21}, a_{31})$, $Q(a_{12}, a_{22}, a_{32})$, $R(a_{13}, a_{23}, a_{33})$ in the three dimensional Euclidean space. It is enough to find a point in the interior of the triangle PQR whose coordinates are all positive, all negative, or all zero.

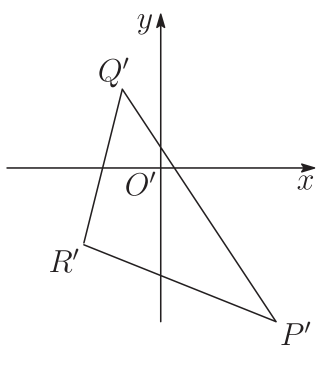
Let O', P', Q', R' be the projections of O, P, Q, R onto the xy-plane. Recall that points P', Q' and R' lie on the fourth, second and third quadrant respectively.

Case 1: O' is in the exterior or on the boundary of the triangle $P'Q'R'$.



Denote by S' the intersection of the segments $P'Q'$ and $O'R'$, and let S be the point on the segment PQ whose projection is S'. Recall that the z-coordinate of the point S is negative, since the z-coordinate of the points P' and Q' are both negative. Thus any point in the interior of the segment SR sufficiently close to S has coordinates all of which are negative, and we are done.

Case 2: O' is in the interior of the triangle $P'Q'R'$.



Let T be the point on the plane PQR whose projection is O'. If $T = O$, we are done again. Suppose T has negative (resp. positive) z-coordinate. Let U be a point in the interior of the triangle PQR, sufficiently close to T, whose x-coordinates and y-coordinates are both negative (resp. positive). Then the coordinates of U are all negative (resp. positive), and we are done.

A2. Find all nondecreasing functions $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ such that

(i) $f(0) = 0$, $f(1) = 1$;

(ii) $f(a) + f(b) = f(a)f(b) + f(a + b - ab)$ for all real numbers a, b such that $a < 1 < b$.

Solution. Let $g(x) = f(x + 1) - 1$. Then g is nondecreasing, $g(0) = 0$, $g(-1) = -1$, and $g\big(-(a-1)(b-1)\big) = -g(a-1)g(b-1)$ for $a < 1 < b$. Thus $g(-xy) = -g(x)g(y)$ for $x < 0 < y$, or $g(yz) = -g(y)g(-z)$ for $y, z > 0$. Vice versa, if g satisfies those conditions, then f satisfies the given conditions.

Case 1: If $g(1) = 0$, then $g(z) = 0$ for all $z > 0$. Now let $g\colon \mathbb{R} \longrightarrow \mathbb{R}$ be any nondecreasing function such that $g(-1) = -1$ and $g(x) = 0$ for all $x \geq 0$. Then g satisfies the required conditions.

Case 2: If $g(1) > 0$, putting $y = 1$ yields

$$g(-z) = -\frac{g(z)}{g(1)} \tag{$*$}$$

for all $z > 0$. Hence $g(yz) = g(y)g(z)/g(1)$ for all $y, z > 0$. Let $h(x) = g(x)/g(1)$. Then h is nondecreasing, $h(0) = 0$, $h(1) = 1$, and $h(xy) = h(x)h(y)$. It follows that $h(x^q) = h(x)^q$ for any $x > 0$ and any rational number q. Since h is nondecreasing, there exists a nonnegative number k such that $h(x) = x^k$ for all $x > 0$. Putting $g(1) = c$, we have $g(x) = cx^k$ for all $x > 0$. Furthermore $(*)$ implies $g(-x) = -x^k$ for all $x > 0$. Now let $k \geq 0$, $c > 0$ and

$$g(x) = \begin{cases} cx^k, & \text{if } x > 0; \\ 0, & \text{if } x = 0; \\ -(-x)^k, & \text{if } x < 0. \end{cases}$$

Then g is nondecreasing, $g(0) = 0$, $g(-1) = -1$, and $g(-xy) = -g(x)g(y)$ for $x < 0 < y$. Hence g satisfies the required conditions.

We obtain all solutions for f by the re-substitution $f(x) = g(x - 1) + 1$. In Case 1, we have any nondecreasing function f satisfying

$$f(x) = \begin{cases} 1, & \text{if } x \geq 1; \\ 0, & \text{if } x = 0. \end{cases}$$

In Case 2, we obtain

$$f(x) = \begin{cases} c(x-1)^k + 1, & \text{if } x > 1; \\ 1, & \text{if } x = 1; \\ -(1-x)^k + 1, & \text{if } x < 1, \end{cases}$$

where $c > 0$ and $k \geq 0$.

A3. Consider pairs of sequences of positive real numbers

$$a_1 \geq a_2 \geq a_3 \geq \cdots, \qquad b_1 \geq b_2 \geq b_3 \geq \cdots$$

and the sums

$$A_n = a_1 + \cdots + a_n, \quad B_n = b_1 + \cdots + b_n; \qquad n = 1, 2, \ldots.$$

For any pair define $c_i = \min\{a_i, b_i\}$ and $C_n = c_1 + \cdots + c_n$, $n = 1, 2, \ldots$.

(1) Does there exist a pair $(a_i)_{i\geq 1}$, $(b_i)_{i\geq 1}$ such that the sequences $(A_n)_{n\geq 1}$ and $(B_n)_{n\geq 1}$ are unbounded while the sequence $(C_n)_{n\geq 1}$ is bounded?

(2) Does the answer to question (1) change by assuming additionally that $b_i = 1/i$, $i = 1, 2, \ldots$?

Justify your answer.

Solution. (1) Yes.

Let (c_i) be an arbitrary sequence of positive numbers such that $c_i \geq c_{i+1}$ and $\sum_{i=1}^{\infty} c_i < \infty$. Let (k_m) be a sequence of integers satisfying $1 = k_1 < k_2 < k_3 < \cdots$ and $(k_{m+1} - k_m)c_{k_m} \geq 1$.

Now we define the sequences (a_i) and (b_i) as follows. For n odd and $k_n \leq i < k_{n+1}$, define $a_i = c_{k_n}$ and $b_i = c_i$. Then we have $A_{k_{n+1}-1} \geq A_{k_n-1} + 1$. For n even and $k_n \leq i < k_{n+1}$, define $a_i = c_i$ and $b_i = c_{k_n}$. Then we have $B_{k_{n+1}-1} \geq B_{k_n-1} + 1$. Thus (A_n) and (B_n) are unbounded and $c_i = \min\{a_i, b_i\}$.

(2) Yes.

Suppose that there is such a pair.

Case 1: $b_i = c_i$ for only finitely many i's.

There exists a sufficiently large I such that $c_i = a_i$ for any $i \geq I$. Therefore

$$\sum_{i\geq I} c_i = \sum_{i\geq I} a_i = \infty,$$

a contradiction.

Case 2: $b_i = c_i$ for infinitely many i's.

Let (k_m) be a sequence of integers satisfying $k_{m+1} \geq 2k_m$ and $b_{k_m} = c_{k_m}$. Then

$$\sum_{k=k_i+1}^{k_{i+1}} c_k \geq (k_{i+1} - k_i)\frac{1}{k_{i+1}} \geq \frac{1}{2}.$$

Thus $\sum_{i=1}^{\infty} c_i = \infty$, a contradiction.

A4. Let n be a positive integer and let $x_1 \le x_2 \le \cdots \le x_n$ be real numbers.

(1) Prove that

$$\left(\sum_{i,j=1}^{n} |x_i - x_j|\right)^2 \le \frac{2(n^2-1)}{3} \sum_{i,j=1}^{n} (x_i - x_j)^2.$$

(2) Show that the equality holds if and only if $x_1, \ldots, x_n$ is an arithmetic sequence.

Solution. (1) Since both sides of the inequality are invariant under any translation of all x_i's, we may assume without loss of generality that $\sum_{i=1}^{n} x_i = 0$.

We have

$$\sum_{i,j=1}^{n} |x_i - x_j| = 2\sum_{i<j}(x_j - x_i) = 2\sum_{i=1}^{n}(2i-n-1)x_i.$$

By the Cauchy-Schwarz inequality, we have

$$\left(\sum_{i,j=1}^{n} |x_i - x_j|\right)^2 \le 4\sum_{i=1}^{n}(2i-n-1)^2 \sum_{i=1}^{n} x_i^2 = 4 \cdot \frac{n(n+1)(n-1)}{3} \sum_{i=1}^{n} x_i^2.$$

On the other hand, we have

$$\sum_{i,j=1}^{n}(x_i - x_j)^2 = n\sum_{i=1}^{n} x_i^2 - \sum_{i=1}^{n} x_i \sum_{j=1}^{n} x_j + n\sum_{j=1}^{n} x_j^2 = 2n\sum_{i=1}^{n} x_i^2.$$

Therefore

$$\left(\sum_{i,j=1}^{n} |x_i - x_j|\right)^2 \le \frac{2(n^2-1)}{3} \sum_{i,j=1}^{n} (x_i - x_j)^2.$$

(2) If the equality holds, then $x_i = k(2i-n-1)$ for some k, which means that $x_1, \ldots, x_n$ is an arithmetic sequence.

On the other hand, suppose that $x_1, \ldots, x_{2n}$ is an arithmetic sequence with common difference d. Then we have

$$x_i = \frac{d}{2}(2i-n-1) + \frac{x_1 + x_n}{2}.$$

Translate x_i's by $-(x_1+x_n)/2$ to obtain $x_i = d(2i-n-1)/2$ and $\sum_{i=1}^{n} x_i = 0$, from which the equality follows.

A5. Let $\mathbb{R}^+$ be the set of all positive real numbers. Find all functions $f\colon \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ that satisfy the following conditions:

(i) $f(xyz) + f(x) + f(y) + f(z) = f(\sqrt{xy})f(\sqrt{yz})f(\sqrt{zx})$ for all $x, y, z \in \mathbb{R}^+$;

(ii) $f(x) < f(y)$ for all $1 \le x < y$.

Solution 1. We claim that $f(x) = x^\lambda + x^{-\lambda}$, where λ is an arbitrary positive real number.

Lemma. There exists a unique function $g\colon [1, \infty) \longrightarrow [1, \infty)$ such that

$$f(x) = g(x) + \frac{1}{g(x)}.$$

Proof. Put $x = y = z = 1$ in the given functional equation

$$f(xyz) + f(x) + f(y) + f(z) = f(\sqrt{xy})f(\sqrt{yz})f(\sqrt{zx})$$

to obtain $4f(1) = f(1)^3$. Since $f(1) > 0$, we have $f(1) = 2$.

Define the function $A\colon [1, \infty) \longrightarrow [2, \infty)$ by $A(x) = x + 1/x$. Since f is strictly increasing on $[1, \infty)$ and A is bijective, the function g is uniquely determined. ∎

Since A is strictly increasing, we see that g is also strictly increasing. Since $f(1) = 2$, we have $g(1) = 1$.

We put $(x, y, z) = (t, t, 1/t), (t^2, 1, 1)$ to obtain $f(t) = f(1/t)$ and $f(t^2) = f(t)^2 - 2$. Put $(x, y, z) = (s/t, t/s, st), (s^2, 1/s^2, t^2)$ to obtain

$$f(st) + f\left(\frac{t}{s}\right) = f(s)f(t) \quad \text{and} \quad f(st)f\left(\frac{t}{s}\right) = f(s^2) + f(t^2) = f(s)^2 + f(t)^2 - 4.$$

Let $1 \le x \le y$. We will show that $g(xy) = g(x)g(y)$. We have

$$\begin{aligned} f(xy) + f\left(\frac{y}{x}\right) &= \left(g(x) + \frac{1}{g(x)}\right)\left(g(y) + \frac{1}{g(y)}\right) \\ &= \left(g(x)g(y) + \frac{1}{g(x)g(y)}\right) + \left(\frac{g(x)}{g(y)} + \frac{g(y)}{g(x)}\right), \end{aligned}$$

and

$$\begin{aligned} f(xy)f\left(\frac{y}{x}\right) &= \left(g(x) + \frac{1}{g(x)}\right)^2 + \left(g(y) + \frac{1}{g(y)}\right)^2 - 4 \\ &= \left(g(x)g(y) + \frac{1}{g(x)g(y)}\right)\left(\frac{g(x)}{g(y)} + \frac{g(y)}{g(x)}\right). \end{aligned}$$

Thus

$$\left\{f(xy), f\left(\frac{y}{x}\right)\right\} = \left\{g(x)g(y) + \frac{1}{g(x)g(y)}, \frac{g(x)}{g(y)} + \frac{g(y)}{g(x)}\right\} = \left\{A\big(g(x)g(y)\big), A\left(\frac{g(y)}{g(x)}\right)\right\}.$$

Since $f(xy) = A\bigl(g(xy)\bigr)$ and A is bijective, it follows that either $g(xy) = g(x)g(y)$ or $g(xy) = g(y)/g(x)$. Since $xy \geq y$ and g is increasing, we have $g(xy) = g(x)g(y)$.

Fix a real number $\varepsilon > 1$ and suppose that $g(\varepsilon) = \varepsilon^\lambda$. Since $g(\varepsilon) > 1$, we have $\lambda > 0$. Using the multiplicity of g, we may easily see that $g(\varepsilon^q) = \varepsilon^{q\lambda}$ for all rationals $q \in [0, \infty)$. Since g is strictly increasing, $g(\varepsilon^t) = \varepsilon^{t\lambda}$ for all $t \in [0, \infty)$, that is, $g(x) = x^\lambda$ for all $x \geq 1$.

For all $x \geq 1$, we have $f(x) = x^\lambda + x^{-\lambda}$. Recalling that $f(t) = f(1/t)$, we have $f(x) = x^\lambda + x^{-\lambda}$ for $0 < x < 1$ as well.

Now we must check that for any $\lambda > 0$, the function $f(x) = x^\lambda + x^{-\lambda}$ satisfies the two given conditions. The condition (i) is satisfied because

$$\begin{aligned} f(\sqrt{xy})f(\sqrt{yz})f(\sqrt{zx}) &= \bigl((xy)^{\lambda/2} + (xy)^{-\lambda/2}\bigr)\bigl((yz)^{\lambda/2} + (yz)^{-\lambda/2}\bigr)\bigl((zx)^{\lambda/2} + (zx)^{-\lambda/2}\bigr) \\ &= (xyz)^\lambda + x^\lambda + y^\lambda + z^\lambda + x^{-\lambda} + y^{-\lambda} + z^{-\lambda} + (xyz)^{-\lambda} \\ &= f(xyz) + f(x) + f(y) + f(z). \end{aligned}$$

The condition (ii) is also satisfied because $1 \leq x < y$ implies

$$f(y) - f(x) = (y^\lambda - x^\lambda)\left(1 - \frac{1}{(xy)^\lambda}\right) > 0.$$

Solution 2. We can a find positive real number λ such that $f(e) = \exp(\lambda) + \exp(-\lambda)$ since the function $B\colon [0, \infty) \longrightarrow [2, \infty)$ defined by $B(x) = \exp(x) + \exp(-x)$ is bijective.

Since $f(t)^2 = f(t^2) + 2$ and $f(x) > 0$, we have

$$f\left(\exp\left(\frac{1}{2^n}\right)\right) = \exp\left(\frac{\lambda}{2^n}\right) + \exp\left(-\frac{\lambda}{2^n}\right)$$

for all nonnegative integers n.

Since $f(st) = f(s)f(t) - f(t/s)$, we have

$$f\left(\exp\left(\frac{m+1}{2^n}\right)\right) = f\left(\exp\left(\frac{1}{2^n}\right)\right)f\left(\exp\left(\frac{m}{2^n}\right)\right) - f\left(\exp\left(\frac{m-1}{2^n}\right)\right) \qquad (*)$$

for all nonnegative integers m and n.

From $(*)$ and $f(1) = 2$, we obtain by induction that

$$f\left(\exp\left(\frac{m}{2^n}\right)\right) = \exp\left(\frac{m\lambda}{2^n}\right) + \exp\left(-\frac{m\lambda}{2^n}\right)$$

for all nonnegative integers m and n.

Since f is increasing on $[1, \infty)$, we have $f(x) = x^\lambda + x^{-\lambda}$ for $x \geq 1$.

We can prove that $f(x) = x^\lambda + x^{-\lambda}$ for $0 < x < 1$ and that this function satisfies the given conditions in the same manner as in the first solution.

A6. Let n be a positive integer and let $(x_1, \ldots, x_n)$, $(y_1, \ldots, y_n)$ be two sequences of positive real numbers. Suppose $(z_2, \ldots, z_{2n})$ is a sequence of positive real numbers such that

$$z_{i+j}^2 \geq x_i y_j \qquad \text{for all } 1 \leq i, j \leq n.$$

Let $M = \max\{z_2, \ldots, z_{2n}\}$. Prove that

$$\left(\frac{M + z_2 + \cdots + z_{2n}}{2n}\right)^2 \geq \left(\frac{x_1 + \cdots + x_n}{n}\right)\left(\frac{y_1 + \cdots + y_n}{n}\right).$$

Solution. Let $X = \max\{x_1, \ldots, x_n\}$ and $Y = \max\{y_1, \ldots, y_n\}$. By replacing x_i by $x_i' = x_i/X$, y_i by $y_i' = y_i/Y$, and z_i by $z_i' = z_i/\sqrt{XY}$, we may assume that $X = Y = 1$. Now we will prove that

$$M + z_2 + \cdots + z_{2n} \geq x_1 + \cdots + x_n + y_1 + \cdots + y_n, \tag{$*$}$$

so

$$\frac{M + z_2 + \cdots + z_{2n}}{2n} \geq \frac{1}{2}\left(\frac{x_1 + \cdots + x_n}{n} + \frac{y_1 + \cdots + y_n}{n}\right)$$

which implies the desired result by the AM-GM inequality.

To prove $(*)$, we will show that for any $r \geq 0$, the number of terms greater that r on the left hand side is at least the number of such terms on the right hand side. Then the kth largest term on the left hand side is greater than or equal to the kth largest term on the right hand side for each k, proving $(*)$. If $r \geq 1$, then there are no terms greater than r on the right hand side. So suppose $r < 1$. Let $A = \{1 \leq i \leq n \mid x_i > r\}$, $a = |A|$, $B = \{1 \leq i \leq n \mid y_i > r\}$, $b = |B|$. Since $\max\{x_1, \ldots, x_n\} = \max\{y_1, \ldots, y_n\} = 1$, both a and b are at least 1. Now $x_i > r$ and $y_j > r$ implies $z_{i+j} \geq \sqrt{x_i y_j} > r$, so

$$C = \{2 \leq i \leq 2n \mid z_i > r\} \supset A + B = \{\alpha + \beta \mid \alpha \in A,\ \beta \in B\}.$$

However, we know that $|A + B| \geq |A| + |B| - 1$, because if $A = \{i_1, \ldots, i_a\}$, $i_1 < \cdots < i_a$ and $B = \{j_1, \ldots, j_b\}$, $j_1 < \cdots < j_b$, then the $a + b - 1$ numbers $i_1 + j_1$, $i_1 + j_2$, $\ldots$, $i_1 + j_b$, $i_2 + j_b$, $\ldots$, $i_a + j_b$ are all distinct and belong to $A + B$. Hence $|C| \geq a + b - 1$. In particular, $|C| \geq 1$ so $z_k > r$ for some k. Then $M > r$, so the left hand side of $(*)$ has at least $a + b$ terms greater than r. Since $a + b$ is the number of terms greater than r on the right hand side, we have proved $(*)$.

Combinatorics

C1. Let A be a 101-element subset of the set $S = \{1, 2, \ldots, 1000000\}$. Prove that there exist numbers $t_1, t_2, \ldots, t_{100}$ in S such that the sets

$$A_j = \{x + t_j \mid x \in A\}, \qquad j = 1, 2, \ldots, 100$$

are pairwise disjoint.

Solution 1. Consider the set $D = \{x - y \mid x, y \in A\}$. There are at most $101 \times 100 + 1 = 10101$ elements in D. Two sets $A + t_i$ and $A + t_j$ have nonempty intersection if and only if $t_i - t_j$ is in D. So we need to choose the 100 elements in such a way that we do not use a difference from D.

Now select these elements by induction. Choose one element arbitrarily. Assume that k elements, $k \leq 99$, are already chosen. An element x that is already chosen prevents us from selecting any element from the set $x + D$. Thus after k elements are chosen, at most $10101k \leq 999999$ elements are forbidden. Hence we can select one more element.

Comment. The size $|S| = 10^6$ is unnecessarily large. The following statement is true:

> If A is a k-element subset of $S = \{1, \ldots, n\}$ and m is a positive integer such that $n > (m-1)\left(\binom{k}{2} + 1\right)$, then there exist $t_1, \ldots, t_m \in S$ such that the sets $A_j = \{x + t_j \mid x \in A\}$, $j = 1, \ldots, m$ are pairwise disjoint.

Solution 2. We give a solution to the generalised version.

Consider the set $B = \big\{|x - y| \bigm| x, y \in A\big\}$. Clearly, $|B| \leq \binom{k}{2} + 1$.

It suffices to prove that there exist $t_1, \ldots, t_m \in S$ such that $|t_i - t_j| \notin B$ for every distinct i and j. We will select $t_1, \ldots, t_m$ inductively.

Choose 1 as t_1, and consider the set $C_1 = S \setminus (B + t_1)$. Then we have $|C_1| \geq n - \left(\binom{k}{2} + 1\right) > (m-2)\left(\binom{k}{2} + 1\right)$.

For $1 \leq i < m$, suppose that $t_1, \ldots, t_i$ and C_i are already defined and that $|C_i| > (m - i - 1)\left(\binom{k}{2} + 1\right) \geq 0$. Choose the least element in C_i as t_{i+1} and consider the set $C_{i+1} = C_i \setminus (B + t_{i+1})$. Then

$$|C_{i+1}| \geq |C_i| - \left(\binom{k}{2} + 1\right) > (m - i - 2)\left(\binom{k}{2} + 1\right) \geq 0.$$

Clearly, $t_1, \ldots, t_m$ satisfy the desired condition.

C2. Let D_1, …, D_n be closed discs in the plane. (A closed disc is the region limited by a circle, taken jointly with this circle.) Suppose that every point in the plane is contained in at most 2003 discs D_i. Prove that there exists a disc D_k which intersects at most $7 \cdot 2003 - 1$ other discs D_i.

Solution. Pick a disc S with the smallest radius, say s. Subdivide the plane into seven regions as in Figure 1, that is, subdivide the complement of S into six congruent regions T_1, …, T_6.

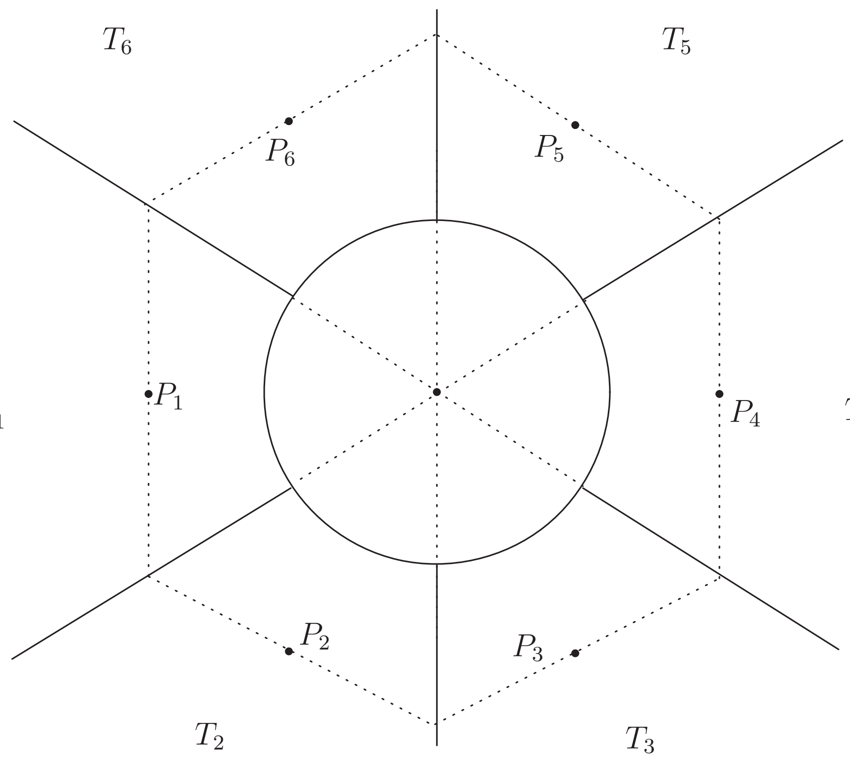


Figure 1

Since s is the smallest radius, any disc different from S whose centre lies inside S contains the centre O of the disc S. Therefore the number of such discs is less than or equal to 2002.

We will show that if a disc D_k has its centre inside T_i and intersects S, then D_k contains P_i, where P_i is the point such that $OP_i = \sqrt{3}\, s$ and OP_i bisects the angle formed by the two half-lines that bound T_i.

Subdivide T_i into U_i and V_i as in Figure 2.

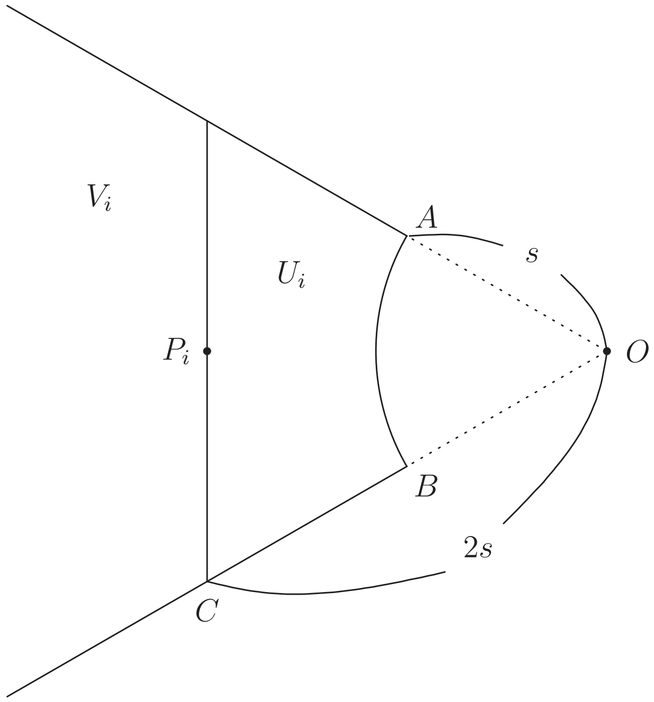


Figure 2

The region U_i is contained in the disc with radius s and centre P_i. Thus, if the centre of D_k is inside U_i, then D_k contains P_i.

Suppose that the centre of D_k is inside V_i. Let Q be the centre of D_k and let R be the intersection of OQ and the boundary of S. Since D_k intersects S, the radius of D_k is greater than QR. Since $\angle QP_iR \geq \angle CP_iB = 60^\circ$ and $\angle P_iRO \geq \angle P_iBO = 120^\circ$, we have $\angle QP_iR \geq \angle P_iRQ$. Hence $QR \geq QP_i$ and so D_k contains P_i.

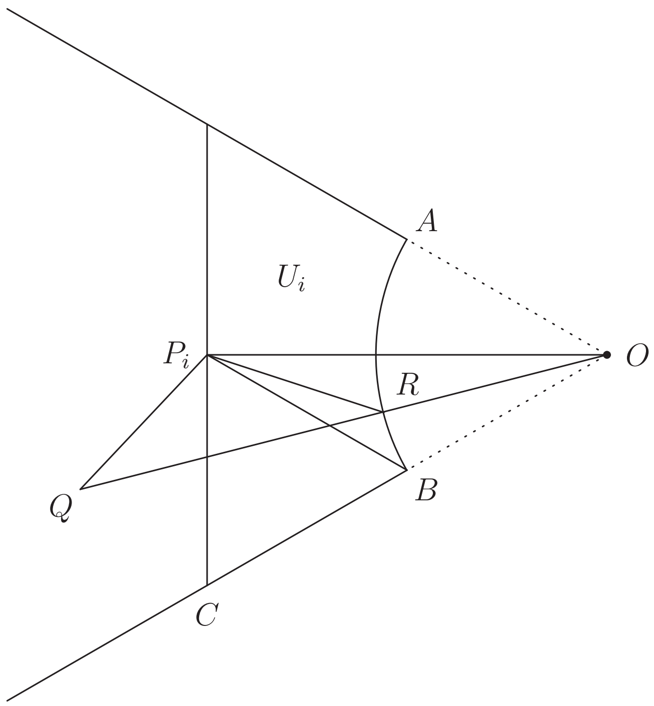


Figure 3

For $i = 1, \ldots, 6$, the number of discs D_k having their centres inside T_i and intersecting S is less than or equal to 2003. Consequently, the number of discs D_k that intersect S is less than or equal to $2002 + 6 \cdot 2003 = 7 \cdot 2003 - 1$.

C3. Let $n \geq 5$ be a given integer. Determine the greatest integer k for which there exists a polygon with n vertices (convex or not, with non-selfintersecting boundary) having k internal right angles.

Solution. We will show that the greatest integer k satisfying the given condition is equal to 3 for $n = 5$, and $\lfloor 2n/3 \rfloor + 1$ for $n \geq 6$.

Assume that there exists an n-gon having k internal right angles. Since all other $n - k$ angles are less than $360°$, we have

$$(n-k) \cdot 360° + k \cdot 90° > (n-2) \cdot 180°,$$

or $k < (2n+4)/3$. Since k and n are integers, we have $k \leq \lfloor 2n/3 \rfloor + 1$.

If $n = 5$, then $\lfloor 2n/3 \rfloor + 1 = 4$. However, if a pentagon has 4 internal right angles, then the other angle is equal to $180°$, which is not appropriate. Figure 1 gives the pentagon with 3 internal right angles, thus the greatest integer k is equal to 3.

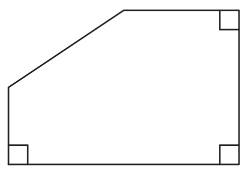


Figure 1

We will construct an n-gon having $\lfloor 2n/3 \rfloor + 1$ internal right angles for each $n \geq 6$. Figure 2 gives the examples for $n = 6, 7, 8$.

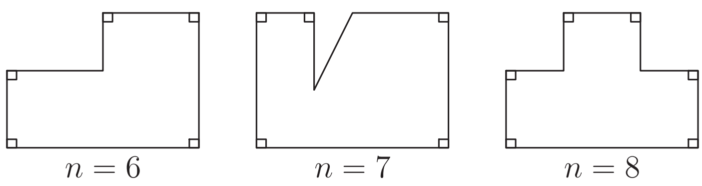


Figure 2

For $n \geq 9$, we will construct examples inductively. Since all internal non-right angles in this construction are greater than $180°$, we can cut off 'a triangle without a vertex' around a non-right angle in order to obtain three more vertices and two more internal right angles as in Figure 3.

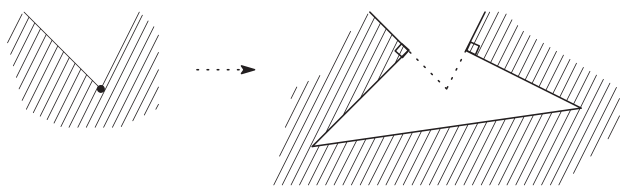


Figure 3

Comment. Here we give two other ways to construct examples.

One way is to add 'a rectangle with a hat' near an internal non-right angle as in Figure 4.

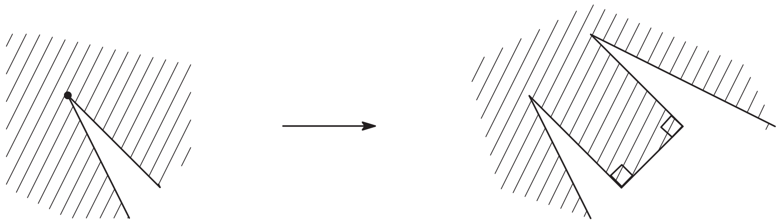
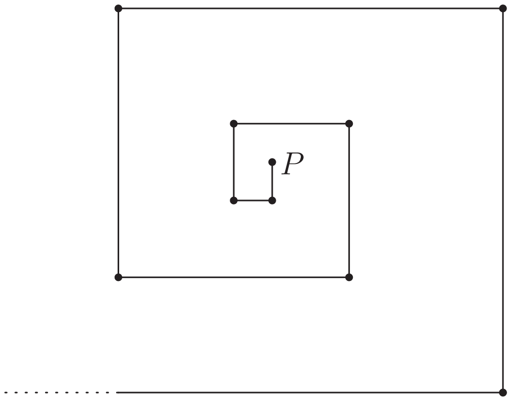
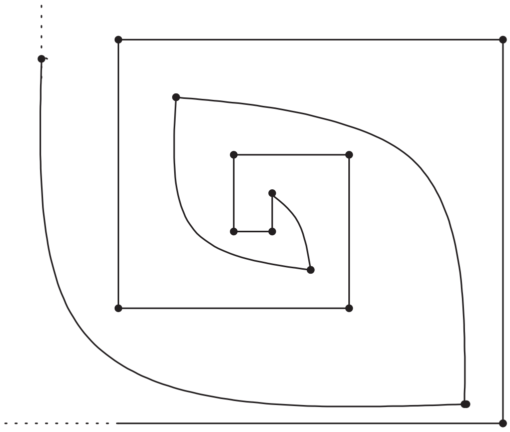


Figure 4

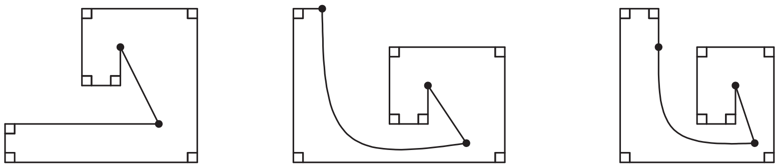
The other way is 'the escaping construction.' First we draw right angles in spiral.



Then we 'escape' from the point P.



The followings are examples for $n = 9, 10, 11$. The angles around the black points are not right.



$n = 9$ $n = 10$ $n = 11$

The 'escaping lines' are not straight in these figures. However, in fact, we can make them straight when we draw sufficiently large figures.

C4. Let $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ be real numbers. Let $A = (a_{ij})_{1\le i,j\le n}$ be the matrix with entries

$$a_{ij} = \begin{cases} 1, & \text{if } x_i + y_j \ge 0; \\ 0, & \text{if } x_i + y_j < 0. \end{cases}$$

Suppose that B is an $n \times n$ matrix with entries 0, 1 such that the sum of the elements in each row and each column of B is equal to the corresponding sum for the matrix A. Prove that $A = B$.

Solution 1. Let $B = (b_{ij})_{1\le i,j\le n}$. Define $S = \sum_{1\le i,j\le n}(x_i + y_j)(a_{ij} - b_{ij})$.

On one hand, we have

$$S = \sum_{i=1}^{n} x_i \left(\sum_{j=1}^{n} a_{ij} - \sum_{j=1}^{n} b_{ij} \right) + \sum_{j=1}^{n} y_j \left(\sum_{i=1}^{n} a_{ij} - \sum_{i=1}^{n} b_{ij} \right) = 0.$$

On the other hand, if $x_i + y_j \ge 0$, then $a_{ij} = 1$, which implies $a_{ij} - b_{ij} \ge 0$; if $x_i + y_j < 0$, then $a_{ij} = 0$, which implies $a_{ij} - b_{ij} \le 0$. Therefore $(x_i + y_j)(a_{ij} - b_{ij}) \ge 0$ for every i and j.

Thus we have $(x_i + y_j)(a_{ij} - b_{ij}) = 0$ for every i and j. In particular, if $a_{ij} = 0$, then $x_i + y_j < 0$ and so $a_{ij} - b_{ij} = 0$. This means that $a_{ij} \ge b_{ij}$ for every i and j.

Since the sum of the elements in each row of B is equal to the corresponding sum for A, we have $a_{ij} = b_{ij}$ for every i and j.

Solution 2. Let $B = (b_{ij})_{1\le i,j\le n}$. Suppose that $A \ne B$, that is, there exists (i_0, j_0) such that $a_{i_0j_0} \ne b_{i_0j_0}$. We may assume without loss of generality that $a_{i_0j_0} = 0$ and $b_{i_0j_0} = 1$.

Since the sum of the elements in the i_0-th row of B is equal to that in A, there exists j_1 such that $a_{i_0j_1} = 1$ and $b_{i_0j_1} = 0$. Similarly there exists i_1 such that $a_{i_1j_1} = 0$ and $b_{i_1j_1} = 1$. Let us define i_k and j_k inductively in this way so that $a_{i_kj_k} = 0$, $b_{i_kj_k} = 1$, $a_{i_kj_{k+1}} = 1$, $b_{i_kj_{k+1}} = 0$.

Because the size of the matrix is finite, there exist s and t such that $s \ne t$ and $(i_s, j_s) = (i_t, j_t)$.

Since $a_{i_kj_k} = 0$ implies $x_{i_k} + y_{j_k} < 0$ by definition, we have $\sum_{k=s}^{t-1}(x_{i_k} + y_{j_k}) < 0$. Similarly, since $a_{i_kj_{k+1}} = 1$ implies $x_{i_k} + y_{j_{k+1}} \ge 0$, we have $\sum_{k=s}^{t-1}(x_{i_k} + y_{j_{k+1}}) \ge 0$. However, since $j_s = j_t$, we have

$$\sum_{k=s}^{t-1}(x_{i_k} + y_{j_{k+1}}) = \sum_{k=s}^{t-1} x_{i_k} + \sum_{k=s+1}^{t} y_{j_k} = \sum_{k=s}^{t-1} x_{i_k} + \sum_{k=s}^{t-1} y_{j_k} = \sum_{k=s}^{t-1}(x_{i_k} + y_{j_k}).$$

This is a contradiction.

C5. Every point with integer coordinates in the plane is the centre of a disc with radius 1/1000.

(1) Prove that there exists an equilateral triangle whose vertices lie in different discs.

(2) Prove that every equilateral triangle with vertices in different discs has side-length greater than 96.

Solution 1. (1) Define $f\colon \mathbb{Z} \longrightarrow [0,1)$ by $f(x) = x\sqrt{3} - \lfloor x\sqrt{3} \rfloor$. By the pigeonhole principle, there exist distinct integers x_1 and x_2 such that $\bigl|f(x_1) - f(x_2)\bigr| < 0.001$. Put $a = |x_1 - x_2|$. Then the distance either between $\bigl(a, a\sqrt{3}\bigr)$ and $\bigl(a, \lfloor a\sqrt{3} \rfloor\bigr)$ or between $\bigl(a, a\sqrt{3}\bigr)$ and $\bigl(a, \lfloor a\sqrt{3} \rfloor + 1\bigr)$ is less than 0.001. Therefore the points $(0,0)$, $(2a,0)$, $\bigl(a, a\sqrt{3}\bigr)$ lie in different discs and form an equilateral triangle.

(2) Suppose that $P'Q'R'$ is a triangle such that $P'Q' = Q'R' = R'P' = l \leq 96$ and P', Q', R' lie in discs with centres P, Q, R, respectively. Then

$$l - 0.002 \leq PQ, QR, RP \leq l + 0.002.$$

Since PQR is not an equilateral triangle, we may assume that $PQ \neq QR$. Therefore

$$\begin{aligned} |PQ^2 - QR^2| &= (PQ + QR)|PQ - QR| \\ &\leq \bigl((l + 0.002) + (l + 0.002)\bigr)\bigl((l + 0.002) - (l - 0.002)\bigr) \\ &\leq 2 \cdot 96.002 \cdot 0.004 \\ &< 1. \end{aligned}$$

However, $PQ^2 - QR^2 \in \mathbb{Z}$. This is a contradiction.

Solution 2. We give another solution to (2).

Lemma. Suppose that ABC and $A'B'C'$ are equilateral triangles and that A, B, C and A', B', C' lie anticlockwise. If $AA', BB' \leq r$, then $CC' \leq 2r$.

Proof. Let α, β, γ; α', β', γ' be the complex numbers corresponding to A, B, C; A', B', C'. Then

$$\gamma = \omega\beta + (1 - \omega)\alpha \quad \text{and} \quad \gamma' = \omega\beta' + (1 - \omega)\alpha',$$

where $\omega = \bigl(1 + \sqrt{3}\, i\bigr)/2$. Therefore

$$\begin{aligned} CC' = |\gamma - \gamma'| &= \bigl|\omega(\beta - \beta') + (1 - \omega)(\alpha - \alpha')\bigr| \\ &\leq |\omega||\beta - \beta'| + |1 - \omega||\alpha - \alpha'| = BB' + AA' \\ &\leq 2r. \end{aligned}$$

∎

Suppose that P, Q, R lie on discs with radius r and centres P', Q', R', respectively, and that PQR is an equilateral triangle. Let R'' be the point such that $P'Q'R''$ is an equilateral triangle and P', Q', R' lie anticlockwise. It follows from the lemma that $RR'' \le 2r$, and so $R'R'' \le RR' + RR'' \le r + 2r = 3r$ by the triangle inequality.

Put $\overrightarrow{P'Q'} = \begin{pmatrix} m \\ n \end{pmatrix}$ and $\overrightarrow{P'R'} = \begin{pmatrix} s \\ t \end{pmatrix}$, where m, n, s, t are integers. We may suppose that $m, n \ge 0$. Then we have

$$\sqrt{\left(\frac{m - n\sqrt{3}}{2} - s\right)^2 + \left(\frac{n + m\sqrt{3}}{2} - t\right)^2} \le 3r.$$

Setting $a = 2t - n$ and $b = m - 2s$, we obtain

$$\sqrt{\left(a - m\sqrt{3}\right)^2 + \left(b - n\sqrt{3}\right)^2} \le 6r.$$

Since $\left|a - m\sqrt{3}\right| \ge 1/\left|a + m\sqrt{3}\right|$, $\left|b - n\sqrt{3}\right| \ge 1/\left|b + n\sqrt{3}\right|$ and $|a| \le m\sqrt{3} + 6r$, $|b| \le n\sqrt{3} + 6r$, we have

$$\sqrt{\frac{1}{\left(2m\sqrt{3} + 6r\right)^2} + \frac{1}{\left(2n\sqrt{3} + 6r\right)^2}} \le 6r.$$

Since $1/x^2 + 1/y^2 \ge 8/(x+y)^2$ for all positive real numbers x and y, it follows that

$$\frac{2\sqrt{2}}{2\sqrt{3}(m+n) + 12r} \le 6r.$$

As $P'Q' = \sqrt{m^2 + n^2} \ge (m+n)/\sqrt{2}$, we have

$$\frac{2\sqrt{2}}{2\sqrt{6}\,P'Q' + 12r} \le 6r.$$

Therefore

$$P'Q' \ge \frac{1}{6\sqrt{3}\,r} - \sqrt{6}\,r.$$

Finally we obtain

$$PQ \ge P'Q' - 2r \ge \frac{1}{6\sqrt{3}\,r} - \sqrt{6}\,r - 2r.$$

For $r = 1/1000$, we have $PQ \ge 96.22\cdots > 96$.

C6. Let $f(k)$ be the number of integers n that satisfy the following conditions:

(i) $0 \le n < 10^k$, so n has exactly k digits (in decimal notation), with leading zeroes allowed;

(ii) the digits of n can be permuted in such a way that they yield an integer divisible by 11.

Prove that $f(2m) = 10f(2m-1)$ for every positive integer m.

Solution 1. We use the notation $[a_{k-1}a_{k-2}\cdots a_0]$ to indicate the positive integer with digits $a_{k-1}, a_{k-2}, \ldots, a_0$.

The following fact is well-known:

$$[a_{k-1}a_{k-2}\cdots a_0] \equiv i \pmod{11} \iff \sum_{l=0}^{k-1}(-1)^l a_l \equiv i \pmod{11}.$$

Fix $m \in \mathbb{N}$ and define the sets A_i and B_i as follows:

- A_i is the set of all integers n with the following properties:
  (1) $0 \le n < 10^{2m}$, i.e., n has $2m$ digits;
  (2) the right $2m-1$ digits of n can be permuted so that the resulting integer is congruent to i modulo 11.
- B_i is the set of all integers n with the following properties:
  (1) $0 \le n < 10^{2m-1}$, i.e., n has $2m-1$ digits;
  (2) the digits of n can be permuted so that the resulting integer is congruent to i modulo 11.

It is clear that $f(2m) = |A_0|$ and $f(2m-1) = |B_0|$. Since $\underbrace{99\cdots9}_{2m} \equiv 0 \pmod{11}$, we have

$$n \in A_i \iff \underbrace{99\cdots9}_{2m} - n \in A_{-i}.$$

Hence

$$|A_i| = |A_{-i}|. \tag{1}$$

Since $\underbrace{99\cdots9}_{2m-1} \equiv 9 \pmod{11}$, we have

$$n \in B_i \iff \underbrace{99\cdots9}_{2m-1} - n \in B_{9-i}.$$

Thus

$$|B_i| = |B_{9-i}|. \tag{2}$$

For any $2m$-digit integer $n = [ja_{2m-2}\cdots a_0]$, we have

$$n \in A_i \iff [a_{2m-2}\cdots a_0] \in B_{i-j}.$$

Hence

$$|A_i| = |B_i| + |B_{i-1}| + \cdots + |B_{i-9}|.$$

Since $B_i = B_{i+11}$, this can be written as

$$|A_i| = \sum_{k=0}^{10} |B_k| - |B_{i+1}|, \tag{3}$$

hence

$$|A_i| = |A_j| \iff |B_{i+1}| = |B_{j+1}|. \tag{4}$$

From (1), (2), and (4), we obtain $|A_i| = |A_0|$ and $|B_i| = |B_0|$. Substituting this into (3) yields $|A_0| = 10|B_0|$, and so $f(2m) = 10f(2m-1)$.

Comment. This solution works for all even bases b, and the result is $f(2m) = bf(2m-1)$.

Solution 2. We will use the notation in Solution 1. For a $2m$-tuple $(a_0, \ldots, a_{2m-1})$ of integers, we consider the following property:

$$(a_0, \ldots, a_{2m-1}) \text{ can be permuted so that } \sum_{l=0}^{2m-1} (-1)^l a_l \equiv 0 \pmod{11}. \tag{$*$}$$

It is easy to verify that

$$(a_0, \ldots, a_{2m-1}) \text{ satisfies } (*) \iff (a_0 + k, \ldots, a_{2m-1} + k) \text{ satisfies } (*) \tag{1}$$

for all integers k, and that

$$(a_0, \ldots, a_{2m-1}) \text{ satisfies } (*) \iff (ka_0, \ldots, ka_{2m-1}) \text{ satisfies } (*) \tag{2}$$

for all integers $k \not\equiv 0 \pmod{11}$.

For an integer k, denote by $\langle k \rangle$ the nonnegative integer less than 11 congruent to k modulo 11.

For a fixed $j \in \{0, 1, \ldots, 9\}$, let k be the unique integer such that $k \in \{1, 2, \ldots, 10\}$ and $(j+1)k \equiv 1 \pmod{11}$.

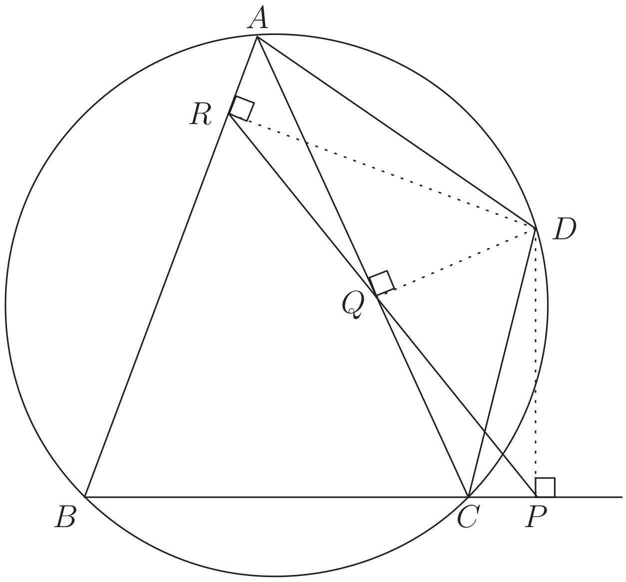
Suppose that $[a_{2m-1} \cdots a_1 j] \in A_0$, that is, $(a_{2m-1}, \ldots, a_1, j)$ satisfies $(*)$. From (1) and (2), it follows that $\big((a_{2m-1}+1)k - 1, \ldots, (a_1+1)k - 1, 0\big)$ also satisfies $(*)$. Putting $b_i = \big\langle (a_i+1)k \big\rangle - 1$, we have $[b_{2m-1} \cdots b_1] \in B_0$.

For any $j \in \{0, 1, \ldots, 9\}$, we can reconstruct $[a_{2m-1} \ldots a_1 j]$ from $[b_{2m-1} \cdots b_1]$. Hence we have $|A_0| = 10|B_0|$, and so $f(2m) = 10f(2m-1)$.

Geometry

G1. Let $ABCD$ be a cyclic quadrilateral. Let P, Q, R be the feet of the perpendiculars from D to the lines BC, CA, AB, respectively. Show that $PQ = QR$ if and only if the bisectors of $\angle ABC$ and $\angle ADC$ are concurrent with AC.

Solution 1.



It is well-known that P, Q, R are collinear (Simson's theorem). Moreover, since $\angle DPC$ and $\angle DQC$ are right angles, the points D, P, Q, C are concyclic and so $\angle DCA = \angle DPQ = \angle DPR$. Similarly, since D, Q, R, A are concyclic, we have $\angle DAC = \angle DRP$. Therefore $\triangle DCA \sim \triangle DPR$.

Likewise, $\triangle DAB \sim \triangle DQP$ and $\triangle DBC \sim \triangle DRQ$. Then

$$\frac{DA}{DC} = \frac{DR}{DP} = \frac{DB \cdot \frac{QR}{BC}}{DB \cdot \frac{PQ}{BA}} = \frac{QR}{PQ} \cdot \frac{BA}{BC}.$$

Thus $PQ = QR$ if and only if $DA/DC = BA/BC$.

Now the bisectors of the angles ABC and ADC divide AC in the ratios of BA/BC and DA/DC, respectively. This completes the proof.

Solution 2. Suppose that the bisectors of $\angle ABC$ and $\angle ADC$ meet AC at L and M, respectively. Since $AL/CL = AB/CB$ and $AM/CM = AD/CD$, the bisectors in question meet on AC if and only if $AB/CB = AD/CD$, that is, $AB \cdot CD = CB \cdot AD$. We will prove that $AB \cdot CD = CB \cdot AD$ is equivalent to $PQ = QR$.

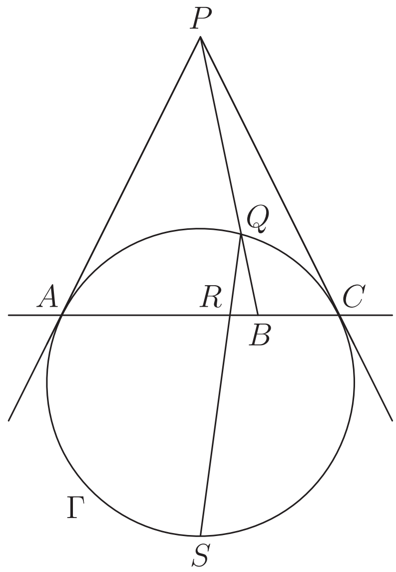
Because $DP \perp BC$, $DQ \perp AC$, $DR \perp AB$, the circles with diameters DC and DA contain the pairs of points P, Q and Q, R, respectively. It follows that $\angle PDQ$ is equal to γ or $180^\circ - \gamma$, where $\gamma = \angle ACB$. Likewise, $\angle QDR$ is equal to α or $180^\circ - \alpha$, where $\alpha = \angle CAB$. Then, by the law of sines, we have $PQ = CD \sin\gamma$ and $QR = AD \sin\alpha$. Hence the condition $PQ = QR$ is equivalent to $CD/AD = \sin\alpha/\sin\gamma$.

On the other hand, $\sin\alpha/\sin\gamma = CB/AB$ by the law of sines again. Thus $PQ = QR$ if and only if $CD/AD = CB/AB$, which is the same as $AB \cdot CD = CB \cdot AD$.

Comment. Solution 2 shows that this problem can be solved without the knowledge of Simson's theorem.

G2. Three distinct points A, B, C are fixed on a line in this order. Let Γ be a circle passing through A and C whose centre does not lie on the line AC. Denote by P the intersection of the tangents to Γ at A and C. Suppose Γ meets the segment PB at Q. Prove that the intersection of the bisector of $\angle AQC$ and the line AC does not depend on the choice of Γ.

Solution 1.



Suppose that the bisector of $\angle AQC$ intersects the line AC and the circle Γ at R and S, respectively, where S is not equal to Q.

Since $\triangle APC$ is an isosceles triangle, we have $AB : BC = \sin\angle APB : \sin\angle CPB$. Likewise, since $\triangle ASC$ is an isosceles triangle, we have $AR : RC = \sin\angle ASQ : \sin\angle CSQ$.

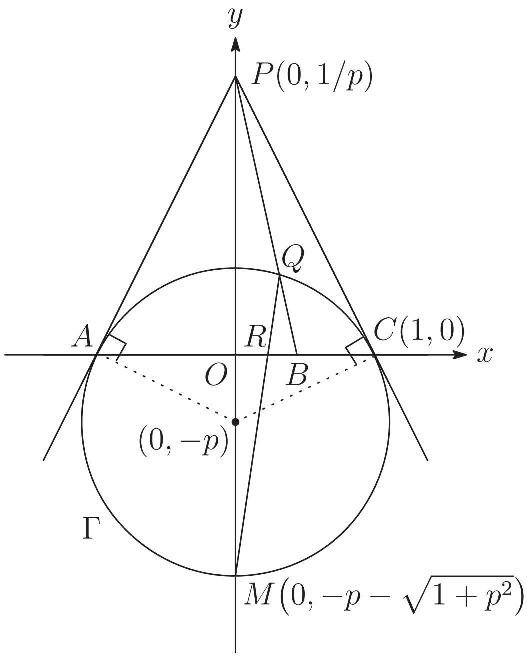
Applying the sine version of Ceva's theorem to the triangle PAC and Q, we obtain

$$\sin\angle APB : \sin\angle CPB = \sin\angle PAQ \sin\angle QCA : \sin\angle PCQ \sin\angle QAC.$$

The tangent theorem shows that $\angle PAQ = \angle ASQ = \angle QCA$ and $\angle PCQ = \angle CSQ = \angle QAC$.

Hence $AB : BC = AR^2 : RC^2$, and so R does not depend on Γ.

Solution 2.



Let R be the intersection of the bisector of the angle AQC and the line AC.

We may assume that $A(-1,0)$, $B(b,0)$, $C(1,0)$, and $\Gamma\colon x^2+(y+p)^2=1+p^2$. Then $P(0,1/p)$.

Let M be the midpoint of the largest arc AC. Then $M\big(0,-p-\sqrt{1+p^2}\big)$. The points Q, R, M are collinear, since $\angle AQR=\angle CQR$.

Because $PB\colon y=-x/pb+1/p$, computation shows that

$$Q\left(\frac{(1+p^2)b-pb\sqrt{(1+p^2)(1-b^2)}}{1+p^2b^2},\frac{-p(1-b^2)+\sqrt{(1+p^2)(1-b^2)}}{1+p^2b^2}\right),$$

so we have

$$\frac{QP}{BQ}=\frac{\sqrt{1+p^2}}{p\sqrt{1-b^2}}.$$

Since

$$\frac{MO}{PM}=\frac{p+\sqrt{1+p^2}}{\frac{1}{p}+p+\sqrt{1+p^2}}=\frac{p}{\sqrt{1+p^2}},$$

we obtain

$$\frac{OR}{RB}=\frac{MO}{PM}\cdot\frac{QP}{BQ}=\frac{p}{\sqrt{1+p^2}}\cdot\frac{\sqrt{1+p^2}}{p\sqrt{1-b^2}}=\frac{1}{\sqrt{1-b^2}}.$$

Therefore R does not depend on p or Γ.

G3. Let ABC be a triangle and let P be a point in its interior. Denote by D, E, F the feet of the perpendiculars from P to the lines BC, CA, AB, respectively. Suppose that

$$AP^2 + PD^2 = BP^2 + PE^2 = CP^2 + PF^2.$$

Denote by I_A, I_B, I_C the excentres of the triangle ABC. Prove that P is the circumcentre of the triangle $I_AI_BI_C$.

Solution. Since the given condition implies

$$0 = (BP^2 + PE^2) - (CP^2 + PF^2) = (BP^2 - PF^2) - (CP^2 - PE^2) = BF^2 - CE^2,$$

we may put $x = BF = CE$. Similarly we may put $y = CD = AF$ and $z = AE = BD$.

If one of three points D, E, F does not lie on the sides of the triangle ABC, then this contradicts the triangle inequality. Indeed, if, for example, B, C, D lie in this order, we have $AB + BC = (x + y) + (z - y) = x + z = AC$, a contradiction. Thus all three points lie on the sides of the triangle ABC.

Putting $a = BC$, $b = CA$, $c = AB$ and $s = (a + b + c)/2$, we have $x = s - a$, $y = s - b$, $z = s - c$. Since $BD = s - c$ and $CD = s - b$, we see that D is the point at which the excircle of the triangle ABC opposite to A meets BC. Similarly E and F are the points at which the excircle opposite to B and C meet CA and AB, respectively. Since both PD and I_AD are perpendicular to BC, the three points P, D, I_A are collinear. Analogously P, E, I_B are collinear and P, F, I_C are collinear.

The three points I_A, C, I_B are collinear and the triangle PI_AI_B is isosceles because $\angle PI_AC = \angle PI_BC = \angle C/2$. Likewise we have $PI_A = PI_C$ and so $PI_A = PI_B = PI_C$. Thus P is the circumcentre of the triangle $I_AI_BI_C$.

Comment 1. The conclusion is true even if the point P lies outside the triangle ABC.

Comment 2. In fact, the common value of $AP^2 + PD^2$, $BP^2 + PE^2$, $CP^2 + PF^2$ is equal to $8R^2 - s^2$, where R is the circumradius of the triangle ABC and $s = (BC + CA + AB)/2$. We can prove this as follows:

Observe that the circumradius of the triangle $I_AI_BI_C$ is equal to $2R$ since its orthic triangle is ABC. It follows that $PD = PI_A - DI_A = 2R - r_A$, where r_A is the radius of the excircle of the triangle ABC opposite to A. Putting r_B and r_C in a similar manner, we have $PE = 2R - r_B$ and $PF = 2R - r_C$. Now we have

$$AP^2 + PD^2 = AE^2 + PE^2 + PD^2 = (s - c)^2 + (2R - r_B)^2 + (2R - r_A)^2.$$

Since

$$\begin{aligned}
(2R - r_A)^2 &= 4R^2 - 4Rr_A + r_A^2 \\
&= 4R^2 - 4 \cdot \frac{abc}{4\,\text{area}(\triangle ABC)} \cdot \frac{\text{area}(\triangle ABC)}{s - a} + \left(\frac{\text{area}(\triangle ABC)}{s - a}\right)^2 \\
&= 4R^2 + \frac{s(s - b)(s - c) - abc}{s - a} \\
&= 4R^2 + bc - s^2
\end{aligned}$$

and we can obtain $(2R - r_B)^2 = 4R^2 + ca - s^2$ in a similar way, it follows that

$$AP^2 + PD^2 = (s - c)^2 + (4R^2 + ca - s^2) + (4R^2 + bc - s^2) = 8R^2 - s^2.$$

G4. Let Γ_1, Γ_2, Γ_3, Γ_4 be distinct circles such that Γ_1, Γ_3 are externally tangent at P, and Γ_2, Γ_4 are externally tangent at the same point P. Suppose that Γ_1 and Γ_2; Γ_2 and Γ_3; Γ_3 and Γ_4; Γ_4 and Γ_1 meet at A, B, C, D, respectively, and that all these points are different from P.

Prove that

$$\frac{AB \cdot BC}{AD \cdot DC} = \frac{PB^2}{PD^2}.$$

Solution 1.

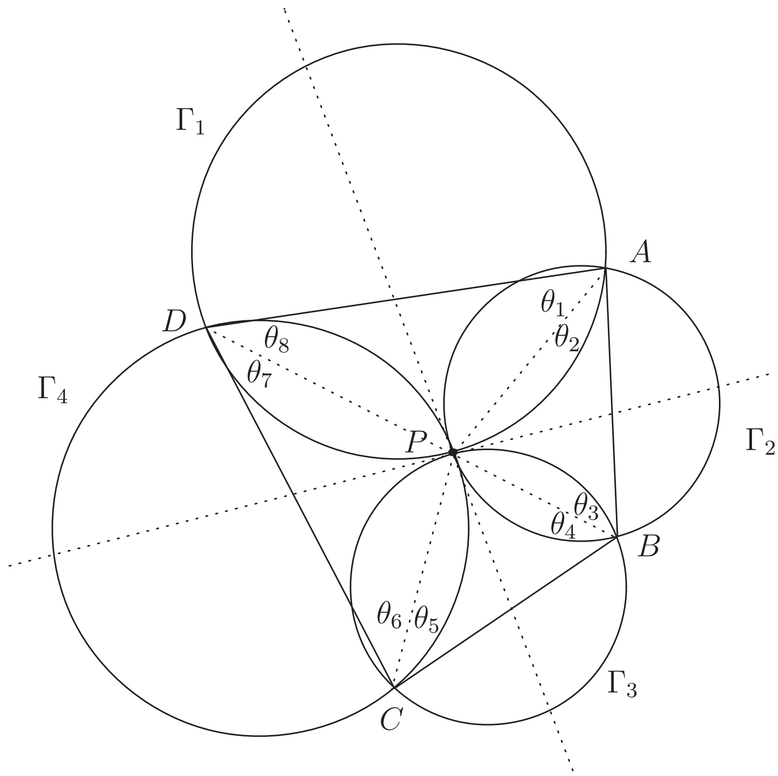


Figure 1

Let Q be the intersection of the line AB and the common tangent of Γ_1 and Γ_3. Then

$$\angle APB = \angle APQ + \angle BPQ = \angle PDA + \angle PCB.$$

Define $\theta_1, \ldots, \theta_8$ as in Figure 1. Then

$$\theta_2 + \theta_3 + \angle APB = \theta_2 + \theta_3 + \theta_5 + \theta_8 = 180°. \tag{1}$$

Similarly, $\angle BPC = \angle PAB + \angle PDC$ and

$$\theta_4 + \theta_5 + \theta_2 + \theta_7 = 180°. \tag{2}$$

Multiply the side-lengths of the triangles PAB, PBC, PCD, PAD by $PC \cdot PD$, $PD \cdot PA$, $PA \cdot PB$, $PB \cdot PC$, respectively, to get the new quadrilateral $A'B'C'D'$ as in Figure 2.

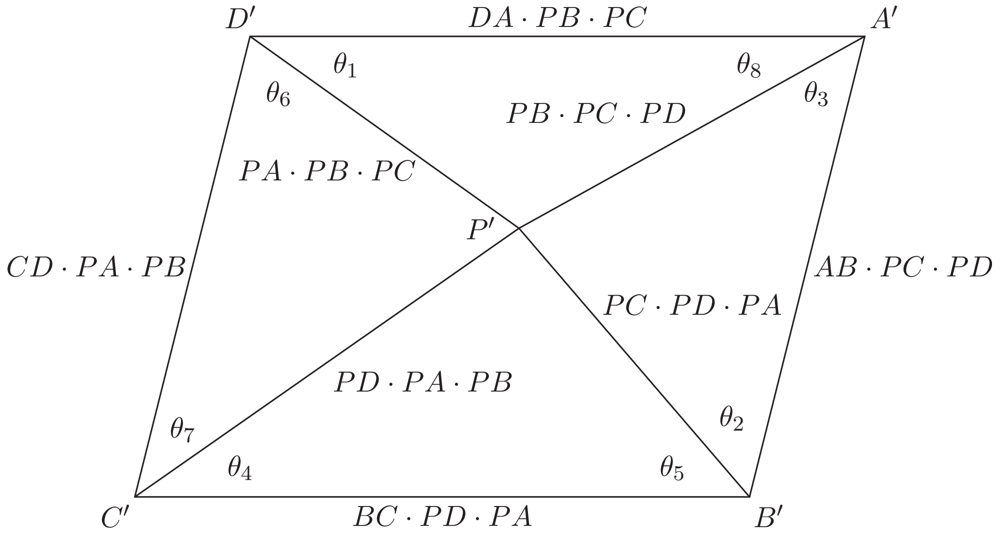
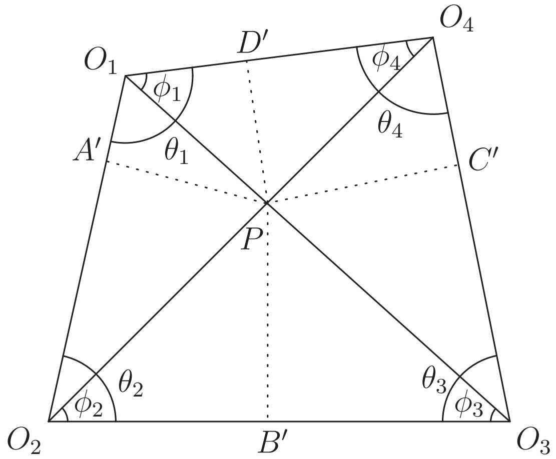


Figure 2

(1) and (2) show that $A'D' \parallel B'C'$ and $A'B' \parallel C'D'$. Thus the quadrilateral $A'B'C'D'$ is a parallelogram. It follows that $A'B' = C'D'$ and $A'D' = C'B'$, that is, $AB \cdot PC \cdot PD = CD \cdot PA \cdot PB$ and $AD \cdot PB \cdot PC = BC \cdot PA \cdot PD$, from which we see that

$$\frac{AB \cdot BC}{AD \cdot DC} = \frac{PB^2}{PD^2}.$$

Solution 2. Let O_1, O_2, O_3, O_4 be the centres of Γ_1, Γ_2, Γ_3, Γ_4, respectively, and let A', B', C', D' be the midpoints of PA, PB, PC, PD, respectively. Since Γ_1, Γ_3 are externally tangent at P, it follows that O_1, O_3, P are collinear. Similarly we see that O_2, O_4, P are collinear.



Put $\theta_1 = \angle O_4O_1O_2$, $\theta_2 = \angle O_1O_2O_3$, $\theta_3 = \angle O_2O_3O_4$, $\theta_4 = \angle O_3O_4O_1$ and $\phi_1 = \angle PO_1O_4$, $\phi_2 = \angle PO_2O_3$, $\phi_3 = \angle PO_3O_2$, $\phi_4 = \angle PO_4O_1$. By the law of sines, we have

$$O_1O_2 : O_1O_3 = \sin\phi_3 : \sin\theta_2, \qquad O_3O_4 : O_2O_4 = \sin\phi_2 : \sin\theta_3,$$
$$O_3O_4 : O_1O_3 = \sin\phi_1 : \sin\theta_4, \qquad O_1O_2 : O_2O_4 = \sin\phi_4 : \sin\theta_1.$$

Since the segment PA is the common chord of Γ_1 and Γ_2, the segment PA' is the altitude from P to O_1O_2. Similarly PB', PC', PD' are the altitudes from P to O_2O_3, O_3O_4, O_4O_1, respectively. Then O_1, A', P, D' are concyclic. So again by the law of sines, we have

$$D'A' : PD' = \sin\theta_1 : \sin\phi_1.$$

Likewise we have

$$A'B' : PB' = \sin\theta_2 : \sin\phi_2, \quad B'C' : PB' = \sin\theta_3 : \sin\phi_3, \quad C'D' : PD' = \sin\theta_4 : \sin\phi_4.$$

Since $A'B' = AB/2$, $B'C' = BC/2$, $C'D' = CD/2$, $D'A' = DA/2$, $PB' = PB/2$, $PD' = PD/2$, we have

$$\begin{aligned}\frac{AB \cdot BC}{AD \cdot DC} \cdot \frac{PD^2}{PB^2} &= \frac{A'B' \cdot B'C'}{A'D' \cdot D'C'} \cdot \frac{PD'^2}{PB'^2} = \frac{\sin\theta_2 \sin\theta_3 \sin\phi_4 \sin\phi_1}{\sin\phi_2 \sin\phi_3 \sin\theta_4 \sin\theta_1} \\ &= \frac{O_1O_3}{O_1O_2} \cdot \frac{O_2O_4}{O_3O_4} \cdot \frac{O_1O_2}{O_2O_4} \cdot \frac{O_3O_4}{O_1O_3} = 1\end{aligned}$$

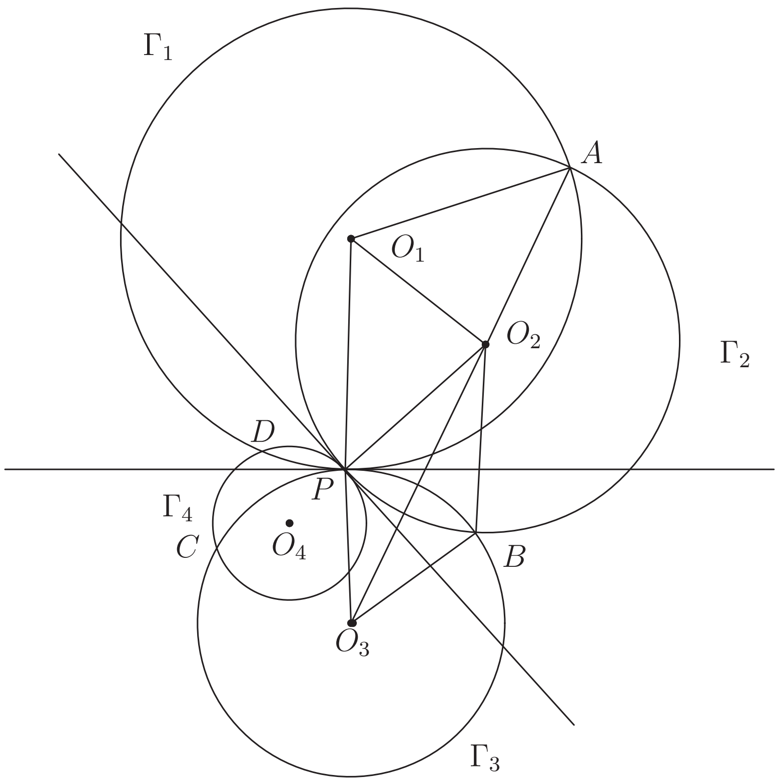
and the conclusion follows.

Comment. It is not necessary to assume that Γ_1, Γ_3 and Γ_2, Γ_4 are *externally* tangent. We may change the first sentence in the problem to the following:

> Let Γ_1, Γ_2, Γ_3, Γ_4 be distinct circles such that Γ_1, Γ_3 are tangent at P, and Γ_2, Γ_4 are tangent at the same point P.

The following two solutions are valid for the changed version.

Solution 3.



Let O_i and r_i be the centre and the signed radius of Γ_i, $i = 1, 2, 3, 4$. We may assume that $r_1 > 0$. If O_1, O_3 are in the same side of the common tangent, then we have $r_3 > 0$; otherwise we have $r_3 < 0$.

Put $\theta = \angle O_1PO_2$. We have $\angle O_iPO_{i+1} = \theta$ or $180° - \theta$, which shows that

$$\sin \angle O_iPO_{i+1} = \sin\theta. \tag{1}$$

Since $PB \perp O_2O_3$ and $\triangle PO_2O_3 \equiv \triangle BO_2O_3$, we have

$$\frac{1}{2} \cdot \frac{1}{2} \cdot O_2O_3 \cdot PB = \text{area}(\triangle PO_2O_3) = \frac{1}{2} \cdot PO_2 \cdot PO_3 \cdot \sin\theta = \frac{1}{2}|r_2||r_3|\sin\theta.$$

It follows that

$$PB = \frac{2|r_2||r_3|\sin\theta}{O_2O_3}. \tag{2}$$

Because the triangle O_2AB is isosceles, we have

$$AB = 2|r_2|\sin\frac{\angle AO_2B}{2}. \tag{3}$$

Since $\angle O_1O_2P = \angle O_1O_2A$ and $\angle O_3O_2P = \angle O_3O_2B$, we have

$$\sin(\angle AO_2B/2) = \sin\angle O_1O_2O_3.$$

Therefore, keeping in mind that

$$\begin{aligned}\frac{1}{2} \cdot O_1O_2 \cdot O_2O_3 \cdot \sin\angle O_1O_2O_3 = \text{area}(\triangle O_1O_2O_3) &= \frac{1}{2} \cdot O_1O_3 \cdot PO_2 \cdot \sin\theta \\ &= \frac{1}{2}|r_1 - r_3||r_2|\sin\theta,\end{aligned}$$

we have

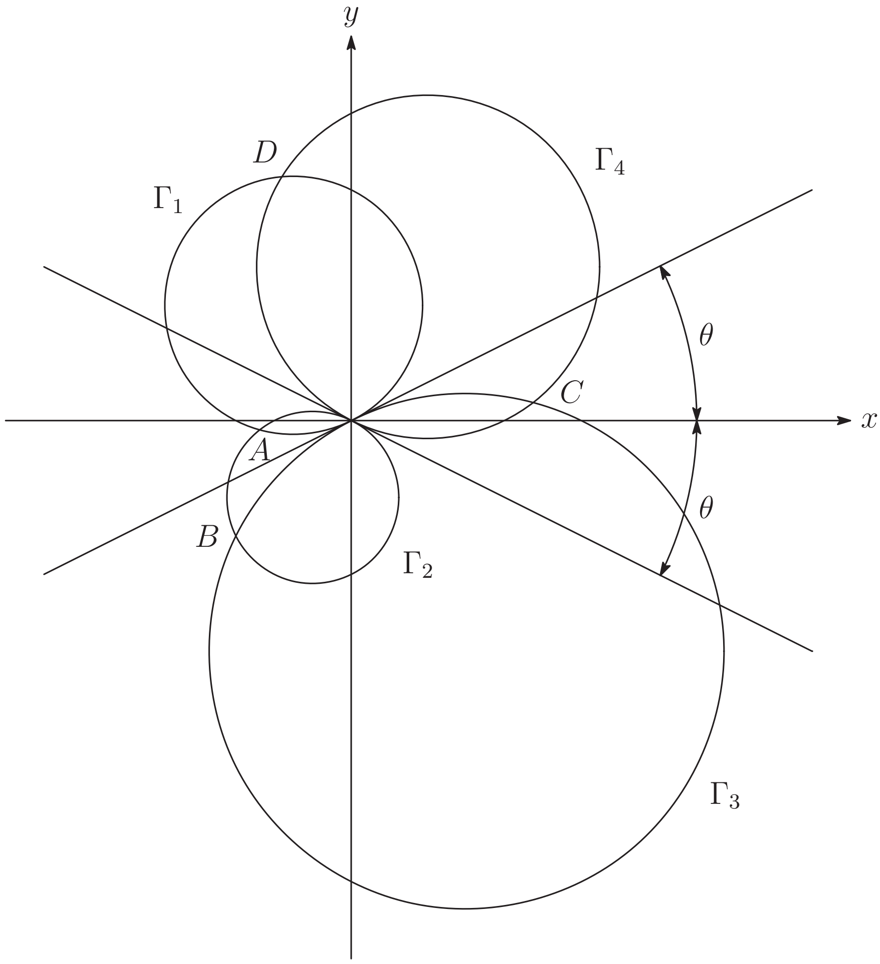
$$AB = 2|r_2|\frac{|r_1 - r_3||r_2|\sin\theta}{O_1O_2 \cdot O_2O_3}$$

by (3).

Likewise, by (1), (2), (4), we can obtain the lengths of PD, BC, CD, DA and compute as follows:

$$\begin{aligned}\frac{AB \cdot BC}{CD \cdot DA} &= \frac{2|r_1 - r_3|r_2^2\sin\theta}{O_1O_2 \cdot O_2O_3} \cdot \frac{2|r_2 - r_4|r_3^2\sin\theta}{O_2O_3 \cdot O_3O_4} \cdot \frac{O_3O_4 \cdot O_4O_1}{2|r_1 - r_3|r_4^2\sin\theta} \cdot \frac{O_4O_1 \cdot O_1O_2}{2|r_2 - r_4|r_1^2\sin\theta} \\ &= \left(\frac{2|r_2||r_3|\sin\theta}{O_2O_3}\right)^2\left(\frac{O_4O_1}{2|r_4||r_1|\sin\theta}\right)^2 \\ &= \frac{PB^2}{PD^2}.\end{aligned}$$

Solution 4. Let l_1 be the common tangent of the circles Γ_1 and Γ_3 and let l_2 be that of Γ_2 and Γ_4. Set the coordinate system as in the following figure.



We may assume that

$$\Gamma_1\colon x^2+y^2+2ax\sin\theta-2ay\cos\theta=0, \qquad \Gamma_2\colon x^2+y^2+2bx\sin\theta+2by\cos\theta=0,$$
$$\Gamma_3\colon x^2+y^2-2cx\sin\theta+2cy\cos\theta=0, \qquad \Gamma_4\colon x^2+y^2-2dx\sin\theta-2dy\cos\theta=0.$$

Simple computation shows that

$$A\left(-\frac{4ab(a+b)\sin\theta\cos^2\theta}{a^2+b^2+2ab\cos 2\theta}, -\frac{4ab(a-b)\sin^2\theta\cos\theta}{a^2+b^2+2ab\cos 2\theta}\right),$$
$$B\left(\frac{4bc(b-c)\sin\theta\cos^2\theta}{b^2+c^2-2bc\cos 2\theta}, -\frac{4bc(b+c)\sin^2\theta\cos\theta}{b^2+c^2-2bc\cos 2\theta}\right),$$
$$C\left(\frac{4cd(c+d)\sin\theta\cos^2\theta}{c^2+d^2+2cd\cos 2\theta}, \frac{4cd(c-d)\sin^2\theta\cos\theta}{c^2+d^2+2cd\cos 2\theta}\right),$$
$$D\left(-\frac{4da(d-a)\sin\theta\cos^2\theta}{d^2+a^2-2da\cos 2\theta}, \frac{4da(d+a)\sin^2\theta\cos\theta}{d^2+a^2-2da\cos 2\theta}\right).$$

Slightly long computation shows that

$$\begin{aligned}
AB &= \frac{4b^2|a+c|\sin\theta\cos\theta}{\sqrt{(a^2+b^2+2ab\cos 2\theta)(b^2+c^2-2bc\cos 2\theta)}},\\
BC &= \frac{4c^2|b+d|\sin\theta\cos\theta}{\sqrt{(b^2+c^2-2bc\cos 2\theta)(c^2+d^2+2cd\cos 2\theta)}},\\
CD &= \frac{4d^2|c+a|\sin\theta\cos\theta}{\sqrt{(c^2+d^2+2cd\cos 2\theta)(d^2+a^2-2da\cos 2\theta)}},\\
DA &= \frac{4a^2|d+b|\sin\theta\cos\theta}{\sqrt{(d^2+a^2-2da\cos 2\theta)(a^2+b^2+2ab\cos 2\theta)}},
\end{aligned}$$

which implies

$$\frac{AB\cdot BC}{AD\cdot DC} = \frac{b^2c^2(d^2+a^2-2da\cos 2\theta)}{d^2a^2(b^2+c^2-2bc\cos 2\theta)}.$$

On the other hand, we have

$$MB = \frac{4|b||c|\sin\theta\cos\theta}{\sqrt{b^2+c^2-2bc\cos 2\theta}} \quad \text{and} \quad MD = \frac{4|d||a|\sin\theta\cos\theta}{\sqrt{d^2+a^2-2da\cos 2\theta}},$$

which implies

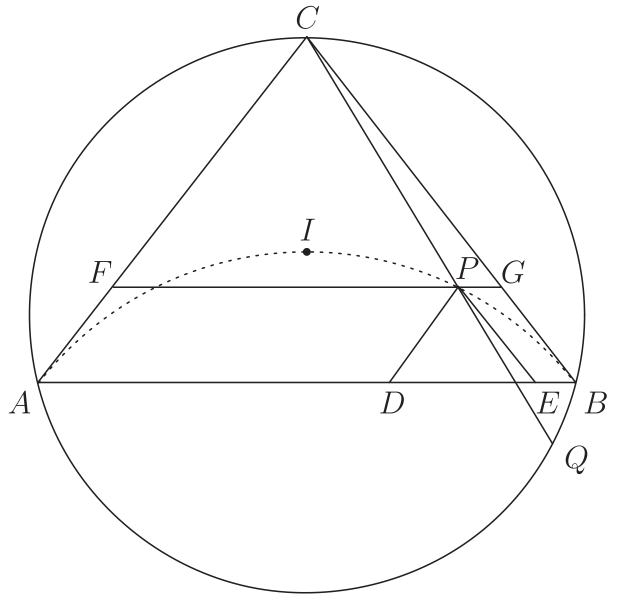
$$\frac{MB^2}{MD^2} = \frac{b^2c^2(d^2+a^2-2da\cos 2\theta)}{d^2a^2(b^2+c^2-2bc\cos 2\theta)}.$$

Hence we obtain

$$\frac{AB\cdot BC}{AD\cdot DC} = \frac{MB^2}{MD^2}.$$

G5. Let ABC be an isosceles triangle with $AC = BC$, whose incentre is I. Let P be a point on the circumcircle of the triangle AIB lying inside the triangle ABC. The lines through P parallel to CA and CB meet AB at D and E, respectively. The line through P parallel to AB meets CA and CB at F and G, respectively. Prove that the lines DF and EG intersect on the circumcircle of the triangle ABC.

Solution 1.



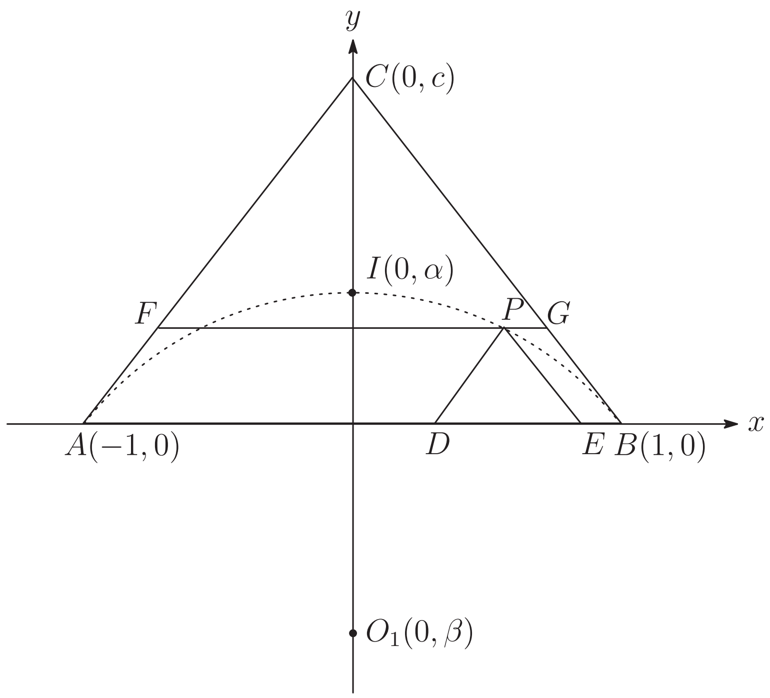
The corresponding sides of the triangles PDE and CFG are parallel. Therefore, if DF and EG are not parallel, then they are homothetic, and so DF, EG, CP are concurrent at the centre of the homothety. This observation leads to the following claim:

Claim. Suppose that CP meets again the circumcircle of the triangle ABC at Q. Then Q is the intersection of DF and EG.

Proof. Since $\angle AQP = \angle ABC = \angle BAC = \angle PFC$, it follows that the quadrilateral $AQPF$ is cyclic, and so $\angle FQP = \angle PAF$. Since $\angle IBA = \angle CBA/2 = \angle CAB/2 = \angle IAC$, the circumcircle of the triangle AIB is tangent to CA at A, which implies that $\angle PAF = \angle DBP$. Since $\angle QBD = \angle QCA = \angle QPD$, it follows that the quadrilateral $DQBP$ is cyclic, and so $\angle DBP = \angle DQP$. Thus $\angle FQP = \angle PAF = \angle DBP = \angle DQP$, which implies that F, D, Q are collinear. Analogously we obtain that G, E, Q are collinear. ∎

Hence the lines DF, EG, CP meet the circumcircle of the triangle ABC at the same point.

Solution 2.



Set the coordinate system so that $A(-1,0)$, $B(1,0)$, $C(0,c)$. Suppose that $I(0,\alpha)$.

Since

$$\text{area}(\triangle ABC) = \frac{1}{2}(AB + BC + CA)\alpha,$$

we obtain

$$\alpha = \frac{c}{1+\sqrt{1+c^2}}.$$

Suppose that $O_1(0,\beta)$ is the centre of the circumcircle Γ_1 of the triangle AIB. Since

$$(\beta-\alpha)^2 = O_1I^2 = O_1A^2 = 1+\beta^2,$$

we have $\beta = -1/c$ and so $\Gamma_1 \colon x^2 + (y+1/c)^2 = 1 + (1/c)^2$.

Let $P(p,q)$. Since $D(p - q/c, 0)$, $E(p + q/c, 0)$, $F(q/c - 1, q)$, $G(-q/c + 1, q)$, it follows that the equations of the lines DF and EG are

$$y = \frac{q}{\frac{2q}{c} - p - 1}\left(x - \left(p - \frac{q}{c}\right)\right) \quad \text{and} \quad y = \frac{q}{-\frac{2q}{c} - p + 1}\left(x - \left(p + \frac{q}{c}\right)\right),$$

respectively. Therefore the intersection Q of these lines is $\big((q-c)p/(2q-c), q^2/(2q-c)\big)$.

Let $O_2(0,\gamma)$ be the circumcentre of the triangle ABC. Then $\gamma = (c^2-1)/2c$ since $1+\gamma^2 = O_2A^2 = O_2C^2 = (\gamma - c)^2$.

Note that $p^2 + (q+1/c)^2 = 1 + (1/c)^2$ since $P(p,q)$ is on the circle Γ_1. It follows that

$$O_2Q^2 = \left(\frac{q-c}{2q-c}\right)^2 p^2 + \left(\frac{q^2}{2q-c} - \frac{c^2-1}{2c}\right)^2 = \left(\frac{c^2+1}{2c}\right)^2 = O_2C^2,$$

which shows that Q is on the circumcircle of the triangle ABC.

Comment. The point P can be any point on the circumcircle of the triangle AIB other than A and B; that is, P need not lie inside the triangle ABC.

G6. Each pair of opposite sides of a convex hexagon has the following property:

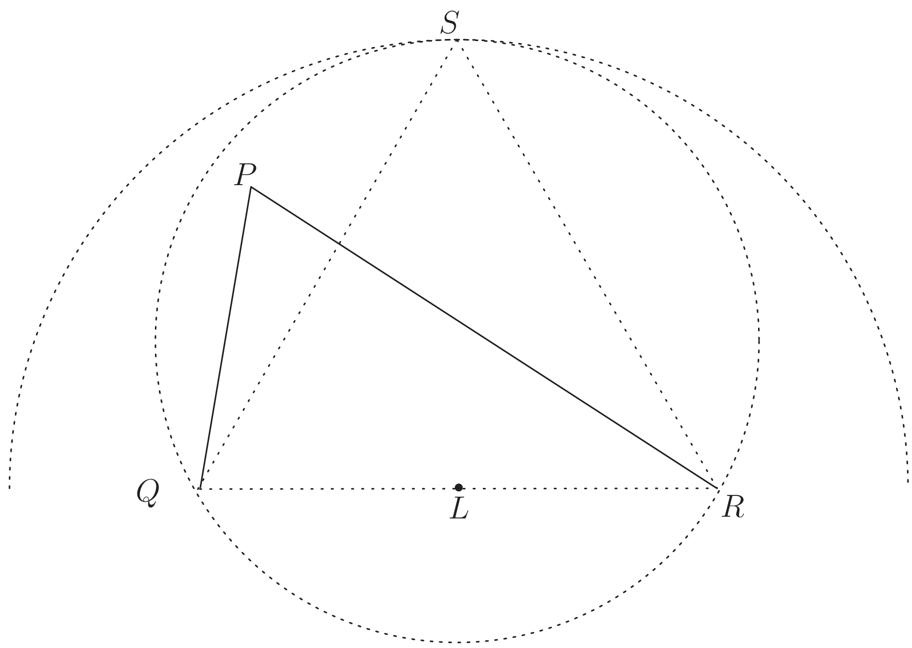
> the distance between their midpoints is equal to $\sqrt{3}/2$ times the sum of their lengths.

Prove that all the angles of the hexagon are equal.

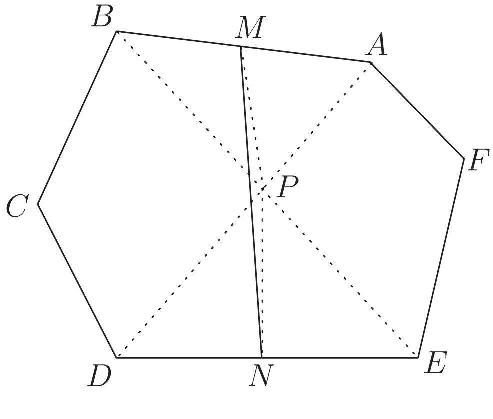
Solution 1. We first prove the following lemma:

Lemma. Consider a triangle PQR with $\angle QPR \geq 60^\circ$. Let L be the midpoint of QR. Then $PL \leq \sqrt{3}\,QR/2$, with equality if and only if the triangle PQR is equilateral.

Proof.



Let S be the point such that the triangle QRS is equilateral, where the points P and S lie in the same half-plane bounded by the line QR. Then the point P lies inside the circumcircle of the triangle QRS, which lies inside the circle with centre L and radius $\sqrt{3}\,QR/2$. This completes the proof of the lemma. ∎



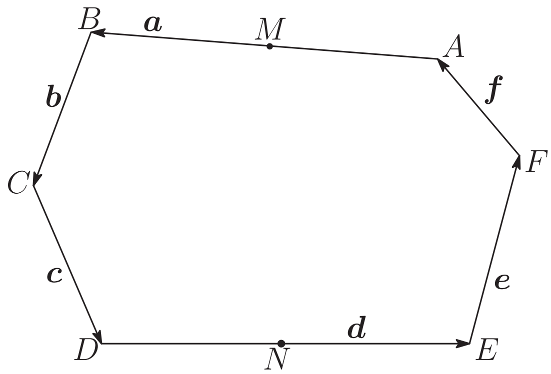
The main diagonals of a convex hexagon form a triangle though the triangle can be degenerated. Thus we may choose two of these three diagonals that form an angle greater than or equal to $60°$. Without loss of generality, we may assume that the diagonals AD and BE of the given hexagon $ABCDEF$ satisfy $\angle APB \geq 60°$, where P is the intersection of these diagonals. Then, using the lemma, we obtain

$$MN = \frac{\sqrt{3}}{2}(AB + DE) \geq PM + PN \geq MN,$$

where M and N are the midpoints of AB and DE, respectively. Thus it follows from the lemma that the triangles ABP and DEP are equilateral.

Therefore the diagonal CF forms an angle greater than or equal to $60°$ with one of the diagonals AD and BE. Without loss of generality, we may assume that $\angle AQF \geq 60°$, where Q is the intersection of AD and CF. Arguing in the same way as above, we infer that the triangles AQF and CQD are equilateral. This implies that $\angle BRC = 60°$, where R is the intersection of BE and CF. Using the same argument as above for the third time, we obtain that the triangles BCR and EFR are equilateral. This completes the solution.

Solution 2. Let $ABCDEF$ be the given hexagon and let $\boldsymbol{a} = \overrightarrow{AB}$, $\boldsymbol{b} = \overrightarrow{BC}$, $\ldots$, $\boldsymbol{f} = \overrightarrow{FA}$.



Let M and N be the midpoints of the sides AB and DE, respectively. We have

$$\overrightarrow{MN} = \frac{1}{2}\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c} + \frac{1}{2}\boldsymbol{d} \quad \text{and} \quad \overrightarrow{MN} = -\frac{1}{2}\boldsymbol{a} - \boldsymbol{f} - \boldsymbol{e} - \frac{1}{2}\boldsymbol{d}.$$

Thus we obtain

$$\overrightarrow{MN} = \frac{1}{2}(\boldsymbol{b} + \boldsymbol{c} - \boldsymbol{e} - \boldsymbol{f}). \tag{1}$$

From the given property, we have

$$\overrightarrow{MN} = \frac{\sqrt{3}}{2}\big(|\boldsymbol{a}| + |\boldsymbol{d}|\big) \geq \frac{\sqrt{3}}{2}|\boldsymbol{a} - \boldsymbol{d}|. \tag{2}$$

Set $\boldsymbol{x} = \boldsymbol{a} - \boldsymbol{d}$, $\boldsymbol{y} = \boldsymbol{c} - \boldsymbol{f}$, $\boldsymbol{z} = \boldsymbol{e} - \boldsymbol{b}$. From (1) and (2), we obtain

$$|\boldsymbol{y} - \boldsymbol{z}| \geq \sqrt{3}\,|\boldsymbol{x}|. \tag{3}$$

Similarly we see that

$$|\boldsymbol{z} - \boldsymbol{x}| \geq \sqrt{3}\,|\boldsymbol{y}|, \tag{4}$$
$$|\boldsymbol{x} - \boldsymbol{y}| \geq \sqrt{3}\,|\boldsymbol{z}|. \tag{5}$$

Note that

$$\begin{aligned}
(3) &\iff |\boldsymbol{y}|^2 - 2\boldsymbol{y}\cdot\boldsymbol{z} + |\boldsymbol{z}|^2 \ge 3|\boldsymbol{x}|^2,\\
(4) &\iff |\boldsymbol{z}|^2 - 2\boldsymbol{z}\cdot\boldsymbol{x} + |\boldsymbol{x}|^2 \ge 3|\boldsymbol{y}|^2,\\
(5) &\iff |\boldsymbol{x}|^2 - 2\boldsymbol{x}\cdot\boldsymbol{y} + |\boldsymbol{y}|^2 \ge 3|\boldsymbol{z}|^2.
\end{aligned}$$

By adding up the last three inequalities, we obtain

$$-|\boldsymbol{x}|^2 - |\boldsymbol{y}|^2 - |\boldsymbol{z}|^2 - 2\boldsymbol{y}\cdot\boldsymbol{z} - 2\boldsymbol{z}\cdot\boldsymbol{x} - 2\boldsymbol{x}\cdot\boldsymbol{y} \ge 0,$$

or $-|\boldsymbol{x}+\boldsymbol{y}+\boldsymbol{z}|^2 \ge 0$. Thus $\boldsymbol{x}+\boldsymbol{y}+\boldsymbol{z} = \boldsymbol{0}$ and the equalities hold in all inequalities above. Hence we conclude that

$$\begin{aligned}
&\boldsymbol{x}+\boldsymbol{y}+\boldsymbol{z} = \boldsymbol{0},\\
|\boldsymbol{y}-\boldsymbol{z}| &= \sqrt{3}\,|\boldsymbol{x}|, \quad \boldsymbol{a} \parallel \boldsymbol{d} \parallel \boldsymbol{x},\\
|\boldsymbol{z}-\boldsymbol{x}| &= \sqrt{3}\,|\boldsymbol{y}|, \quad \boldsymbol{c} \parallel \boldsymbol{f} \parallel \boldsymbol{y},\\
|\boldsymbol{x}-\boldsymbol{y}| &= \sqrt{3}\,|\boldsymbol{z}|, \quad \boldsymbol{e} \parallel \boldsymbol{b} \parallel \boldsymbol{z}.
\end{aligned}$$

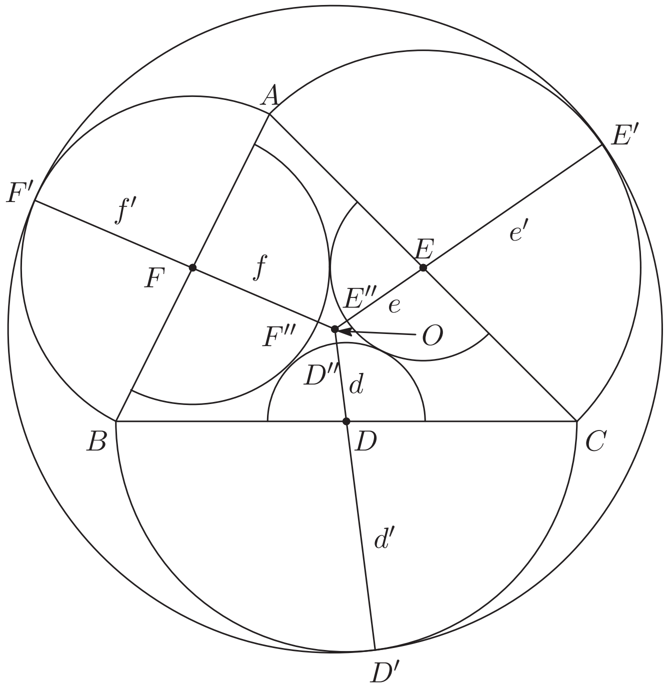
Suppose that PQR is the triangle such that $\overrightarrow{PQ} = \boldsymbol{x}$, $\overrightarrow{QR} = \boldsymbol{y}$, $\overrightarrow{RP} = \boldsymbol{z}$. We may assume $\angle QPR \ge 60^\circ$, without loss of generality. Let L be the midpoint of QR, then $PL = |\boldsymbol{z}-\boldsymbol{x}|/2 = \sqrt{3}\,|\boldsymbol{y}|/2 = \sqrt{3}\,QR/2$. It follows from the lemma in Solution 1 that the triangle PQR is equilateral. Thus we have $\angle ABC = \angle BCD = \cdots = \angle FAB = 120^\circ$.

Comment. We have obtained the complete characterisation of the hexagons satisfying the given property. They are all obtained from an equilateral triangle by cutting its 'corners' at the same height.

G7. Let ABC be a triangle with semiperimeter s and inradius r. The semicircles with diameters BC, CA, AB are drawn on the outside of the triangle ABC. The circle tangent to all three semicircles has radius t. Prove that

$$\frac{s}{2} < t \leq \frac{s}{2} + \left(1 - \frac{\sqrt{3}}{2}\right) r.$$

Solution 1.



Let O be the centre of the circle and let D, E, F be the midpoints of BC, CA, AB, respectively. Denote by D', E', F' the points at which the circle is tangent to the semicircles. Let d', e', f' be the radii of the semicircles. Then all of DD', EE', FF' pass through O, and $s = d' + e' + f'$.

Put

$$d = \frac{s}{2} - d' = \frac{-d' + e' + f'}{2}, \quad e = \frac{s}{2} - e' = \frac{d' - e' + f'}{2}, \quad f = \frac{s}{2} - f' = \frac{d' + e' - f'}{2}.$$

Note that $d + e + f = s/2$. Construct smaller semicircles inside the triangle ABC with radii d, e, f and centres D, E, F. Then the smaller semicircles touch each other, since $d + e = f' = DE$, $e + f = d' = EF$, $f + d = e' = FD$. In fact, the points of tangency are the points where the incircle of the triangle DEF touches its sides.

Suppose that the smaller semicircles cut DD', EE', FF' at D'', E'', F'', respectively. Since these semicircles do not overlap, the point O is outside the semicircles. Therefore $D'O > D'D''$, and so $t > s/2$. Put $g = t - s/2$.

Clearly, $OD'' = OE'' = OF'' = g$. Therefore the circle with centre O and radius g touches all of the three mutually tangent semicircles.

Claim. We have

$$\frac{1}{d^2} + \frac{1}{e^2} + \frac{1}{f^2} + \frac{1}{g^2} = \frac{1}{2}\left(\frac{1}{d} + \frac{1}{e} + \frac{1}{f} + \frac{1}{g}\right)^2.$$

Proof. Consider a triangle PQR and let $p = QR$, $q = RP$, $r = PQ$. Then

$$\cos\angle QPR = \frac{-p^2+q^2+r^2}{2qr}$$

and

$$\sin\angle QPR = \frac{\sqrt{(p+q+r)(-p+q+r)(p-q+r)(p+q-r)}}{2qr}.$$

Since

$$\cos\angle EDF = \cos(\angle ODE + \angle ODF) = \cos\angle ODE\cos\angle ODF - \sin\angle ODE\sin\angle ODF,$$

we have

$$\begin{aligned}\frac{d^2+de+df-ef}{(d+e)(d+f)} = {} & \frac{(d^2+de+dg-eg)(d^2+df+dg-fg)}{(d+g)^2(d+e)(d+f)} \\ & - \frac{4dg\sqrt{(d+e+g)(d+f+g)ef}}{(d+g)^2(d+e)(d+f)},\end{aligned}$$

which simplifies to

$$(d+g)\left(\frac{1}{d}+\frac{1}{e}+\frac{1}{f}+\frac{1}{g}\right) - 2\left(\frac{d}{g}+1+\frac{g}{d}\right) = -2\sqrt{\frac{(d+e+g)(d+f+g)}{ef}}.$$

Squaring and simplifying, we obtain

$$\begin{aligned}\left(\frac{1}{d}+\frac{1}{e}+\frac{1}{f}+\frac{1}{g}\right)^2 &= 4\left(\frac{1}{de}+\frac{1}{df}+\frac{1}{dg}+\frac{1}{ef}+\frac{1}{eg}+\frac{1}{fg}\right) \\ &= 2\left(\left(\frac{1}{d}+\frac{1}{e}+\frac{1}{f}+\frac{1}{g}\right)^2 - \left(\frac{1}{d^2}+\frac{1}{e^2}+\frac{1}{f^2}+\frac{1}{g^2}\right)\right),\end{aligned}$$

from which the conclusion follows. ∎

Solving for the smaller value of g, i.e., the larger value of $1/g$, we obtain

$$\begin{aligned}\frac{1}{g} &= \frac{1}{d}+\frac{1}{e}+\frac{1}{f}+\sqrt{2\left(\frac{1}{d}+\frac{1}{e}+\frac{1}{f}\right)^2 - 2\left(\frac{1}{d^2}+\frac{1}{e^2}+\frac{1}{f^2}\right)} \\ &= \frac{1}{d}+\frac{1}{e}+\frac{1}{f}+2\sqrt{\frac{d+e+f}{def}}.\end{aligned}$$

Comparing the formulas area$(\triangle DEF)$ = area$(\triangle ABC)/4 = rs/4$ and area$(\triangle DEF) = \sqrt{(d+e+f)def}$, we have

$$\frac{r}{2} = \frac{2}{s}\sqrt{(d+e+f)def} = \sqrt{\frac{def}{d+e+f}}.$$

All we have to prove is that

$$\frac{r}{2g} \geq \frac{1}{2-\sqrt{3}} = 2+\sqrt{3}.$$

Since

$$\frac{r}{2g} = \sqrt{\frac{def}{d+e+f}}\left(\frac{1}{d}+\frac{1}{e}+\frac{1}{f}+2\sqrt{\frac{d+e+f}{def}}\right) = \frac{x+y+z}{\sqrt{xy+yz+zx}}+2,$$

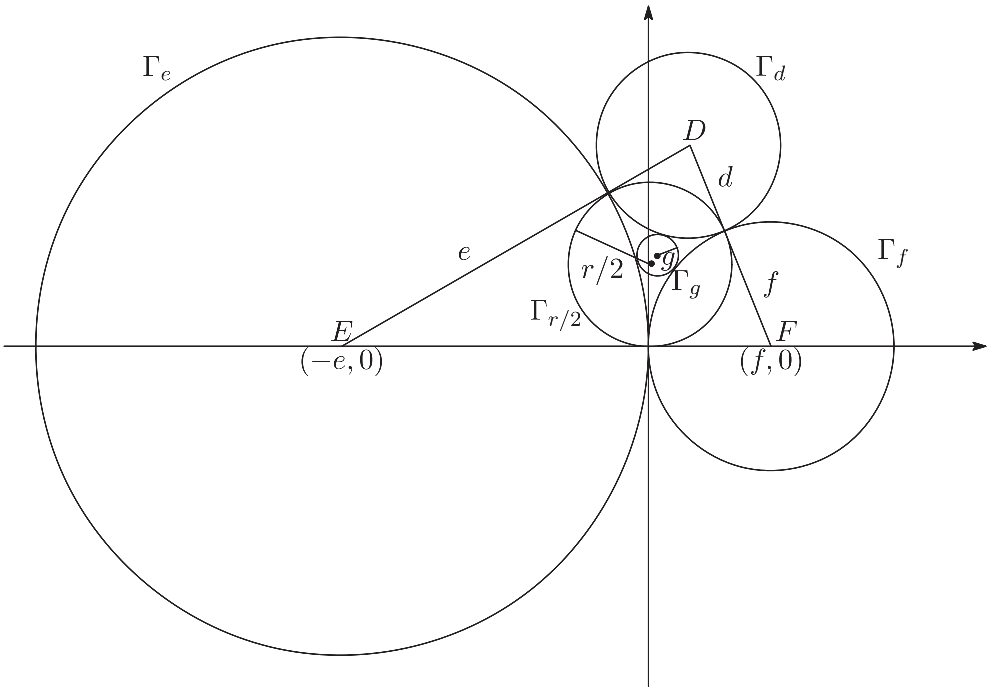
where $x = 1/d$, $y = 1/e$, $z = 1/f$, it suffices to prove that

$$\frac{(x+y+z)^2}{xy+yz+zx} \geq 3.$$

This inequality is true because

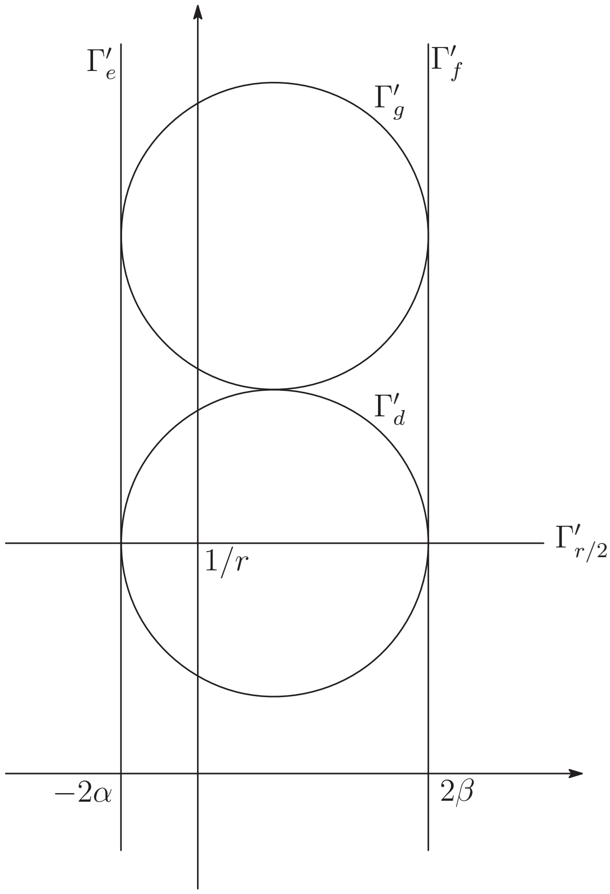
$$(x+y+z)^2 - 3(xy+yz+zx) = \frac{1}{2}\big((x-y)^2+(y-z)^2+(z-x)^2\big) \geq 0.$$

Solution 2. We prove that $t > s/2$ in the same way as in Solution 1. Put $g = t - s/2$.



Now set the coordinate system so that $E(-e,0)$, $F(f,0)$, and the y-coordinate of D is positive. Let Γ_d, Γ_e, Γ_f, Γ_g be the circles with radii d, e, f, g and centres D, E, F, O, respectively. Let $\Gamma_{r/2}$ be the incircle of the triangle DEF. Note that the radius of $\Gamma_{r/2}$ is $r/2$.

Now consider the inversion with respect to the circle with radius 1 and centre $(0,0)$.



Let Γ'_d, Γ'_e, Γ'_f, Γ'_g, $\Gamma'_{r/2}$ be the images of Γ_d, Γ_e, Γ_f, Γ_g, $\Gamma_{r/2}$, respectively. Set $\alpha = 1/4e$, $\beta = 1/4f$ and $R = \alpha + \beta$. The equations of the lines Γ'_e, Γ'_f and $\Gamma'_{r/2}$ are $x = -2\alpha$, $x = 2\beta$ and $y = 1/r$, respectively. Both of the radii of the circles Γ'_d and Γ'_g are R, and their centres are $(-\alpha + \beta, 1/r)$ and $(-\alpha + \beta, 1/r + 2R)$, respectively.

Let D be the distance between $(0, 0)$ and the centre of Γ'_g. Then we have

$$2g = \frac{1}{D - R} - \frac{1}{D + R} = \frac{2R}{D^2 - R^2},$$

which shows $g = R/(D^2 - R^2)$.

What we have to show is $g \le \big(1 - \sqrt{3}/2\big)r$, that is $\big(4 + 2\sqrt{3}\big)g \le r$. This is verified by the following computation:

$$\begin{aligned}
r - \big(4 + 2\sqrt{3}\big)g = r - \big(4 + 2\sqrt{3}\big)\frac{R}{D^2 - R^2} &= \frac{r}{D^2 - R^2}\left((D^2 - R^2) - \big(4 + 2\sqrt{3}\big)\frac{1}{r}R\right) \\
&= \frac{r}{D^2 - R^2}\left(\left(\frac{1}{r} + 2R\right)^2 + (\alpha - \beta)^2 - R^2 - \big(4 + 2\sqrt{3}\big)\frac{1}{r}R\right) \\
&= \frac{r}{D^2 - R^2}\left(3\left(R - \frac{1}{\sqrt{3}\,r}\right)^2 + (\alpha - \beta)^2\right) \\
&\ge 0.
\end{aligned}$$

Number Theory

N1. Let m be a fixed integer greater than 1. The sequence $x_0,\ x_1,\ x_2,\ \ldots$ is defined as follows:

$$x_i = \begin{cases} 2^i, & \text{if } 0 \le i \le m-1; \\ \sum_{j=1}^{m} x_{i-j}, & \text{if } i \ge m. \end{cases}$$

Find the greatest k for which the sequence contains k consecutive terms divisible by m.

Solution. Let r_i be the remainder of $x_i \bmod m$. Then there are at most m^m types of m-consecutive blocks in the sequence (r_i). So, by the pigeonhole principle, some type reappears. Since the definition formula works forward and backward, the sequence (r_i) is purely periodic.

Now the definition formula backward $x_i = x_{i+m} - \sum_{j=1}^{m-1} x_{i+j}$ applied to the block $(r_0, \ldots, r_{m-1})$ produces the m-consecutive block $\underbrace{0, \ldots, 0}_{m-1}, 1$. Together with the pure periodicity, we see that $\max k \ge m-1$.

On the other hand, if there are m-consecutive zeroes in (r_i), then the definition formula and the pure periodicity force $r_i = 0$ for any $i \ge 0$, a contradiction. Thus $\max k = m-1$.

N2. Each positive integer a undergoes the following procedure in order to obtain the number $d = d(a)$:

(i) move the last digit of a to the first position to obtain the number b;

(ii) square b to obtain the number c;

(iii) move the first digit of c to the end to obtain the number d.

(All the numbers in the problem are considered to be represented in base 10.) For example, for $a = 2003$, we get $b = 3200$, $c = 10240000$, and $d = 02400001 = 2400001 = d(2003)$.

Find all numbers a for which $d(a) = a^2$.

Solution. Let a be a positive integer for which the procedure yields $d = d(a) = a^2$. Further assume that a has $n+1$ digits, $n \geq 0$.

Let s be the last digit of a and f the first digit of c. Since $(* \cdots * s)^2 = a^2 = d = * \cdots * f$ and $(s * \cdots *)^2 = b^2 = c = f * \cdots *$, where the stars represent digits that are unimportant at the moment, f is both the last digit of the square of a number that ends in s and the first digit of the square of a number that starts in s.

The square $a^2 = d$ must have either $2n+1$ or $2n+2$ digits. If $s = 0$, then $n \neq 0$, b has n digits, its square c has at most $2n$ digits, and so does d, a contradiction. Thus the last digit of a is not 0.

Consider now, for example, the case $s = 4$. Then f must be 6, but this is impossible, since the squares of numbers that start in 4 can only start in 1 or 2, which is easily seen from

$$160 \cdots 0 = (40 \cdots 0)^2 \leq (4 * \cdots *)^2 < (50 \cdots 0)^2 = 250 \cdots 0.$$

Thus s cannot be 4.

The following table gives all possibilities:

s	1	2	3	4	5	6	7	8	9
$f =$ last digit of $(\cdots s)^2$	1	4	9	6	5	6	9	4	1
$f =$ first digit of $(s \cdots)^2$	1, 2, 3	4, 5, 6, 7, 8	9, 1	1, 2	2, 3	3, 4	4, 5, 6	6, 7, 8	8, 9

Thus $s = 1$, $s = 2$, or $s = 3$ and in each case $f = s^2$. When s is 1 or 2, the square $c = b^2$ of the $(n+1)$-digit number b which starts in s has $2n+1$ digits. Moreover, when $s = 3$, the square $c = b^2$ either has $2n+1$ digits and starts in 9 or has $2n+2$ digits and starts in 1. However the latter is impossible since $f = s^2 = 9$. Thus c must have $2n+1$ digits.

Let $a = 10x + s$, where x is an n-digit number (in case $x = 0$ we set $n = 0$). Then

$$\begin{aligned} b &= 10^n s + x, \\ c &= 10^{2n} s^2 + 2 \cdot 10^n sx + x^2, \\ d &= 10(c - 10^{m-1} f) + f = 10^{2n+1} s^2 + 20 \cdot 10^n sx + 10x^2 - 10^m f + f, \end{aligned}$$

where m is the number of digits of c. However, we already know that m must be $2n+1$ and $f = s^2$, so

$$d = 20 \cdot 10^n sx + 10x^2 + s^2$$

and the equality $a^2 = d$ yields

$$x = 2s \cdot \frac{10^n - 1}{9},$$

i.e.,

$$a = \underbrace{6 \cdots 6}_{n}3 \quad \text{or} \quad a = \underbrace{4 \cdots 4}_{n}2 \quad \text{or} \quad a = \underbrace{2 \cdots 2}_{n}1,$$

for $n \geq 0$. The first two possibilities must be rejected for $n \geq 1$, since $a^2 = d$ would have $2n + 2$ digits, which means that c would have to have at least $2n + 2$ digits, but we already know that c must have $2n + 1$ digits. Thus the only remaining possibilities are

$$a = 3 \quad \text{or} \quad a = 2 \quad \text{or} \quad a = \underbrace{2 \cdots 2}_{n}1,$$

for $n \geq 0$. It is easily seen that they all satisfy the requirements of the problem.

N3. Determine all pairs of positive integers (a, b) such that

$$\frac{a^2}{2ab^2 - b^3 + 1}$$

is a positive integer.

Solution. Let (a, b) be a pair of positive integers satisfying the condition. Because $k = a^2/(2ab^2 - b^3 + 1) > 0$, we have $2ab^2 - b^3 + 1 > 0$, $a > b/2 - 1/2b^2$, and hence $a \geq b/2$. Using this, we infer from $k \geq 1$, or $a^2 \geq b^2(2a - b) + 1$, that $a^2 > b^2(2a - b) \geq 0$. Hence

$$a > b \quad \text{or} \quad 2a = b. \tag{$*$}$$

Now consider the two solutions a_1, a_2 to the equation

$$a^2 - 2kb^2a + k(b^3 - 1) = 0 \tag{$\sharp$}$$

for fixed positive integers k and b, and assume that one of them is an integer. Then the other is also an integer because $a_1 + a_2 = 2kb^2$. We may assume that $a_1 \geq a_2$, and we have $a_1 \geq kb^2 > 0$. Furthermore, since $a_1a_2 = k(b^3 - 1)$, we get

$$0 \leq a_2 = \frac{k(b^3 - 1)}{a_1} \leq \frac{k(b^3 - 1)}{kb^2} < b.$$

Together with $(*)$, we conclude that $a_2 = 0$ or $a_2 = b/2$ (in the latter case b must be even).

If $a_2 = 0$, then $b^3 - 1 = 0$, and hence $a_1 = 2k$, $b = 1$.

If $a_2 = b/2$, then $k = b^2/4$ and $a_1 = b^4/2 - b/2$.

Therefore the only possibilities are

$$(a, b) = (2l, 1) \quad \text{or} \quad (l, 2l) \quad \text{or} \quad (8l^4 - l, 2l)$$

for some positive integer l. All of these pairs satisfy the given condition.

Comment 1. An alternative way to see $(*)$ is as follows: Fix $a \geq 1$ and consider the function $f_a(b) = 2ab^2 - b^3 + 1$. Then f_a is increasing on $[0, 4a/3]$ and decreasing on $[4a/3, \infty)$. We have

$$\begin{aligned} f_a(a) &= a^3 + 1 > a^2, \\ f_a(2a - 1) &= 4a^2 - 4a + 2 > a^2, \\ f_a(2a + 1) &= -4a^2 - 4a < 0. \end{aligned}$$

Hence if $b \geq a$ and $a^2/f_a(b)$ is a positive integer, then $b = 2a$.

Indeed, if $a \leq b \leq 4a/3$, then $f_a(b) \geq f_a(a) > a^2$, and so $a^2/f_a(b)$ is not an integer, a contradiction, and if $b > 4a/3$, then

(i) if $b \geq 2a + 1$, then $f_a(b) \leq f_a(2a + 1) < 0$, a contradiction;

(ii) if $b \leq 2a - 1$, then $f_a(b) \geq f_a(2a - 1) > a^2$, and so $a^2/f_a(b)$ is not an integer, a contradiction.

Comment 2. There are several alternative solutions to this problem. Here we sketch three of them.

1. The discriminant D of the equation ($\sharp$) is the square of some integer $d \geq 0$: $D = (2b^2k - b)^2 + 4k - b^2 = d^2$. If $e = 2b^2k - b = d$, we have $4k = b^2$ and $a = 2b^2k - b/2, b/2$. Otherwise, the clear estimation $|d^2 - e^2| \geq 2e - 1$ for $d \neq e$ implies $|4k - b^2| \geq 4b^2k - 2b - 1$. If $4k - b^2 > 0$, this implies $b = 1$. The other case yields no solutions.

2. Assume that $b \neq 1$ and let $s = \gcd(2a, b^3 - 1)$, $2a = su$, $b^3 - 1 = st'$, and $2ab^2 - b^3 + 1 = st$. Then $t + t' = ub^2$ and $\gcd(u, t) = 1$. Together with $st \mid a^2$, we have $t \mid s$. Let $s = rt$. Then the problem reduces to the following lemma:

> **Lemma.** Let b, r, t, t', u be positive integers satisfying $b^3 - 1 = rtt'$ and $t + t' = ub^2$. Then $r = 1$. Furthermore, either one of t or t' or u is 1.

The lemma is proved as follows. We have $b^3 - 1 = rt(ub^2 - t) = rt'(ub^2 - t')$. Since $rt^2 \equiv rt'^2 \equiv 1 \pmod{b^2}$, if $rt^2 \neq 1$ and $rt'^2 \neq 1$, then $t, t' > b/\sqrt{r}$. It is easy to see that

$$r\frac{b}{\sqrt{r}}\left(ub^2 - \frac{b}{\sqrt{r}}\right) \geq b^3 - 1,$$

unless $r = u = 1$.

3. With the same notation as in the previous solution, since $rt^2 \mid (b^3 - 1)^2$, it suffices to prove the following lemma:

> **Lemma.** Let $b \geq 2$. If a positive integer $x \equiv 1 \pmod{b^2}$ divides $(b^3 - 1)^2$, then $x = 1$ or $x = (b^3 - 1)^2$ or $(b, x) = (4, 49)$ or $(4, 81)$.

To prove this lemma, let p, q be positive integers with $p > q > 0$ satisfying $(b^3 - 1)^2 = (pb^2 + 1)(qb^2 + 1)$. Then

$$b^4 = 2b + p + q + pqb^2. \tag{1}$$

A natural observation leads us to multiply (1) by $qb^2 - 1$. We get

$$\big(q(pq - b^2) + 1\big)b^4 = p - (q + 2b)(qb^2 - 1).$$

Together with the simple estimation

$$-3 < \frac{p - (q + 2b)(qb^2 - 1)}{b^4} < 1,$$

the conclusion of the lemma follows.

Comment 3. The problem was originally proposed in the following form:

> Let a, b be relatively prime positive integers. Suppose that $a^2/(2ab^2 - b^3 + 1)$ is a positive integer greater than 1. Prove that $b = 1$.

N4. Let b be an integer greater than 5. For each positive integer n, consider the number

$$x_n = \underbrace{11\cdots1}_{n-1}\underbrace{22\cdots2}_{n}5,$$

written in base b.

Prove that the following condition holds if and only if $b = 10$:

> there exists a positive integer M such that for any integer n greater than M, the number x_n is a perfect square.

Solution. For $b = 6, 7, 8, 9$, the number 5 is congruent to no square numbers modulo b, and hence x_n is not a square. For $b = 10$, we have $x_n = \big((10^n + 5)/3\big)^2$ for all n. By algebraic calculation, it is easy to see that $x_n = (b^{2n} + b^{n+1} + 3b - 5)/(b - 1)$.

Consider now the case $b \geq 11$ and put $y_n = (b - 1)x_n$. Assume that the condition in the problem is satisfied. Then it follows that $y_n y_{n+1}$ is a perfect square for $n > M$. Since $b^{2n} + b^{n+1} + 3b - 5 < (b^n + b/2)^2$, we infer

$$y_n y_{n+1} < \left(b^n + \frac{b}{2}\right)^2 \left(b^{n+1} + \frac{b}{2}\right)^2 = \left(b^{2n+1} + \frac{b^{n+1}(b+1)}{2} + \frac{b^2}{4}\right)^2. \tag{1}$$

On the other hand, we can prove by computation that

$$y_n y_{n+1} > \left(b^{2n+1} + \frac{b^{n+1}(b+1)}{2} - b^3\right)^2. \tag{2}$$

From (1) and (2), we conclude that for all integers $n > M$, there is an integer a_n such that

$$y_n y_{n+1} = \left(b^{2n+1} + \frac{b^{n+1}(b+1)}{2} + a_n\right)^2 \quad \text{and} \quad -b^3 < a_n < \frac{b^2}{4}. \tag{3}$$

It follows that $b^n \mid \big(a_n^2 - (3b-5)^2\big)$, and thus $a_n = \pm(3b - 5)$ for all sufficiently large n. Substituting in (3), we obtain $a_n = 3b - 5$ and

$$8(3b - 5)b + b^2(b + 1)^2 = 4b^3 + 4(3b - 5)(b^2 + 1). \tag{4}$$

The left hand side of the equation (4) is divisible by b. The other side is a polynomial in b with integral coefficients and its constant term is -20. Hence b must divide 20. Since $b \geq 11$, we conclude that $b = 20$, but then $x_n \equiv 5 \pmod 8$ and hence x_n is not a square.

Comment. Here is a shorter solution using a limit argument:

Assume that x_n is a square for all $n > M$, where M is a positive integer.

For $n > M$, take $y_n = \sqrt{x_n} \in \mathbb{N}$. Clearly,

$$\lim_{n\to\infty} \frac{\frac{b^{2n}}{b-1}}{x_n} = 1.$$

Hence

$$\lim_{n\to\infty} \frac{\frac{b^n}{\sqrt{b-1}}}{y_n} = 1.$$

On the other hand,

$$(by_n + y_{n+1})(by_n - y_{n+1}) = b^2 x_n - x_{n+1} = b^{n+2} + 3b^2 - 2b - 5. \tag{$*$}$$

These equations imply

$$\lim_{n\to\infty} (by_n - y_{n+1}) = \frac{b\sqrt{b-1}}{2}.$$

As $by_n - y_{n+1}$ is an integer, there exists $N > M$ such that $by_n - y_{n+1} = b\sqrt{b-1}/2$ for any $n > N$. This means that $b - 1$ is a perfect square.

If b is odd, then $\sqrt{b-1}/2$ is an integer and so b divides $b\sqrt{b-1}/2$. Hence using $(*)$, we obtain $b \mid 5$. This is a contradiction.

If b is even, then $b/2$ divides 5. Hence $b = 10$.

In the case $b = 10$, we have $x_n = \big((10^n + 5)/3\big)^2$ for $n \geq 1$.

N5. An integer n is said to be *good* if $|n|$ is not the square of an integer. Determine all integers m with the following property:

> m can be represented, in infinitely many ways, as a sum of three distinct good integers whose product is the square of an odd integer.

Solution. Assume that m is expressed as $m = u+v+w$ and uvw is an odd perfect square. Then u, v, w are odd and because $uvw \equiv 1 \pmod 4$, exactly two or none of them are congruent to 3 modulo 4. In both cases, we have $m = u+v+w \equiv 3 \pmod 4$.

Conversely, we prove that $4k+3$ has the required property. To prove this, we look for representations of the form

$$4k+3 = xy + yz + zx.$$

In any such representations, the product of the three summands is a perfect square. Setting $x = 1+2l$ and $y = 1-2l$, we have $z = 2l^2 + 2k + 1$ from above. Then

$$\begin{aligned}
xy &= 1 - 4l^2 = f(l), \\
yz &= -4l^3 + 2l^2 - (4k+2)l + 2k + 1 = g(l), \\
zx &= 4l^3 + 2l^2 + (4k+2)l + 2k + 1 = h(l).
\end{aligned}$$

The numbers $f(l)$, $g(l)$, $h(l)$ are odd for each integer l and their product is a perfect square, as noted above. They are distinct, except for finitely many l. It remains to note that $|g(l)|$ and $|h(l)|$ are not perfect squares for infinitely many l (note that $|f(l)|$ is not a perfect square, unless $l = 0$).

Choose distinct prime numbers p, q such that $p, q > 4k+3$ and pick l such that

$$\begin{aligned}
1 + 2l &\equiv 0 \pmod p, & \qquad 1 + 2l &\not\equiv 0 \pmod{p^2}, \\
1 - 2l &\equiv 0 \pmod q, & \qquad 1 - 2l &\not\equiv 0 \pmod{q^2}.
\end{aligned}$$

We can choose such l by the Chinese remainder theorem. Then $2l^2 + 2k + 1$ is not divisible by p, because $p > 4k+3$. Hence $|h(l)|$ is not a perfect square. Similarly, $|g(l)|$ is not a perfect square.

N6. Let p be a prime number. Prove that there exists a prime number q such that for every integer n, the number $n^p - p$ is not divisible by q.

Solution. Since $(p^p - 1)/(p-1) = 1 + p + p^2 + \cdots + p^{p-1} \equiv p + 1 \pmod{p^2}$, we can get at least one prime divisor of $(p^p - 1)/(p-1)$ which is not congruent to 1 modulo p^2. Denote such a prime divisor by q. This q is what we wanted. The proof is as follows. Assume that there exists an integer n such that $n^p \equiv p \pmod{q}$. Then we have $n^{p^2} \equiv p^p \equiv 1 \pmod{q}$ by the definition of q. On the other hand, from Fermat's little theorem, $n^{q-1} \equiv 1 \pmod{q}$, because q is a prime. Since $p^2 \nmid q - 1$, we have $(p^2, q-1) \mid p$, which leads to $n^p \equiv 1 \pmod{q}$. Hence we have $p \equiv 1 \pmod{q}$. However, this implies $1 + p + \cdots + p^{p-1} \equiv p \pmod{q}$. From the definition of q, this leads to $p \equiv 0 \pmod{q}$, a contradiction.

Comment 1. First, students will come up, perhaps, with the idea that q has to be of the form $pk + 1$. Then,

$$\exists n \quad n^p \equiv p \pmod{q} \iff p^k \equiv 1 \pmod{q},$$

i.e.,

$$\forall n \quad n^p \not\equiv p \pmod{q} \iff p^k \not\equiv 1 \pmod{q}.$$

So, we have to find such q. These observations will take you quite naturally to the idea of taking a prime divisor of $p^p - 1$. Therefore the idea of the solution is not so tricky or technical.

Comment 2. The prime q satisfies the required condition if and only if q remains a prime in $k = \mathbb{Q}(\sqrt[p]{p})$. By applying Chebotarev's density theorem to the Galois closure of k, we see that the set of such q has the density $1/p$. In particular, there are infinitely many q satisfying the required condition. This gives an alternative solution to the problem.

N7. The sequence $a_0, a_1, a_2, \ldots$ is defined as follows:

$$a_0 = 2, \qquad a_{k+1} = 2a_k^2 - 1 \quad \text{for } k \geq 0.$$

Prove that if an odd prime p divides a_n, then 2^{n+3} divides $p^2 - 1$.

Solution. By induction, we show that

$$a_n = \frac{\left(2+\sqrt{3}\right)^{2^n} + \left(2-\sqrt{3}\right)^{2^n}}{2}.$$

Case 1: $x^2 \equiv 3 \pmod{p}$ has an integer solution

Let m be an integer such that $m^2 \equiv 3 \pmod{p}$. Then $(2+m)^{2^n} + (2-m)^{2^n} \equiv 0 \pmod{p}$. Therefore $(2+m)(2-m) \equiv 1 \pmod{p}$ shows that $(2+m)^{2^{n+1}} \equiv -1 \pmod{p}$ and that $2+m$ has the order 2^{n+2} modulo p. This implies $2^{n+2} \mid (p-1)$ and so $2^{n+3} \mid (p^2-1)$.

Case 2: otherwise

Similarly, we see that there exist integers a, b satisfying $\left(2+\sqrt{3}\right)^{2^{n+1}} = -1 + pa + pb\sqrt{3}$. Furthermore, since $\left(\left(1+\sqrt{3}\right)a_{n-1}\right)^2 = (a_n+1)(2+\sqrt{3})$, there exist integers a', b' satisfying $\left(\left(1+\sqrt{3}\right)a_{n-1}\right)^{2^{n+2}} = -1 + pa' + pb'\sqrt{3}$.

Let us consider the set $S = \{i + j\sqrt{3} \mid 0 \leq i, j \leq p-1,\ (i,j) \neq (0,0)\}$. Let $I = \left\{a + b\sqrt{3} \mid a \equiv b \equiv 0 \pmod{p}\right\}$. We claim that for each $i + j\sqrt{3} \in S$, there exists an $i' + j'\sqrt{3} \in S$ satisfying $\left(i + j\sqrt{3}\right)\left(i' + j'\sqrt{3}\right) - 1 \in I$. In fact, since $i^2 - 3j^2 \not\equiv 0 \pmod{p}$ (otherwise 3 is a square mod p), we can take an integer k satisfying $k(i^2 - 3j^2) - 1 \in I$. Then $i' + j'\sqrt{3}$ with $i' + j'\sqrt{3} - k\left(i - j\sqrt{3}\right) \in I$ will do. Now the claim together with the previous observation implies that the minimal r with $\left(\left(1+\sqrt{3}\right)a_{n-1}\right)^r - 1 \in I$ is equal to 2^{n+3}. The claim also implies that a map $f\colon S \longrightarrow S$ satisfying $\left(i + j\sqrt{3}\right)\left(1+\sqrt{3}\right)a_{n-1} - f\left(i + j\sqrt{3}\right) \in I$ for any $i + j\sqrt{3} \in S$ exists and is bijective. Thus $\prod_{x \in S} x = \prod_{x \in S} f(x)$, so

$$\left(\prod_{x \in S} x\right)\left(\left(\left(1+\sqrt{3}\right)a_{n-1}\right)^{p^2-1} - 1\right) \in I.$$

Again, by the claim, we have $\left(\left(1+\sqrt{3}\right)a_{n-1}\right)^{p^2-1} - 1 \in I$. Hence $2^{n+3} \mid (p^2-1)$.

Comment 1. Not only Case 2 but also Case 1 can be treated by using $\left(1+\sqrt{3}\right)a_{n-1}$. In fact, we need not divide into cases: in any case, the element $\left(1+\sqrt{3}\right)a_{n-1} = \left(1+\sqrt{3}\right)/\sqrt{2}$ of the multiplicative group $\mathbb{F}_{p^2}^{\times}$ of the finite field $\mathbb{F}_{p^2}$ having p^2 elements has the order 2^{n+3}, which suffices (in Case 1, the number $\left(1+\sqrt{3}\right)a_{n-1}$ even belongs to the subgroup $\mathbb{F}_p^{\times}$ of $\mathbb{F}_{p^2}^{\times}$, so $2^{n+3} \mid (p-1)$).

Comment 2. The numbers a_k are the numerators of the approximation to $\sqrt{3}$ obtained by using the Newton method with $f(x) = x^2 - 3$, $x_0 = 2$. More precisely,

$$x_{k+1} = \frac{x_k + \frac{3}{x_k}}{2}, \qquad x_k = \frac{a_k}{d_k},$$

where

$$d_k = \frac{\left(2+\sqrt{3}\right)^{2^k} - \left(2-\sqrt{3}\right)^{2^k}}{2\sqrt{3}}.$$

Comment 3. Define $f_n(x)$ inductively by

$$f_0(x) = x, \qquad f_{k+1}(x) = f_k(x)^2 - 2 \quad \text{for } k \geq 0.$$

Then the condition $p \mid a_n$ can be read that the $\bmod p$ reduction of the minimal polynomial f_n of the algebraic integer $\alpha = \zeta_{2^{n+2}} + \zeta_{2^{n+2}}^{-1}$ over $\mathbb{Q}$ has the root $2a_0$ in $\mathbb{F}_p$, where $\zeta_{2^{n+2}}$ is a primitive 2^{n+2}-th root of 1. Thus the conclusion $(p^2-1) \mid 2^{n+3}$ of the problem is a part of the decomposition theorem in the class field theory applied to the abelian extension $\mathbb{Q}(\alpha)$, which asserts that a prime p is completely decomposed in $\mathbb{Q}(\alpha)$ (equivalently, f_n has a root $\bmod p$) if and only if the class of p in $(\mathbb{Z}/2^{n+2}\mathbb{Z})^\times$ belongs to its subgroup $\{1, -1\}$. Thus the problem illustrates a result in the class field theory.

N8. Let p be a prime number and let A be a set of positive integers that satisfies the following conditions:

(i) the set of prime divisors of the elements in A consists of $p-1$ elements;

(ii) for any nonempty subset of A, the product of its elements is not a perfect p-th power.

What is the largest possible number of elements in A?

Solution. The answer is $(p-1)^2$. For simplicity, let $r = p-1$. Suppose that the prime numbers $p_1, \ldots, p_r$ are distinct. Define

$$B_i = \left\{p_i, p_i^{p+1}, p_i^{2p+1}, \ldots, p_i^{(r-1)p+1}\right\},$$

and let $B = \bigcup_{i=1}^r B_i$. Then B has r^2 elements and clearly satisfies (i) and (ii).

Now suppose that $|A| \geq r^2+1$ and that A satisfies (i) and (ii). We will show that a (nonempty) product of elements in A is a perfect p-th power. This will complete the proof.

Let $p_1, \ldots, p_r$ be distinct prime numbers for which each $t \in A$ can be written as $t = p_1^{a_1} \cdots p_r^{a_r}$. Take $t_1, \ldots, t_{r^2+1} \in A$, and for each i, let $v_i = (a_{i1}, a_{i2}, \ldots, a_{ir})$ denote the vector of exponents of prime divisors of t_i. We would like to show that a (nonempty) sum of v_i is the zero vector modulo p.

We shall show that the following system of congruence equations has a nonzero solution:

$$F_1 = \sum_{i=1}^{r^2+1} a_{i1}x_i^r \equiv 0 \pmod{p},$$

$$F_2 = \sum_{i=1}^{r^2+1} a_{i2}x_i^r \equiv 0 \pmod{p},$$

$$\vdots$$

$$F_r = \sum_{i=1}^{r^2+1} a_{ir}x_i^r \equiv 0 \pmod{p}.$$

If $(x_1, \ldots, x_{r^2+1})$ is a nonzero solution to the above system, then, since $x_i^r \equiv 0$ or $1 \pmod{p}$, a sum of vectors v_i is the zero vector modulo p.

In order to find a nonzero solution to the above system, it is enough to show that the following congruence equation has a nonzero solution:

$$F = F_1^r + F_2^r + \cdots + F_r^r \equiv 0 \pmod{p}. \tag{$*$}$$

In fact, because each F_i^r is 0 or 1 modulo p, the nonzero solution to this equation $(*)$ has to satisfy $F_i^r \equiv 0$ for $1 \leq i \leq r$.

We will show that the number of the solutions to the equation $(*)$ is divisible by p. Then since $(0, 0, \ldots, 0)$ is a trivial solution, there exists a nonzero solution to $(*)$ and we are done.

We claim that

$$\sum F^r(x_1, \ldots, x_{r^2+1}) \equiv 0 \pmod{p},$$

where the sum is over the set of all vectors $(x_1, \ldots, x_{r^2+1})$ in the vector space $\mathbb{F}_p^{r^2+1}$ over the finite field $\mathbb{F}_p$. By Fermat's little theorem, this claim evidently implies that the number of solutions to the equation $(*)$ is divisible by p.

We prove the claim. In each monomial in F^r, there are at most r^2 variables, and therefore at least one of the variables is absent. Suppose that the monomial is of the form $bx_{i_1}^{\alpha_1}x_{i_2}^{\alpha_2}\cdots x_{i_k}^{\alpha_k}$, where $1 \le k \le r^2$. Then $\sum bx_{i_1}^{\alpha_1}x_{i_2}^{\alpha_2}\cdots x_{i_k}^{\alpha_k}$, where the sum is over the same set as above, is equal to $p^{r^2+1-k}\sum_{x_{i_1},\ldots,x_{i_k}} bx_{i_1}^{\alpha_1}x_{i_2}^{\alpha_2}\cdots x_{i_k}^{\alpha_k}$, which is divisible by p. This proves the claim.

Comment. In general, if we replace $p-1$ in (i) with any positive integer d, the answer is $(p-1)d$. In fact, if $k > (p-1)d$, then the constant term of the element $(1-g_1)\cdots(1-g_k)$ of the group algebra $\mathbb{Q}_p(\zeta_p)\big[(\mathbb{Z}/p\mathbb{Z})^d\big]$ can be evaluated p-adically so we see that it is not equal to 1. Here $g_1, \ldots, g_k \in (\mathbb{Z}/p\mathbb{Z})^d$, $\mathbb{Q}_p$ is the p-adic number field, and ζ_p is a primitive p-th root of 1. This also gives an alternative solution to the problem.

Algebra

A1. Find all monic integer polynomials $p(x)$ of degree two for which there exists an integer polynomial $q(x)$ such that $p(x)q(x)$ is a polynomial having all coefficients ± 1.

First solution. We show that the only polynomials $p(x)$ with the required property are $x^2 \pm x \pm 1$, $x^2 \pm 1$ and $x^2 \pm 2x + 1$.

Let $f(x)$ be any polynomial of degree n having all coefficients ± 1. Suppose that z is a root of $f(x)$ with $|z| > 1$. Then

$$|z|^n = |\pm z^{n-1} \pm z^{n-2} \pm \cdots \pm 1| \leq |z|^{n-1} + |z|^{n-2} + \cdots + 1 = \frac{|z|^n - 1}{|z| - 1}.$$

This leads to $|z|^n(|z| - 2) \leq -1$; hence $|z| < 2$. Thus, all the roots of $f(x) = 0$ have absolute value less than 2.

Clearly, a polynomial $p(x)$ with the required properties must be of the form $p(x) = x^2 + ax \pm 1$ for some integer a. Let x_1 and x_2 be its roots (not necessarily distinct). As $x_1 x_2 = \pm 1$, we may assume that $|x_1| \geq 1$ and $|x_2| \leq 1$. Since x_1, x_2 are also roots of $p(x)q(x)$, a polynomial with coefficients ± 1, we have $|x_1| < 2$, and so $|a| = |x_1 + x_2| \leq |x_1| + |x_2| < 2 + 1$. Thus, $a \in \{\pm 2, \pm 1, 0\}$.

If $a = \pm 1$, then $q(x) = 1$ leads to a solution.

If $a = 0$, then $q(x) = x + 1$ leads to a solution.

If $a = \pm 2$, both polynomials $x^2 \pm 2x - 1$ have one root of absolute value greater than 2, so they cannot satisfy the requirement. Finally, the polynomials $p(x) = x^2 \pm 2x + 1$ do have the required property with $q(x) = x \mp 1$, respectively.

Comment. By a “root” we may mean a “complex root,” and then nothing requires clarification. But complex numbers need not be mentioned at all, because $p(x) = x^2 + ax \pm 1$ has real roots if $|a| \geq 2$; and the cases of $|a| \leq 1$ must be handled separately anyway.

The proposer remarks that even if $p(x)q(x)$ is allowed to have zero coefficients, the conclusion $|z| < 2$ about its roots holds true. However, extra solutions appear: x^2 and $x^2 \pm x$.

Second solution. Suppose that the polynomials $p(x) = a_0 + a_1x + x^2$ and $q(x) = b_0 + b_1x + \cdots + b_nx^n$ are such that $p(x)q(x) = c_0 + c_1x + \cdots + c_{n+2}x^{n+2}$ with all $c_k = \pm 1$. Then $|a_0| = |b_0| = |b_n| = 1$ and

$$a_0b_1 = c_1 - a_1b_0, \qquad a_0b_k = c_k - a_1b_{k-1} - b_{k-2} \quad \text{for } k = 2, \ldots, n,$$

whence

$$|b_1| \geq |a_1| - 1, \qquad |b_k| \geq |a_1b_{k-1}| - |b_{k-2}| - 1 \quad \text{for } k = 2, \ldots, n.$$

Assume $|a_1| \geq 3$. Then clearly $q(x)$ cannot be a constant, so $n \geq 1$, and we get

$$|b_1| \geq 2, \qquad |b_k| \geq 3|b_{k-1}| - |b_{k-2}| - 1 \quad \text{for } k = 2, \ldots, n.$$

Recasting the last inequality into

$$|b_k| - |b_{k-1}| \geq 2|b_{k-1}| - |b_{k-2}| - 1 \geq 2\big(|b_{k-1}| - |b_{k-2}|\big) - 1$$

we see that the sequence $d_k = |b_k| - |b_{k-1}|$ $(k = 1, \ldots, n)$ obeys the recursive estimate $d_k \geq 2d_{k-1} - 1$ for $k \geq 2$. As $d_1 = |b_1| - 1 \geq 1$, this implies by obvious induction $d_k \geq 1$ for $k = 1, \ldots, n$. Equivalently, $|b_k| \geq |b_{k-1}| + 1$ for $k = 2, \ldots, n$, and hence $|b_n| \geq |b_0| + n$, in contradiction to $|b_0| = |b_n| = 1$, $n \geq 1$.

It follows that $p(x)$ must be of the form $a_0 + a_1x + x^2$ with $|a_0| = 1$, $|a_1| \leq 2$. If $|a_1| \leq 1$ or $|a_1| = 2$ and $a_0 = 1$, then the corresponding $q(x)$ exists; see the eight examples in the first solution.

We are left with the case $|a_1| = 2$, $a_0 = -1$. Assume $q(x)$ exists. There is no loss of generality in assuming that $b_0 = 1$ and $a_1 = 2$ (if $b_0 = -1$, replace $q(x)$ by $-q(x)$; and if $a_1 = -2$, replace $q(x)$ by $q(-x)$). With $b_0 = 1$, $a_0 = -1$, $a_1 = 2$ the initial recursion formulas become

$$b_1 = 2 - c_1, \qquad b_k = 2b_{k-1} + b_{k-2} - c_k \quad \text{for } k = 2, \ldots, n.$$

Therefore $b_1 \geq 1$, $b_2 \geq 2b_1 + 1 - c_2 \geq 2$, and induction shows that $b_k \geq 2$ for $k = 2, \ldots, n$, again in contradiction with $|b_n| = 1$. So there are no "good" trinomials $p(x)$ except the eight mentioned above.

A2. Let $\mathbb{R}^+$ denote the set of positive real numbers. Determine all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$f(x)f(y) = 2f\bigl(x + yf(x)\bigr)$$

for all positive real numbers x and y.

Solution. The answer is the constant function $f(x) = 2$ which clearly satisfies the equation.

First, we show that a function f satisfying the equation is nondecreasing. Indeed, suppose that $f(x) < f(z)$ for some positive real numbers $x > z$. Set $y = (x - z)(f(z) - f(x)) > 0$, so that $x + yf(x) = z + yf(z)$. The equation now implies

$$f(x)f(y) = 2f(x + yf(x)) = 2f(z + yf(z)) = f(z)f(y),$$

therefore $f(x) = f(z)$, a contradiction. Thus, f is nondecreasing.

Assume now that f is not strictly increasing, that is, $f(x) = f(z)$ holds for some positive real numbers $x > z$. If y belongs to the interval $(0, (x - z)/f(x)]$ then $z < z + yf(z) \leq x$. Since f is nondecreasing, we obtain

$$f(z) \leq f(z + yf(z)) \leq f(x) = f(z),$$

leading to $f(z+yf(z)) = f(x)$. Thus, $f(z)f(y) = 2f(z+yf(z)) = 2f(x) = 2f(z)$. Hence, $f(y) = 2$ for all y in the above interval.

But if $f(y_0) = 2$ for some y_0 then

$$2 \cdot 2 = f(y_0)f(y_0) = 2f(y_0 + y_0 f(y_0)) = 2f(3y_0); \quad \text{therefore} \quad f(3y_0) = 2.$$

By obvious induction, we get that $f(3^k y_0) = 2$ for all positive integers k, and so $f(x) = 2$ for all $x \in \mathbb{R}^+$.

Assume now that f is a strictly increasing function. We then conclude that the inequality $f(x)f(y) = 2f(x + yf(x)) > 2f(x)$ holds for all positive real numbers x, y. Thus, $f(y) > 2$ for all $y > 0$. The equation implies

$$2f(x + 1 \cdot f(x)) = f(x)f(1) = f(1)f(x) = 2f(1 + xf(1)) \qquad \text{for } x > 0,$$

and since f is injective, we get $x + 1 \cdot f(x) = 1 + x \cdot f(1)$ leading to the conclusion that $f(x) = x(f(1) - 1) + 1$ for all $x \in \mathbb{R}^+$. Taking a small x (close to zero), we get $f(x) < 2$, which is a contradiction. (Alternatively, one can verify directly that $f(x) = cx + 1$ is not a solution of the given functional equation.)

A3. Four real numbers p, q, r, s satisfy

$$p + q + r + s = 9 \qquad \text{and} \qquad p^2 + q^2 + r^2 + s^2 = 21.$$

Prove that $ab - cd \geq 2$ holds for some permutation (a, b, c, d) of (p, q, r, s).

First solution. Up to a permutation, we may assume that $p \geq q \geq r \geq s$. We first consider the case where $p + q \geq 5$. Then

$$p^2 + q^2 + 2pq \geq 25 = 4 + (p^2 + q^2 + r^2 + s^2) \geq 4 + p^2 + q^2 + 2rs,$$

which is equivalent to $pq - rs \geq 2$.

Assume now that $p + q < 5$; then

$$4 < r + s \leq p + q < 5. \tag{1}$$

Observe that

$$(pq + rs) + (pr + qs) + (ps + qr) = \frac{(p + q + r + s)^2 - (p^2 + q^2 + r^2 + s^2)}{2} = 30.$$

Moreover,

$$pq + rs \geq pr + qs \geq ps + qr,$$

because $(p - s)(q - r) \geq 0$ and $(p - q)(r - s) \geq 0$.

We conclude that $pq + rs \geq 10$. From (1), we get $0 \leq (p + q) - (r + s) < 1$, therefore

$$(p + q)^2 - 2(p + q)(r + s) + (r + s)^2 < 1.$$

Adding this to $(p + q)^2 + 2(p + q)(r + s) + (r + s)^2 = 9^2$ gives

$$(p + q)^2 + (r + s)^2 < 41.$$

Therefore

$$\begin{aligned} 41 = 21 + 2 \cdot 10 &\leq (p^2 + q^2 + r^2 + s^2) + 2(pq + rs) \\ &= (p + q)^2 + (r + s)^2 < 41, \end{aligned}$$

which is a contradiction.

Second solution. We first note that $pq + pr + ps + qr + qs + rs = 30$, as in the first solution. Thus, if $(a,\ b,\ c,\ d)$ is any permutation of (p, q, r, s), then

$$bc + cd + db = 30 - a(b + c + d) = 30 - a(9 - a) = 30 - 9a + a^2,$$

while

$$bc + cd + db \leq b^2 + c^2 + d^2 = 21 - a^2.$$

Hence $30 - 9a + a^2 \leq 21 - a^2$, leading to $a \in [3/2, 3]$. Thus the numbers p, q, r and s are in the interval $[3/2, 3]$.

Assume now that $p \geq q \geq r \geq s$. Note that $q \geq 2$ because otherwise $p = 9 - (q + r + s) \geq 9 - 3q > 9 - 6 = 3$, which is impossible.

Write $x = r - s,\ y = q - r$ and $z = p - q$. On the one hand,

$$\begin{aligned}(p - q)^2 + (p - r)^2 + (p - s)^2 + (q - r)^2 + (q - s)^2 + (r - s)^2 \\ = 3(p^2 + q^2 + r^2 + s^2) - 2(pq + pr + ps + qr + qs + rs) = 3.\end{aligned}$$

On the other hand, this expression equals

$$\begin{aligned}z^2 + (z + y)^2 + &(z + y + x)^2 + y^2 + (y + x)^2 + x^2 \\ &= 3x^2 + 4y^2 + 3z^2 + 4xy + 4yz + 2zx.\end{aligned}$$

Hence,

$$3x^2 + 4y^2 + 3z^2 + 4xy + 4yz + 2zx = 3. \tag{2}$$

Furthermore,

$$pq - rs = q(p - s) + (q - r)s = q(x + y + z) + ys.$$

If $x + y + z \geq 1$ then, in view of $q \geq 2$, we immediately get $pq - rs \geq 2$.

If $x + y + z < 1$ then (2) implies

$$3x^2 + 4y^2 + 3z^2 + 4xy + 4yz + 2zx > 3(x + y + z)^2.$$

It follows that $y^2 > 2xy + 2yz + 4zx \geq 2y(x + z)$, so that $y > 2(x + z)$ and hence $3y > 2(x + y + z)$. The value of the left-hand side of (2) obviously does not exceed $4(x + y + z)^2$, so that $2(x + y + z) \geq \sqrt{3}$. Eventually, $3y > \sqrt{3}$ and recalling that $s \geq 3/2$, we obtain

$$pq - rs = q(x + y + z) + ys \geq 2 \cdot \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{3} \cdot \frac{3}{2} = \frac{3\sqrt{3}}{2} > 2.$$

A4. Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ satisfying the equation

$$f(x+y) + f(x)f(y) = f(xy) + 2xy + 1$$

for all real numbers x and y.

Solution. The solutions are $f(x) = 2x - 1$, $f(x) = -x - 1$ and $f(x) = x^2 - 1$. It is easy to check that these functions indeed satisfy the given equation.

We begin by setting $y = 1$ which gives

$$f(x+1) = af(x) + 2x + 1, \tag{1}$$

where $a = 1 - f(1)$. Then we change y to $y+1$ in the equation and use (1) to expand $f(x+y+1)$ and $f(y+1)$. The result is

$$a\left(f(x+y) + f(x)f(y)\right) + (2y+1)(1+f(x)) = f(x(y+1)) + 2xy + 1,$$

or, using the initial equation again,

$$a\left(f(xy) + 2xy + 1\right) + (2y+1)(1+f(x)) = f(x(y+1)) + 2xy + 1.$$

Let us now set $x = 2t$ and $y = -1/2$ to obtain

$$a\left(f(-t) - 2t + 1\right) = f(t) - 2t + 1.$$

Replacing t by $-t$ yields one more relation involving $f(t)$ and $f(-t)$:

$$a\left(f(t) + 2t + 1\right) = f(-t) + 2t + 1. \tag{2}$$

We now eliminate $f(-t)$ from the last two equations, leading to

$$(1 - a^2)f(t) = 2(1-a)^2 t + a^2 - 1.$$

Note that $a \neq -1$ (or else $8t = 0$ for all t, which is false). If additionally $a \neq 1$ then $1 - a^2 \neq 0$, therefore

$$f(t) = 2\left(\frac{1-a}{1+a}\right)t - 1.$$

Setting $t = 1$ and recalling that $f(1) = 1 - a$, we get $a = 0$ or $a = 3$, which gives the first two solutions.

The case $a = 1$ remains, where (2) yields

$$f(t) = f(-t) \qquad \text{for all } t \in \mathbb{R}. \tag{3}$$

Now set $y = x$ and $y = -x$ in the original equation. In view of (3), we obtain, respectively,

$$f(2x) + f(x)^2 = f(x^2) + 2x^2 + 1, \quad f(0) + f(x)^2 = f(x^2) - 2x^2 + 1.$$

Subtracting gives $f(2x) = 4x^2 + f(0)$. Set $x = 0$ in (1). Since $f(1) = 1 - a = 0$, this yields $f(0) = -1$. Hence $f(2x) = 4x^2 - 1$, i. e. $f(x) = x^2 - 1$ for all $x \in \mathbb{R}$. This completes the solution.

A5. Let x, y and z be positive real numbers such that $xyz \geq 1$. Prove the inequality

$$\frac{x^5 - x^2}{x^5 + y^2 + z^2} + \frac{y^5 - y^2}{y^5 + z^2 + x^2} + \frac{z^5 - z^2}{z^5 + x^2 + y^2} \geq 0.$$

First solution. Standard recasting shows that the given inequality is equivalent to

$$\frac{x^2 + y^2 + z^2}{x^5 + y^2 + z^2} + \frac{x^2 + y^2 + z^2}{y^5 + z^2 + x^2} + \frac{z^2 + x^2 + y^2}{z^5 + x^2 + y^2} \leq 3.$$

In view of the Cauchy-Schwarz inequality and the condition $xyz \geq 1$, we have

$$(x^5 + y^2 + z^2)(yz + y^2 + z^2) \geq \left(x^{5/2}(yz)^{1/2} + y^2 + z^2\right)^2 \geq (x^2 + y^2 + z^2)^2,$$

or

$$\frac{x^2 + y^2 + z^2}{x^5 + y^2 + z^2} \leq \frac{yz + y^2 + z^2}{x^2 + y^2 + z^2}.$$

Taking the cyclic sum and using the fact that $x^2 + y^2 + z^2 \geq yz + zx + xy$ gives

$$\frac{x^2 + y^2 + z^2}{x^5 + y^2 + z^2} + \frac{x^2 + y^2 + z^2}{y^5 + z^2 + x^2} + \frac{x^2 + y^2 + z^2}{z^5 + x^2 + y^2} \leq 2 + \frac{yz + zx + xy}{x^2 + y^2 + z^2} \leq 3,$$

which is exactly what we wished to show.

Comment. The way the Cauchy-Schwarz inequality is used is the crucial point in the solution; it is not at all obvious! The condition $xyz \geq 1$ (which might as well have been $xyz = 1$) allows to transform the expression to a homogeneous form. The smart use of Cauchy-Schwarz inequality has the effect that the common *numerators* of the three fractions become common *denominators* in the transformed expression.

Second solution. We shall prove something more, namely that

$$\frac{x^5}{x^5 + y^2 + z^2} + \frac{y^5}{y^5 + z^2 + x^2} + \frac{z^5}{z^5 + x^2 + y^2} \geq 1, \tag{1}$$

and

$$1 \geq \frac{x^2}{x^5 + y^2 + z^2} + \frac{y^2}{y^5 + z^2 + x^2} + \frac{z^2}{z^5 + x^2 + y^2}. \tag{2}$$

We first prove (1). We have $yz(y^2 + z^2) = y^3 z + yz^3 \leq y^4 + z^4$; the latter inequality holds because $y^4 - y^3 z - yz^3 + z^4 = (y^3 - z^3)(y - z) \geq 0$. Therefore $x(y^4 + z^4) \geq xyz(y^2 + z^2) \geq y^2 + z^2$, or

$$\frac{x^5}{x^5 + y^2 + z^2} \geq \frac{x^5}{x^5 + xy^4 + xz^4} = \frac{x^4}{x^4 + y^4 + z^4}.$$

Taking the cyclic sum, we get the desired inequality.

The proof of (2) is based on exactly the same ideas as in the first solution. From the Cauchy-Schwarz inequality and the fact that $xyz \geq 1$, we have

$$(x^5 + y^2 + z^2)(yz + y^2 + z^2) \geq (x^2 + y^2 + z^2)^2,$$

implying

$$\frac{x^2}{x^5 + y^2 + z^2} \leq \frac{x^2(yz + y^2 + z^2)}{(x^2 + y^2 + z^2)^2}.$$

Taking the cyclic sum, we have

$$\begin{aligned} & \frac{x^2}{x^5 + y^2 + z^2} + \frac{y^2}{y^5 + z^2 + x^2} + \frac{z^2}{z^5 + x^2 + y^2} \\ \leq \quad & \frac{2(x^2y^2 + y^2z^2 + z^2x^2) + x^2yz + y^2zx + z^2xy}{(x^2 + y^2 + z^2)^2} \\ = \quad & \frac{(x^2 + y^2 + z^2)^2 - (x^4 + y^4 + z^4) + (x^2yz + y^2zx + z^2xy)}{(x^2 + y^2 + z^2)^2}. \end{aligned}$$

Thus we need to show that $x^4 + y^4 + z^4 \geq x^2yz + y^2zx + z^2xy$; and this last inequality holds because

$$\begin{aligned} x^4 + y^4 + z^4 &= \frac{x^4 + y^4}{2} + \frac{y^4 + z^4}{2} + \frac{z^4 + x^4}{2} \geq x^2y^2 + y^2z^2 + z^2x^2 \\ &= \frac{x^2y^2 + y^2z^2}{2} + \frac{y^2z^2 + z^2x^2}{2} + \frac{z^2x^2 + x^2y^2}{2} \\ &\geq y^2zx + z^2xy + x^2yz. \end{aligned}$$

Combinatorics

C1. A house has an even number of lamps distributed among its rooms in such a way that there are at least three lamps in every room. Each lamp shares a switch with exactly one other lamp, not necessarily from the same room. Each change in the switch shared by two lamps changes their states simultaneously. Prove that for every initial state of the lamps there exists a sequence of changes in some of the switches at the end of which each room contains lamps which are on as well as lamps which are off.

Solution. Two lamps sharing a switch will be called *twins*. A room will be called *normal* if some of its lamps are on and some are off. We devise an algorithm that increases the number of normal rooms in the house. After several runs of the algorithm we arrive at the state with all rooms normal.

Choose any room R_0 which is not normal, assuming without loss of generality that all lamps in R_0 are off. If there is a pair of twins in R_0, we switch them on and stop. Saying *stop* means that we have achieved what we wanted: there are more normal rooms than before the algorithm started.

So suppose there are no twins in R_0. Choose any lamp $a_0 \in R_0$ and let $b_0 \in R_1$ be its twin. Change their states. After this move room R_0 becomes normal. If R_1 also becomes (or remains) normal, then stop. Otherwise all lamps in R_1 are in equal state; as before we can assume that there are no twins in R_1. Choose any lamp $a_1 \in R_1$ other than b_0 and let $b_1 \in R_2$ be its twin. Change the states of these two twin lamps. If R_2 becomes (or stays) normal, stop.

Proceed in this fashion until a repetition occurs in the sequence $R_0, R_1, R_2, \dots$. Thus assume that the rooms $R_0, R_1, \dots, R_m$ are all distinct, each R_i connected to R_{i+1} through a twin pair $a_i \in R_i$, $b_i \in R_{i+1}$ $(i = 0, \dots, m-1)$, and there is a lamp $a_m \in R_m$ $(a_m \neq b_{m-1})$ which has its twin b_m in some room R_k visited earlier $(0 \leq k \leq m-1)$. If the algorithm did not stop after we entered room R_m, we change the states of the lamps a_m and b_m; room R_m becomes normal.

If $k \geq 1$, then there are two lamps in R_k touched previously, b_{k-1} and a_k. They are the twins of a_{k-1} and b_k, so neither of them can be b_m (twin to a_m). Recall that the moment we entered room R_k the first time, by pressing the b_{k-1} switch, this room became "abnormal" only until we touched lamp a_k. Thus b_{k-1} and a_k are in different states now. Whatever the new state of lamp b_m, room R_k remains normal. Stop.

Finally, if $k = 0$, then $b_m \in R_0$ and $b_m \neq a_0$ because the twin of a_0 is b_0. Each room has at least three lamps, so there is a lamp $c \in R_0$, $c \neq a_0$, $c \neq b_m$. In the first move lamp a_0 was put on while c stayed off. Whatever the new state of b_m, room R_0 stays normal. Stop.

So, indeed, after a single run of this algorithm, the number of normal rooms increases at least by 1. This completes the proof.

Comment. The problem was submitted in the following formulation:

> A school has an even number of students, each of whom attends exactly one of its (finitely many) classes. Each class has at least three students, and each student has exactly one "best friend" in the same school such that, whenever B is A's "best friend", then A is B's "best friend". Furthermore, each student prefers either apple juice over orange juice or orange juice over apple juice, but students change their preferences from time to time. "Best friends", however, will change their preferences (which may or may not be the same) always together, at the same moment.
>
> Whatever preference each student may initially have, prove that there is always a sequence of changes of preferences which will lead to a situation in which no class will have students all of whom have the same preference.

C2. Let k be a fixed positive integer. A company has a special method to sell sombreros. Each customer can convince two persons to buy a sombrero after he/she buys one; convincing someone already convinced does not count. Each of these new customers can convince two others and so on. If each one of the two customers convinced by someone makes at least k persons buy sombreros (directly or indirectly), then that someone wins a free instructional video. Prove that if n persons bought sombreros, then at most $n/(k+2)$ of them got videos.

First solution. Consider the problem in reverse: If w persons won free videos, what is the least number n of persons who bought sombreros? One can easily compute this minimum for small values of w: for $w = 1$ it is $2k + 3$, and for $w = 2$ it is $3k + 5$. These can be rewritten as $n \geq 1 \cdot (k+2) + (k+1)$ and $n \geq 2(k+2) + (k+1)$, leading to the conjecture that

$$n \geq w(k+2) + (k+1). \tag{1}$$

Let us say that a person P *influenced* a person Q if P made Q buy a sombrero directly or indirectly, or if $Q = P$. A *component* is the set of persons influenced by someone who was influenced by no one else but himself. No person from a component influenced another one from a different component. So it suffices to prove (1) for each component. Indeed, if (1) holds for r components of size n_i with w_i winners, $i = 1, \ldots, r$, then

$$n = \sum n_i \geq \sum (w_i(k+2) + (k+1)) = \left(\sum w_i\right)(k+2) + r(k+1),$$

implying (1) for $n = \sum n_i$, $w = \sum w_i$.

Thus one may assume that the whole group is a single component, i. e. all customers were influenced by one person A (directly or indirectly).

Moreover, it suffices to prove (1) for a group G with w winners and of *minimum* size n. Notice that then A is a video winner. If not, imagine him removed from the group. A video winner from the original group is also a winner in the new one. So we have decreased n without changing w, a contradiction.

Under these assumptions, we proceed to prove (1) by induction on $w \geq 1$. For $w = 1$, the group of customers contains a single video winner A, the two persons B and C he/she convinced directly to buy sombreros, and two nonintersecting groups of k persons, the ones persuaded by B and C (directly or indirectly). This makes at least $2k + 3$ persons, as needed.

Assume the claim holds for groups with less than w winners, and consider a group with n winners where everyone was influenced by some person A. Recall that A is a winner. Let B and C be the persons convinced directly by A to buy sombreros. Let n_B be the number of people influenced by B, and w_B the number of video winners among them. Define n_C and w_C analogously.

We have $n_B \geq w_B(k+2)+(k+1)$, by the induction hypothesis if $w_B > 0$ and because A is a winner if $w_B = 0$. Analogously $n_C \geq w_C(k+2)+(k+1)$. Adding the two inequalities gives us $n \geq w(k+2)+(k+1)$, since $n = n_B + n_C + 1$ and $w = w_B + w_C + 1$. This concludes the proof.

Second solution. As in the first solution, we say that a person P *influenced* a person Q if P made Q buy a sombrero directly or indirectly, or if $Q = P$. Likewise, we keep the definition of a component. For brevity, let us write winners instead of video winners.

The components form a partition of the set of people who bought sombreros. It is enough to prove that in each component the fraction of winners is at most $1/(k+2)$.

We will minimise the number of people buying sombreros while keeping the number of winners fixed.

First, we can assume that no winners were convinced (directly) by a nonwinner. Indeed, if a nonwinner P convinced a winner Q, remove all people influenced by P but not by Q and let whoever convinced P (if anyone did) now convince Q. Observe that no winner was removed, hence the new configuration has fewer people, but the same winners.

Thus, indeed, there is no loss of generality in assuming that:

The set of all buyers makes up a single component. (2)

Every winner could have been convinced only by another winner. (3)

Now remove all the winners and consider the new components. We claim that

Each new component has at least $k+1$ persons. (4)

Indeed, let $\mathcal{C}$ be a new component. In view of (2), there is a member C of $\mathcal{C}$ who had been convinced by some removed winner W. Then C must have influenced at least $k+1$ people (including himself), but all the people influenced by C are in $\mathcal{C}$. Therefore $|\mathcal{C}| \geq k+1$.

Now return the winners one by one in such a way that when a winner returns, the people he convinced (directly) are already present. This is possible because of (3). In that way the number of components decreases by one with each winner, thus the number of components with all winners removed is equal to $w+1$, where w is the number of winners. It follows from (4) that the number of nonwinners satisfies the estimate

$$n - w \geq (w+1)(k+1).$$

This implies the desired bound.

C3. In an $m \times n$ rectangular board of mn unit squares, *adjacent* squares are ones with a common edge, and a *path* is a sequence of squares in which any two consecutive squares are adjacent. Each square of the board can be coloured black or white. Let N denote the number of colourings of the board such that there exists at least one black path from the left edge of the board to its right edge, and let M denote the number of colourings in which there exist at least two non-intersecting black paths from the left edge to the right edge. Prove that $N^2 \geq M \cdot 2^{mn}$.

Solution. We generalise the claim to the following. Suppose that a two-sided $m \times n$ board is considered, where some of the squares are transparent and some others are not. Each square must be coloured black or white. However, a transparent square needs to be coloured only on one side; then it looks the same from above and from below. In contrast, a non-transparent square must be coloured on both sides (in the same colour or not).

Let A (respectively B) be the set of colourings of the board with at least one black path from the left edge to the right edge if one looks from above (respectively from below).

Let C be the set of colourings of the board in which there exist two black paths from the left edge to the right edge of the board, one on top and one underneath, not intersecting at any transparent square.

Let D be the set of all colourings of the board.

We claim that

$$|A| \cdot |B| \geq |C| \cdot |D|. \tag{1}$$

Note that this implies the original claim in the case where all squares are transparent: one then has $|A| = |B| = N$, $|C| = M$, $|D| = 2^{mn}$.

We prove (1) by induction on the number k of transparent squares. If $k = 0$ then $|A| = |B| = N \cdot 2^{mn}$, $|C| = N^2$ and $|D| = (2^{mn})^2$, so equality holds in (1). Suppose the claim is true for some k and consider a board with $k+1$ transparent squares. Let A, B, C and D be the sets of colourings of this board as defined above. Choose one transparent square ϑ. Now, convert ϑ into a non-transparent square, and let A', B', C' and D' be the respective sets of colourings of the new board. By the induction hypothesis, we have:

$$|A'| \cdot |B'| \geq |C'| \cdot |D'|. \tag{2}$$

Upon the change made, the number of all colourings doubles. So $|D'| = 2|D|$.

To any given colouring in A, there correspond two colourings in A', obtained by colouring ϑ black and white from below. This is a bijective correspondence, so $|A'| = 2|A|$. Likewise, $|B'| = 2|B|$. In view of (2), it suffices to prove that

$$|C'| \geq 2|C|. \tag{3}$$

Make ϑ transparent again and take any colouring in C. It contains two black paths (one seen from above and one from below) that do not intersect at transparent squares. Being transparent, ϑ can therefore lie on at most one of them, say on the path above. So when we make ϑ non-transparent, let us keep its colour on the side above but colour the side below in the two possible ways. The two colourings obtained will be in C'. It is easy to see that when doing so, different colourings in C give rise to different pairs of colourings in C'. Hence (3) follows, implying (2). As already mentioned, this completes the solution.

Comment. A more direct approach to the problem may go as follows. Consider two $m \times n$ boards instead of one. Let $\mathcal{A}$ denote the set of all colourings of the two boards such that there are at least two non-intersecting black paths from the left edge of the first board to its right edge. Clearly, $|\mathcal{A}| = M \cdot 2^{mn}$: we can colour the first board in M ways and the second board in an arbitrary fashion.

Let $\mathcal{B}$ denote the set of all colourings of the two boards such that there is at least one black path from the left edge of the first board to its right edge, and at least one black path from the left edge of the second board to its right edge. Clearly, $|\mathcal{B}| = N^2$.

It suffices to find an injective function $f : \mathcal{A} \hookrightarrow \mathcal{B}$.

Such an injection can indeed be constructed. However, working it out in all details seems to be a delicate task.

C4. Let $n \geq 3$ be a given positive integer. We wish to label each side and each diagonal of a regular n-gon $P_1, \ldots, P_n$ with a positive integer less than or equal to r so that:

(i) every integer between 1 and r occurs as a label;

(ii) in each triangle $P_iP_jP_k$ two of the labels are equal and greater than the third.

Given these conditions:

(a) Determine the largest positive integer r for which this can be done.

(b) For that value of r, how many such labellings are there?

Solution. A labelling which satisfies condition (ii) will be called *good*. A labelling which satisfies both given conditions (i) and (ii) will be called *very good*. Let us try to understand the structure of good labellings.

Sides and diagonals of the polygon will be called just *edges*. Let AB be an edge with the maximum label m. Let X be any vertex different from A and B. Condition (ii), applied to triangle ABX, implies that one of the segments AX, BX has label m, and the other one has a label smaller than m. Thus we can split all vertices into two disjoint groups 1 and 2; group 1 contains vertices X such that AX has label m (including vertex B) and group 2 contains vertices X such that BX has label m (including vertex A). We claim that the edges labelled m are exactly those which join a vertex of group 1 with a vertex of group 2.

First consider any vertex $X \neq B$ in group 1 and any vertex $Y \neq A$ in group 2. In triangle AXY, we already know that the label of AX (which is m) is larger than the label of AY (which is not m). Therefore the label of XY also has to be equal to m, as we wanted to show.

Now consider any two vertices X, Y in group 1. In triangle AXY, the edges AX and AY have the same label m. So the third edge must have a label smaller than m, as desired. Similarly, any edge joining two vertices in group 2 has a label smaller than m.

We conclude that a good labelling of an n-gon consists of:

• a collection of edges with the maximum label m; they are the ones that go from a vertex of group 1 to a vertex of group 2,

• a good labelling of the polygon determined by the vertices of group 1, and

• a good labelling of the polygon determined by the vertices of group 2.

(a) The greatest possible value of r is $n-1$. We prove this by induction starting with the degenerate cases $n = 1$ and $n = 2$, where the claim is immediate. Assusme it true for values less than n, where $n \geq 3$, and consider any good labelling of an n-gon P.

Its edges are split into two groups 1 and 2; suppose they have k and $n-k$ vertices, respectively. The k-gon P_1 formed by the vertices in group 1 inherits a good labelling. By the induction hypothesis, this good labelling uses at most $k-1$ different labels. Similarly, the $(n-k)$-gon P_2 formed by the vertices in group 2 inherits a good labelling which uses at most $n-k-1$ different labels. The remaining segments, which join a vertex of group 1 with a vertex of group 2, all have the same (maximum) label. Therefore, the total number of different labels in our good labelling is at most $(k-1) + (n-k-1) + 1 = n-1$. This number can be easily achieved, as long as we use different labels in P_1 and P_2.

(b) Let $f(n)$ be the number of very good labellings of an n-gon P with labels $1, \ldots, n-1$. We will show by induction that

$$f(n) = n!(n-1)!/2^{n-1}.$$

This holds for $n = 1$ and $n = 2$. Fix $n \geq 3$ and assume that $f(k) = k!(k-1)!/2^{k-1}$ for $k < n$.

Divide the n vertices into two non-empty groups 1 and 2 in any way. If group 1 is of size k, there are $\binom{n}{k}$ ways of doing that. We must label every edge joining a vertex of group 1 and a vertex of group 2 with the label $n-1$. Now we need to choose which $k-1$ of the remaining labels $1, 2, \ldots, n-2$ will be used to label the k-gon P_1; there are $\binom{n-2}{k-1}$ possible choices. The remaining $n-k-1$ labels will be used to label the $(n-k)$-gon P_2. Finally, there are $f(k)$ very good labellings of P_1 and $f(n-k)$ very good labellings of P_2.

Now we sum the resulting expression over all possible values of k, noticing that we have counted each very good labelling twice, since choosing a set to be group 1 is equivalent to choosing its complement. We have:

$$\begin{aligned}
f(n) &= \frac{1}{2}\sum_{k=1}^{n-1}\binom{n}{k}\binom{n-2}{k-1}f(k)f(n-k) \\
&= \frac{n!(n-1)!}{2(n-1)}\sum_{k=1}^{n-1}\frac{f(k)}{k!(k-1)!}\cdot\frac{f(n-k)}{(n-k)!(n-k-1)!} \\
&= \frac{n!(n-1)!}{2(n-1)}\sum_{k=1}^{n-1}\frac{1}{2^{k-1}}\cdot\frac{1}{2^{n-k-1}} = \frac{n!(n-1)!}{2^{n-1}},
\end{aligned}$$

which is what we wanted to show.

C5. There are n markers, each with one side white and the other side black, aligned in a row so that their white sides are up. In each step, if possible, we choose a marker with the white side up (but not one of the outermost markers), remove it and reverse the closest marker to the left and the closest marker to the right of it. Prove that one can achieve the state with only two markers remaining if and only if $n-1$ is not divisible by 3.

First solution. Given a particular chain of markers, we call white (resp. black) markers the ones with the white (resp. black) side up. Note that the parity of the number of black markers remains unchanged during the game. Hence, if only two markers remain, these markers must have the same colour.

Next, we define an invariant. To a white marker with t black markers to its left we assign the number $(-1)^t$. Only white markers have numbers assigned to them. Let S be the residue class modulo 3 of the sum of all numbers assigned to the white markers.

It is easy to check that S is an invariant under the allowed operations. Suppose, for instance, that a white marker W is removed, with t black markers to the left of it, and that the closest neighbours of W are black. Then S increases by $-(-1)^t+(-1)^{t-1}+(-1)^{t-1}=3(-1)^{t-1}$. The other three cases are analogous.

If the game ends with two black markers, the number S is zero; if it ends with two white markers, then S is 2. Since we start with n white markers and in this case $S\equiv n \pmod 3$, a necessary condition for the game to end is $n\equiv 0,\ 2 \pmod 3$.

If we start with $n\geq 5$ white markers, taking the leftmost allowed white markers in three consecutive moves, we obtain a row of $n-3$ white markers without black markers. Since the goal can be reached for $n=2,\ 3$, we conclude that the game can end with two markers for every positive integer n satisfying $n\equiv 0,\ 2 \pmod 3$.

Second solution. Denote by L the leftmost and by R the rightmost marker, respectively. To start with, note again that the parity of the number of black-side-up markers remains unchanged. Hence, if only two markers remain, these markers must have the same colour up.

We will show by induction on n that the game can be succesfully finished if and only if $n\equiv 0,\ 2 \pmod 3$ and that the upper sides of L and R will be black in the first case and white in the second case.

The statement is clear for $n=2$ and 3. Assume that we finished the game for some n, and denote by k the position of the marker X (counting from the left) that was last removed. Having finished the game, we have also finished the subgames with the k markers from L to X and with the $n-k+1$ markers from X to R (inclusive). Thereby, by the induction hypothesis, before X was removed, the upper side of L had been black if $k \equiv 0 \pmod 3$, and white if $k \equiv 2 \pmod 3$, while the upper side of R had been black if $n-k+1 \equiv 0 \pmod 3$, and white if $n-k+1 \equiv 2 \pmod 3$. Markers R and L were reversed upon the removal of X. Therefore, in the final position, R and L are white if and only if $k \equiv n-k+1 \equiv 0 \pmod 3$, which yields $n \equiv 2 \pmod 3$, and black if and only if $k \equiv n-k+1 \equiv 2 \pmod 3$, which yields $n \equiv 0 \pmod 3$.

On the other hand, a game with n markers can be reduced to a game with $n-3$ markers by removing the second, fourth and third marker in this order. This finishes the induction.

C6. In a mathematical competition in which 6 problems were posed to the participants, every two of these problems were solved by more than 2/5 of the contestants. Moreover, no contestant solved all the 6 problems. Show that there are at least 2 contestants who solved exactly 5 problems each.

First solution. Assume there were n contestants. Let us count the number N of ordered pairs (C, P), where P is a pair of problems solved by contestant C. On the one hand, for every one of the 15 pairs of problems, there are at least $(2n+1)/5$ contestants who solved both problems in the pair. Therefore

$$N \geq 15 \cdot \frac{2n+1}{5} = 6n+3. \tag{1}$$

On the other hand, assume k contestants solved 5 problems. Each of them solved 10 pairs of problems, whereas each of the $n-k$ remaining contestants solved at most 6 pairs of problems. Thus

$$N \leq 10k + 6(n-k) = 6n + 4k. \tag{2}$$

From these two estimates we immediately get $k \geq 1$. If $(2n+1)/5$ were not an integer, there would be, for every pair of problems, at least $(2n+1)/5$ contestants who solved both problems in the pair (rather than $(2n+1)/5$). Then (1) would improve to $N \geq 6n+6$ and this would yield $k \geq 2$. Alternatively, had some student solved less than 4 problems, he would have solved at most 3 pairs of problems (rather than 6), and our second estimate would improve to $N \leq 6n + 4k - 3$, which together with $N \geq 6n+3$ also gives $k \geq 2$.

So we are left with the case where 5 divides $2n+1$ and every contestant has solved 4 or 5 problems. Let us assume $k = 1$ and let us call the contestant who solved 5 problems the 'winner'. We must then have $N = 6n+4$ (the winner solved 10 pairs of problems, and the rest of the contestants solved exactly 6 pairs of problems each). Let us call a pair of problems 'special' if more than $(2n+1)/5$ contestants solved both problems of the pair. If there were more than one special pair of problems, our first estimate would be improved to

$$N \geq 13 \cdot \frac{2n+1}{5} + 2\left(\frac{2n+1}{5} + 1\right) = 6n+5,$$

which is impossible. Similarly, if a special pair of problems exists, no more than $(2n+1)/5 + 1$ contestants could have solved both problems in the pair, because otherwise

$$N \geq 14 \cdot \frac{2n+1}{5} + \left(\frac{2n+1}{5} + 2\right) = 6n+5.$$

Let us now count the number M of pairs (C, P) where the 'tough' problem (the one not solved by the winner) is one of the problems in P. For each of the 5 pairs of problems containing the tough problem, there are either $(2n+1)/5$ or $(2n+1)/5+1$ contestants who solved both problems of the pair. We then get $M = 2n+1$ or $M = 2n+2$; the latter is possible only if there is a special pair of problems and this special pair contains the tough problem. On the other hand, assume m contestants solved the tough problem. Each of them solved 3 other problems and therefore solved 3 pairs of problems containing the tough one. We can then write $M = 3m$. Hence $2n+1 \equiv 0$ or $2 \pmod 3$.

Finally, let us chose one of the problems other than the tough one, say p, and count the number L of pairs (C, P) for which $p \in P$. We can certainly chose p such that the special pair of problems, if it exists, does not contain p. Then we have $L = 2n+1$ (each of the 5 pairs of problems containing p have exactly $(2n+1)/5$ contestants who solved both problems of the pair). On the other hand, if l is the number of contestants, other than the winner, who solved problem p, we have $L = 3l+4$ (the winner solved problem p and other 4 problems, so she solved 4 pairs of problems containg p, and each of the l students who solved p, solved other 3 problems, hence each of them solved 3 pairs of problems containing p). Therefore $2n+1 \equiv 1 \pmod 3$, which is a contradiction.

Second solution. This is basically the same proof as above, written in symbols rather than words. Suppose there were n contestants. Let p_{ij} be the number of contestants who solved both problem i and problem j $(1 \le i < j \le 6)$ and let n_r be the number of contestants who solved exactly r problems $(0 \le r \le 6)$. Clearly, $\sum n_r = n$.

By hypothesis, $p_{ij} \ge (2n+1)/5$ for all $i < j$, and so

$$S = \sum_{i<j} p_{ij} \ge 15 \cdot \frac{2n+1}{5} = 6n+3.$$

A contestant who solved exactly r problems contributes a '1' to $\binom{r}{2}$ summands in this sum (where as usual $\binom{r}{2} = 0$ for $r < 2$). Therefore

$$S = \sum_{r=0}^{6} \binom{r}{2} n_r.$$

Combining this with the previous estimate we obtain

$$3 \le S - 6n = \sum_{r=0}^{6} \left(\binom{r}{2} - 6 \right) n_r, \tag{3}$$

which rewrites as

$$4n_5 + 9n_6 \geq 3 + 6n_0 + 6n_1 + 5n_2 + 3n_3.$$

If no contestant solved all problems, then $n_6 = 0$, and we see from the above that n_5 must be positive. To show that $n_5 \geq 2$, assume the contrary, i. e., $n_5 = 1$. Then all of n_0, n_1, n_2, n_3 must be zero, so that $n_4 = n - 1$. The right equality of (3) reduces to $S = 6n + 4$.

Each one of the 15 summands in $S = \sum p_{ij}$ is at least $(2n+1)/5 = \lambda$. Because $S = 6n + 4$, they cannot be all equal ($6n + 4$ is not divisible by 15); therefore 14 of them are equal to λ and one is $\lambda + 1$.

Let (i_0, j_0) be this specific pair with $p_{i_0 j_0} = \lambda + 1$. The contestant who solved 5 problems will be again called the winner. Assume, without loss of generality, that it was problem 6 at which the winner failed, and that problem 1 is not in the pair (i_0, j_0); that is, $2 \leq i_0 < j_0 \leq 6$. Consider the sums

$$S' = p_{16} + p_{26} + p_{36} + p_{46} + p_{56} \qquad \text{and} \qquad S'' = p_{12} + p_{13} + p_{14} + p_{15} + p_{16}.$$

Suppose that problem 6 has been solved by x contestants (each of them contributes a '3' to S') and problem 1 has been solved by y contestants other than the winner (each of them contributes a '3' to S'', and the winner contributes a '4'). Thus $S' = 3x$ and $S'' = 3y + 4$.

The pair (i_0, j_0) does not appear in the sum S'', which is therefore equal to $5\lambda = 2n + 1$. The sum S' is either 5λ or $5\lambda + 1$. Hence $3x \in \{2n+1,\, 2n+2\}$ and $3y + 4 = 2n + 1$, which is impossible, as examination of remainders (mod 3) shows. Contradiction ends the proof.

Comment. The problem submitted by the proposer consisted of two parts which were found to be two independent problems by the PSC.

Part (a) asked for a proof that if every problem has been solved by more than 2/5 of the contestants then there exists a set of 3 problems solved by more than 1/5 of the contestants and a set of 4 problems solved by more than 1/15 of the contestants.

The arguments needed for a proof of (a) seem rather standard, giving advantage to students who practised those techniques at training courses. This is much less the case with part (b), which was therefore chosen to be Problem **C6** on the shortlist.

The proposer remarks that there exist examples showing the bound 2 can be attained for the number of contestants solving 5 problems, and that the problem would become harder if it asked to find one such example.

C7. Let $n > 1$ be a given integer, and let $a_1, \dots, a_n$ be a sequence of integers such that n divides the sum $a_1 + \cdots + a_n$. Show that there exist permutations σ and τ of $1, 2, \dots, n$ such that $\sigma(i) + \tau(i) \equiv a_i \pmod{n}$ for all $i = 1, \dots, n$.

Solution. Suppose that there exist suitable permutations σ and τ for a certain integer sequence $a_1, \dots, a_n$ of sum zero modulo n. Let $b_1, \dots, b_n$ be another integer sequence with sum divisible by n, and let $b_1, \dots, b_n$ differ modulo n from $a_1, \dots, a_n$ only in two places, i_1 and i_2. Based on the fact that $\sigma(i) + \tau(i) \equiv b_i \pmod{n}$ for each $i \neq i_1, i_2$, one can transform σ and τ into suitable permutations for $b_1, \dots, b_n$. All congruences below are assumed modulo n.

First we construct a three-column rectangular array

$$\begin{array}{ccc}
\sigma(i_1) & -b_{i_1} & \tau(i_1) \\
\sigma(i_2) & -b_{i_2} & \tau(i_2) \\
\sigma(i_3) & -b_{i_3} & \tau(i_3) \\
\vdots & \vdots & \vdots \\
\sigma(i_{p-1}) & -b_{i_{p-1}} & \tau(i_{p-1}) \\
\sigma(i_p) & \boxed{-b_{i_p}} & \boxed{\tau(i_p)} \\
\sigma(i_{p+1}) & -b_{i_{p+1}} & \boxed{\tau(i_{p+1})} \\
\vdots & \vdots & \vdots \\
\boxed{\sigma(i_{q-1})} & -b_{i_{q-1}} & \tau(i_{q-1}) \\
\boxed{\sigma(i_q)} & \boxed{-b_{i_q}} & \tau(i_q)
\end{array}$$

whose rows are some of the ordered triples $T_i = \big(\sigma(i), -b_i, \tau(i)\big)$, $i = 1, \dots, n$. In the first two rows, write the triples T_{i_1} and T_{i_2}, respectively. Since σ and τ are permutations of $1, \dots, n$, there is a unique index i_3 such that $\sigma(i_1) + \tau(i_3) \equiv b_{i_2}$. Write the triple T_{i_3} in row 3. There is a unique i_4 such that $\sigma(i_2) + \tau(i_4) \equiv b_{i_3}$; write the triple T_{i_4} in row 4, and so on. Stop the first moment a number from column 1 occurs in this column twice, as i_p in row p and i_q in row q, where $p < q$.

We claim that $p = 1$ or $p = 2$. Assume on the contrary that $p > 2$ and consider the subarray containing rows p through q. Each of these rows has sum 0 modulo n, because $\sigma(i) + \tau(i) \equiv b_i$ for $i \neq i_1, i_2$, as already mentioned. On the other hand, by construction the sum in each downward right diagonal of the original array is also 0 modulo n. It follows that the six boxed entries add up to 0 modulo n, i. e.

$$-b_{i_p} + \tau(i_p) + \tau(i_{p+1}) + \sigma(i_{q-1}) + \sigma(i_q) - b_{i_q} \equiv 0.$$

Now, $i_p = i_q$ gives $b_{i_q} \equiv \sigma(i_q) + \tau(i_p)$, so that the displayed formula becomes $-b_{i_p} + \tau(i_{p+1}) + \sigma(i_{q-1}) \equiv 0$. And since $\sigma(i_{p-1}) - b_{i_p} + \tau(i_{p+1}) \equiv 0$ by the remark about diagonals, we obtain $\sigma(i_{p-1}) = \sigma(i_{q-1})$. This implies $i_{p-1} = i_{q-1}$, in contradiction with the definition of p and q. Thus $p = 1$ or $p = 2$ indeed.

Now delete the repeating qth row of the array. Then shift cyclically column 1 and column 3 by moving each of their entries one position down and one position up, respectively. The sum in each row of the new array is 0 modulo n, except possibly in the first and the last row ("most" of the new rows used to be diagonals of the initial array). For $p = 1$, the last row sum is also 0 modulo n, in view of $i_p = i_q = i_1$ and $\sigma(i_{q-2}) - b_{i_{q-1}} + \tau(i_q) \equiv 0$ (see the array on the left below). A single change is needed to accomodate the case $p = 2$: in column 3, interchange the top entry $\tau(i_2)$ and the bottom entry $\tau(i_1)$ (see the array on the right). The last row sum becomes 0 modulo n since $i_p = i_q = i_2$.

$$\begin{array}{ccc}
\sigma(i_{q-1}) & -b_{i_1} & \tau(i_2) \\
\sigma(i_1) & -b_{i_2} & \tau(i_3) \\
\sigma(i_2) & -b_{i_3} & \tau(i_4) \\
\vdots & \vdots & \vdots \\
\sigma(i_{q-3}) & -b_{i_{q-2}} & \tau(i_{q-1}) \\
\sigma(i_{q-2}) & -b_{i_{q-1}} & \tau(i_1)
\end{array}
\qquad\qquad
\begin{array}{ccc}
\sigma(i_{q-1}) & -b_{i_1} & \tau(i_1) \\
\sigma(i_1) & -b_{i_2} & \tau(i_3) \\
\sigma(i_2) & -b_{i_3} & \tau(i_4) \\
\vdots & \vdots & \vdots \\
\sigma(i_{q-3}) & -b_{i_{q-2}} & \tau(i_{q-1}) \\
\sigma(i_{q-2}) & -b_{i_{q-1}} & \tau(i_2)
\end{array}$$

$$(p = 1) \qquad\qquad\qquad\qquad (p = 2)$$

For both $p = 1$ and $p = 2$, column 1 and column 3 are permutations the numbers of $\sigma(i_1), \ldots, \sigma(i_{q-1})$ and $\tau(i_1), \ldots, \tau(i_{q-1})$, respectively. So, adding the triples T_i not involved in the construction above, we obtain permutations σ' and τ' of $1, \ldots, n$ in column 1 and column 3 such that $\sigma'(i) + \tau'(i) \equiv b_i$ for all $i \neq i_1$. Finally, the relation $\sigma'(i_1) + \tau'(i_1) \equiv b_{i_1}$ follows from the fact that $\Sigma\big(\sigma'(i) + \tau'(i)\big) \equiv 0 \equiv \Sigma b_i$.

We proved that the statement remains true if we change elements of the original sequence $a_1, \ldots, a_n$ two at a time. However, one can obtain from any given $a_1, \ldots, a_n$ any other zero-sum sequence by changing two elements at a time. (The condition that the sequence has sum zero modulo n is used here again.) And because the claim is true for any constant sequence, the conclusion follows.

C8. Let M be a convex n-gon, $n \geq 4$. Some $n-3$ of its diagonals are coloured green and some other $n-3$ diagonals are coloured red, so that no two diagonals of the same colour meet inside M. Find the maximum possible number of intersection points of green and red diagonals inside M.

Solution. We start with some preliminary observations. It is well-known that $n-3$ is the maximum number of nonintersecting diagonals in a convex n-gon and that any such $n-3$ diagonals partition the n-gon into $n-2$ triangles. It is also known (and not hard to show by induction) that at least two nonadjacent vertices are then left free; that is, there are at least two diagonals cutting off triangles from the n-gon.

Passing to the conditions of the problem, for any diagonal d, denote by $f(d)$ the number of green/red inresections lying on d. Take any pair of green diagonals d, d' and suppose there are k vertices, including the endpoints of d and d', of the part of M between d and d'. The remaining $n-k$ vertices span a convex polygon $A\ldots BC\ldots D$; here A and B are the vertices of M, adjacent to the endpoints of d, outside the "part of M" just mentioned, and C and D are the vertices adjacent to the endpoints of d', also outside that part. (A, B can coincide, as well as C, D.)

Let m be the number of red segments in the polygon $A\ldots BC\ldots D$. Since this $(n-k)$-gon has at most $n-k-3$ nonintersecting diagonals, we get

$$m \leq (n-k-3) + 2;$$

the last '2' comes from the segments AD and BC, which also can be red.

Each one of these m red segments intersects both d and d'. Each one of the remaining $n-3-m$ red segments can meet at most one of d, d'. Hence follows the estimate

$$f(d) + f(d') \leq 2m + (n-3-m) = n - 3 + m \leq n - 3 + (n-k-1) = 2n - k - 4.$$

Now we pair the green diagonals in the following way: we choose any two green diagonals that cut off two triangles from M; they constitute the first pair d_1, d_2. Then we choose two green diagonals that cut off two triangles from the residual $(n-2)$-gon, to make up the second pair d_3, d_4, and so on; d_{2r-1}, d_{2r} are the two diagonals in the r-th pairing. If $n-3$ is odd, the last green diagonal remains unpaired.

The polygon obtained after the r-th pairing has $n-2r$ vertices. Two sides of that polygon are the two diagonals from that pairing; its remaining sides are either sides of M or some of the green diagonals $d_1, \ldots, d_{2r}$. There are at most $2r$ vertices of M outside the part of M between d_{2r-1} and d_{2r}. Thus, denoting by k_r the number of vertices of that part, we have $k_r \geq n - 2r$.

In view of the previous estimates, the number of intersection points on those two diagonals satisfies the inequality

$$f(d_{2r-1}) + f(d_{2r}) \le 2n - k_r - 4 \le n + 2r - 4.$$

If $n-3$ is even, then $d_1, d_2, \ldots, d_{n-3}$ are all the green diagonals; and if $n-3$ is odd, the last unpaired green diagonal can meet at most $n-3$ (i.e., all) red ones. Thus, writing $n - 3 = 2\ell + \varepsilon,\ \varepsilon \in \{0, 1\}$, we conclude that the total number of intersection points does not exceed the sum

$$\begin{aligned}\sum_{r=1}^{\ell}(n + 2r - 4) + \varepsilon \cdot (n - 3) &= \ell(2\ell + \varepsilon - 1) + \ell(\ell + 1) + \varepsilon(2\ell + \varepsilon) \\ &= 3\ell^2 + \varepsilon(3\ell + 1) = \left\lceil \frac{3}{4}(n - 3)^2 \right\rceil,\end{aligned}$$

where $\lceil t \rceil$ is the least integer not less than t. (For $n = 4$ the void sum $\sum_{r=1}^{0}$ evaluates to 0.)

The following example shows that this value can be attained, for both n even and n odd. Let PQ and RS be two sides of M chosen so that the diagonals QR and SP do not meet and, moreover, so that: if U is the part of the boundary of M between Q and R, and V is the part of the boundary of M between S and P $(S, P \notin U,\ Q, R \notin V)$, then the numbers of vertices of M on U and on V differ by at most 1.

Colour in green: the diagonal PR, all diagonals connecting P to vertices on U and all diagonals connecting R to vertices on V.

Colour in red: the diagonal QS, all diagonals connecting Q to vertices on V and all diagonals connecting S to vertices on U.

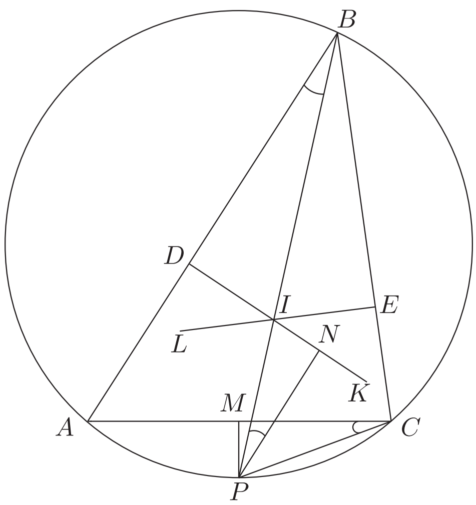
Then equality holds in the estimate above. In conclusion, $\lceil \frac{3}{4}(n - 3)^2 \rceil$ is the greatest number of intersection points available.

Comment. It seems that the easiest way to verify that these examples indeed yield equality in the estimates obtained is to draw a diagram and visualise the process of detaching the corner triangles in appropriate pairings; all inequalities that appear in the arguments above turn into equalities. This is also the way (by inspecting the detaching procedure) in which it is expected that the solver can construct these examples.

Geometry

G1. In a triangle ABC satisfying $AB + BC = 3AC$ the incircle has centre I and touches the sides AB and BC at D and E, respectively. Let K and L be the symmetric points of D and E with respect to I. Prove that the quadrilateral $ACKL$ is cyclic.

Solution. Let P be the other point of intersection of BI with the circumcircle of triangle ABC, let M be the midpoint of AC and N the projection of P to IK. Since $AB + BC = 3AC$, we get $BD = BE = AC$, so $BD = 2CM$. Furthermore, $\angle ABP = \angle ACP$, therefore the triangles DBI and MCP are similar in ratio 2.



It is a known fact that $PA = PI = PC$. Moreover, $\angle NPI = \angle DBI$, so that the triangles PNI and CMP are congruent. Hence $ID = 2PM = 2IN$; i. e. N is the midpoint of IK. This shows that PN is the perpendicular bisector of IK, which gives $PC = PK = PI$. Analogously, $PA = PL = PI$. So P is the centre of the circle through A, K, I, L and C.

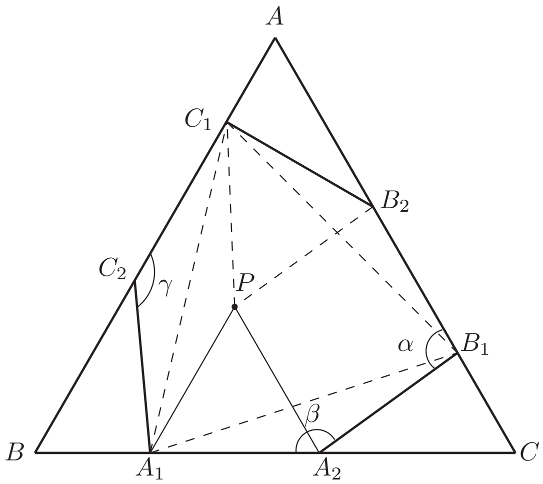
Comment. Variations are possible here. One might for instance *define* N to be the midpoint of IK and apply Ptolemy's theorem to the quadrilateral $BAPC$ and deduce that the triangles NPI and DBI are similar in ratio 2 to conclude that $PN \perp IK$.

G2. Six points are chosen on the sides of an equilateral triangle ABC: A_1, A_2 on BC, B_1, B_2 on CA, and C_1, C_2 on AB, so that they are the vertices of a convex hexagon $A_1A_2B_1B_2C_1C_2$ with equal side lengths. Prove that the lines A_1B_2, B_1C_2 and C_1A_2 are concurrent.

First solution. Let P be the point inside triangle ABC such that the triangle A_1A_2P is equilateral. Note that $A_1P \parallel C_1C_2$ and $A_1P = C_1C_2$, therefore $A_1PC_1C_2$ is a rhombus. Similarly, $A_2PB_2B_1$ is also a rhombus. Hence, the triangle C_1B_2P is equilateral. Let $\alpha = \angle B_2B_1A_2$, $\beta = \angle B_1A_2A_1$ and $\gamma = \angle C_1C_2A_1$. Then α and β are external angles of the triangle CB_1A_2 with $\angle C = 60°$, and hence $\alpha + \beta = 240°$. Note also that $\angle B_2PA_2 = \alpha$ and $\angle C_1PA_1 = \gamma$. So,

$$\alpha + \gamma = 360° - (\angle C_1PB_2 + \angle A_1PA_2) = 240°.$$

Hence, $\beta = \gamma$. Similarly, $\angle C_1B_2B_1 = \beta$. Therefore the triangles $A_1A_2B_1$, $B_1B_2C_1$ and $C_1C_2A_1$ are congruent, which implies that the triangle $A_1B_1C_1$ is equilateral. This shows that B_1C_2, A_1B_2 and C_1A_2 are the perpendicular bisectors of A_1C_1, C_1B_1 and B_1A_1; hence the result.



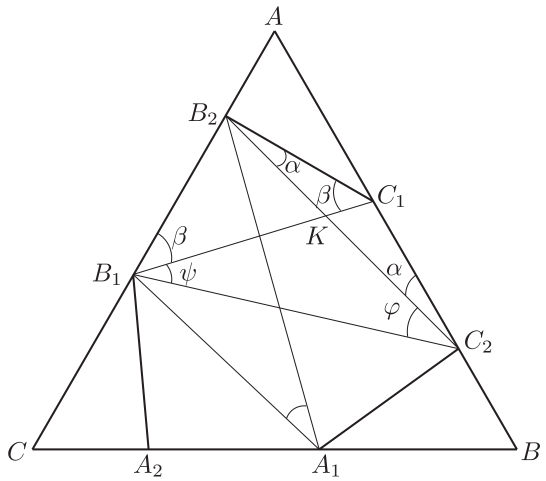
Second solution. Let $\alpha = \angle AC_2B_2$, $\beta = \angle AB_1C_1$ and K be the intersection of B_1C_1 with B_2C_2. The triangles $B_1B_2C_1$ and $B_2C_1C_2$ are isosceles, so $\angle B_1C_1B_2 = \beta$ and $\angle C_2B_2C_1 = \alpha$.

Denoting further $\angle B_1C_2B_2 = \varphi$ and $\angle C_1B_1C_2 = \psi$ we get (from the triangle AB_1C_2) $\alpha + \beta + \varphi + \psi = 120°$; and (from the triangles KB_1C_2, KC_1B_2) $\alpha + \beta = \varphi + \psi$. Then $\alpha + \beta = 60°$, $\angle C_1KB_2 = 120°$, and so the quadrilateral

AB_2KC_1 is cyclic. Hence $\angle KAC_1 = \alpha$ and $\angle B_2AK = \beta$. From $KC_2 = KA = KB_1$ and $\angle B_1KC_2 = 120°$ we get $\varphi = \psi = 30°$.

In the same way, one shows that $\angle B_2A_1B_1 = \angle C_2B_1A_1 = 30°$. It follows that $A_1B_1B_2C_2$ is a cyclic quadrilateral and since its opposite sides A_1C_2 and B_1B_2 have equal lengths, it is an isosceles trapezoid. This implies that A_1B_1 and C_2B_2 are parallel lines, hence $\angle A_1B_1C_2 = \angle B_2C_2B_1 = 30°$.

Thus, B_1C_2 bisects the angle $C_1B_1A_1$. Similarly, by cyclicity, C_1A_2 and A_1B_2 are the bisectors of the angles $A_1C_1B_1$ and $B_1A_1C_1$, therefore they are concurrent.



Third solution. Consider the six vectors of equal lengths, with zero sum:

$$\mathbf{u} = \overrightarrow{B_2C_1},\ \mathbf{u}' = \overrightarrow{C_1C_2},\ \mathbf{v} = \overrightarrow{C_2A_1},\ \mathbf{v}' = \overrightarrow{A_1A_2},\ \mathbf{w} = \overrightarrow{A_2B_1},\ \mathbf{w}' = \overrightarrow{B_1B_2}.$$

Since $\mathbf{u}', \mathbf{v}', \mathbf{w}'$ clearly add up to zero vector, the same is true of $\mathbf{u}, \mathbf{v}, \mathbf{w}$. So $\mathbf{u} + \mathbf{v} = -\mathbf{w}$.

The sum of two vectors of equal lengths is a vector of the same length only if they make an angle of $120°$. This follows e. g. from the parallelogram interpretation of vector addition or from the law of cosines. Therefore the three lines B_2C_1, C_2A_1, A_2B_1 define an equilateral triangle.

Consequently the “corner” triangles AC_1B_2, BA_1C_2, CB_1A_2 are similar, and in fact congruent, as $B_2C_1 = C_2A_1 = A_2B_1$. Thus the whole configuration is invariant under rotation through $120°$ about O, the centre of the triangle ABC.

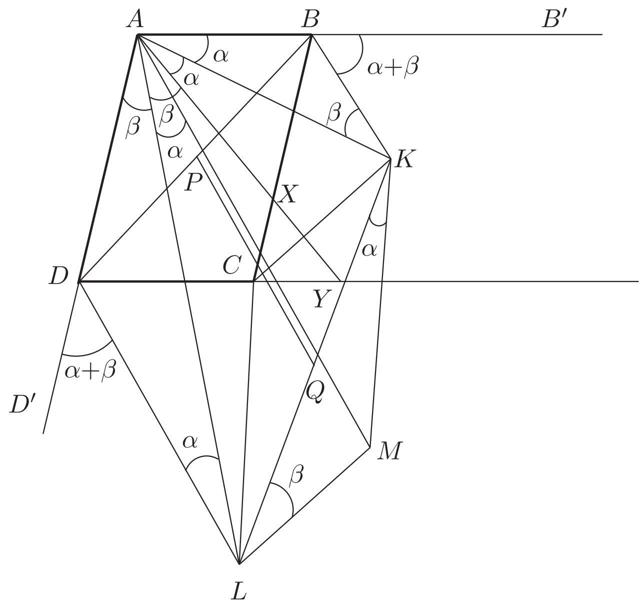
In view of the equalities $\angle B_2C_1C_2 = \angle C_2A_1A_2$ and $\angle A_1A_2B_1 = \angle B_1B_2C_1$ the line B_1C_2 is a symmetry axis of the hexagon $A_1A_2B_1B_2C_1C_2$, so it must pass through the rotation centre O. In conclusion, the three lines in question concur at O.

G3. Let $ABCD$ be a parallelogram. A variable line ℓ passing through the point A intersects the rays BC and DC at points X and Y, respectively. Let K and L be the centres of the excircles of triangles ABX and ADY, touching the sides BX and DY, respectively. Prove that the size of angle KCL does not depend on the choice of the line ℓ.

First solution. Let $\angle BAX = 2\alpha$, $\angle DAY = 2\beta$. The points K and L lie on the internal bisectors of the angles A in triangles ABX, ADY and on the external bisectors of their angles B and D. Taking B' and D' to be any points on the rays AB and AD beyond B and D, we have

$$\angle KAB = \angle KAX = \alpha, \qquad \angle LAD = \angle LAY = \beta,$$
$$\angle KBB' = \frac{1}{2}\angle BAD = \alpha + \beta = \angle LDD', \quad \text{so} \quad \angle AKB = \beta, \quad \angle ALD = \alpha.$$

Let the bisector of angle BAD meet the circumcircle of triangle AKL at a second point M. The vectors $\overrightarrow{BK}$, $\overrightarrow{AM}$, $\overrightarrow{DL}$ are parallel and equally oriented.



Since K and L lie on distinct sides of AM, we see that $AKML$ is a cyclic convex quadrilateral, and hence

$$\angle MKL = \angle MAL = \angle MAD - \angle LAD = \alpha; \qquad \text{likewise,} \quad \angle MLK = \beta.$$

Hence the triangles AKB, KLM, LAD are similar, so $AK \cdot LM = KB \cdot KL$ and $KM \cdot LA = KL \cdot LD$. Applying Ptolemy's theorem to the cyclic quadrilateral

$AKLM$, we obtain

$$AM \cdot KL = AK \cdot LM + KM \cdot LA = (KB + LD) \cdot KL,$$

implying $AM = BK + DL$.

The convex quadrilateral $BKLD$ is a trapezoid. Denoting the midpoints of its sides BD and KL respectively by P and Q, we have

$$2 \cdot PQ = BK + DL = AM;$$

notice that the vector $\overrightarrow{PQ}$ is also parallel to the three vectors mentioned earlier, in particular to $\overrightarrow{AM}$, and equally oriented.

Now, P is also the midpoint of AC. It follows from the last few conclusions that Q is the midpoint of side CM in the triangle ACM. So the segments KL and CM have a common midpoint, which means that $KCLM$ is a parallelogram. Thus, finally,

$$\angle KCL = \angle KML = 180^\circ - (\alpha + \beta) = 180^\circ - \frac{1}{2}\angle BAD,$$

which is a constant value, depending on the parallelogram $ABCD$ alone.

Second solution. Let the line AK meet DC at E, and let the line AL meet BC at F. Denote again $\angle BAX = 2\alpha$, $\angle DAY = 2\beta$. Then $\angle BFA = \beta$. Moreover, $\angle KBF = (1/2)\angle BAD = \alpha + \beta = \angle KAF$. Since the points A and B lie on the same side of the line KF, we infer that $ABKF$ is a cyclic quadrilateral.

Speaking less rigourously, the points A, K, B, F are concyclic. The points E and C lie on the lines AK and BF, and the segment EC is parallel to AB. Therefore the points E, K, C, F lie on a circle, too; this follows easily from an inspection of angles—one just has to consider three cases, according as two, one or none of the points E, C lie(s) on the same side of line KF as the segment AB does.

Analogously, the points F, L, C, E lie on a circle. Clearly C, K, L are three distinct points. It follows that all five points C, E, F, K, L lie on a circle Ω.

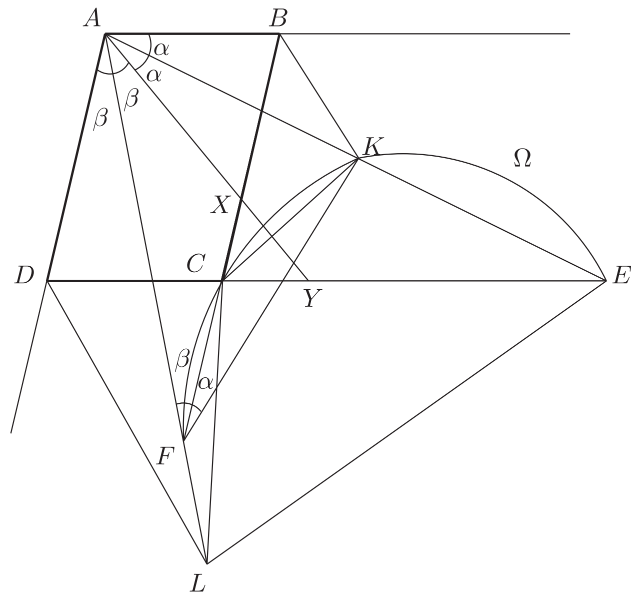
From the cyclic quadrilateral $ABKF$ we have $\angle BFK = \angle BAK = \alpha$, which combined with $\angle BFA = \beta$ implies $\angle KFA = \alpha + \beta$. Since the points A, F, L are in line, $\angle KFL$ is either $\alpha + \beta$ or $180^\circ - (\alpha + \beta)$; and since K, C, F, L are concyclic, also $\angle KCL$ is either $\alpha + \beta$ or $180^\circ - (\alpha + \beta)$.

All that remains is to eliminate one of these two possibilities. To this effect, we will show that the points A and C lie on the same side of the line KL.

Assume without loss of generality that Y, the point where ℓ cuts the ray DC, lies beyond C on that ray. Then so does E.

If also F lies on the ray BC beyond C then Ω does not penetrate the interior of $ABCD$. Hence the line KL does not separate A from C. And if F lies on the segment BC then L lies in the half-plane with edge BC, not containing A. Since K also lies in that half-plane, and since L lies on the opposite side of the line DC than A, this again implies that the line KL does not separate A from C.

Notice that the circle Ω intersects each one of the rays AK, AL at two points (K, E, resp. L, F), possibly coinciding. Thus A lies outside this circle. Knowing that C and A lie on the same side of the line KL, we infer that $\angle KCL > \angle KAL = \alpha + \beta$. This leaves the other possibility as the unique one: $\angle KCL = 180° - (\alpha + \beta)$.



Comment. Alternatively, continuity argument could be applied. If $\angle KCL$ takes on only two values, it must be a constant.

In our attempt to stay within the realm of classical geometry, we were forced to investigate the disposition of the points and lines in question. Notice that the first solution is case-independent.

Other solutions are available by calculation, be it with complex numbers or linear transformations in the coordinate plane; but no one of such approaches seems to be straightforward.

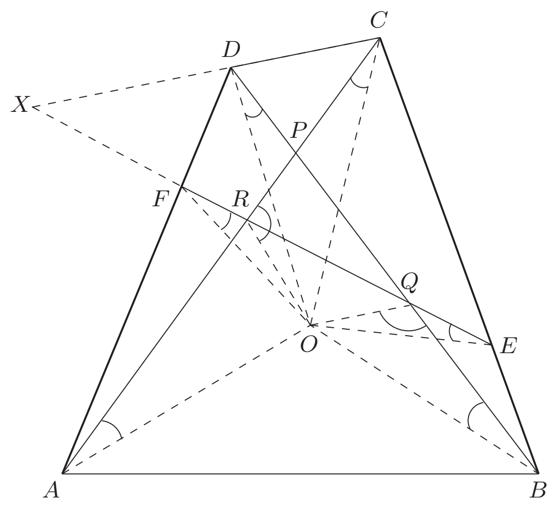
G4. Let $ABCD$ be a fixed convex quadrilateral with $BC = DA$ and BC not parallel to DA. Let two variable points E and F lie on the sides BC and DA, respectively, and satisfy $BE = DF$. The lines AC and BD meet at P, the lines BD and EF meet at Q, the lines EF and AC meet at R. Prove that the circumcircles of triangles PQR, as E and F vary, have a common point other than P.

First solution. Let the perpendicular bisectors of the segments AC and BD meet at O. We show that the circumcircles of triangles PQR pass through O, which is fixed.

It follows from the equalities $OA = OC$, $OB = OD$ and $DA = BC$ that the triangles ODA and OBC are congruent. So the rotation about the point O through the angle BOD takes the point B to D and the point C to A. Since $BE = DF$, the same rotation takes the point E to F. This implies that $OE = OF$ and

$$\angle EOF = \angle BOD = \angle COA \ (= \text{the angle of rotation})\,.$$

These equalities imply that the isosceles triangles EOF, BOD and COA are similar.



Suppose first that the three lines AB, CD and EF are not all parallel. Assume without loss of generality that the lines EF and CD meet at X. From the Menelaus theorem, applied to the triangles ACD and BCD, we obtain

$$\frac{AR}{RC} = \frac{AF}{FD}\cdot\frac{DX}{XC} = \frac{CE}{EB}\cdot\frac{DX}{XC} = \frac{DQ}{QB}\,.$$

In the case $AB \parallel EF \parallel CD$, the quadrilateral $ABCD$ is an isosceles trapezoid, and E, F are the midpoints of its lateral sides. The equality $AR/RC = DQ/QB$ is then obvious.

It follows from the this equality and the similitude of triangles BOD and COA that the triangles BOQ and COR are similar. Thus $\angle BQO = \angle CRO$, which means that the points P, Q, R and O are concyclic.

Second solution. This is just a variation of the preceding proof. As in the first solution, we show that the triangles EOF, BOD and COA are similar. Denote by K, L, M the feet of the perpendiculars from the point O onto the lines EF, BD, AC, respectively. In view of the similarity just mentioned,

$$\frac{OK}{OE} = \frac{OL}{OB} = \frac{OM}{OC} = \lambda \quad \text{and} \quad \angle EOK = \angle BOL = \angle COM = \varphi\,.$$

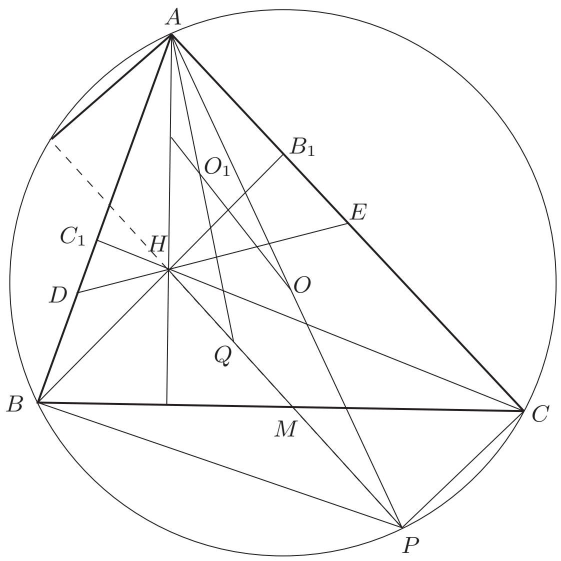
Therefore the rotation about the point O through the angle φ, composed with the homothety with centre O and ratio λ, takes the points B, E, C to the points L, K, M, respectively. This implies that the points L, K, M are collinear. Hence by the theorem about the Simson line we conclude that the circumcircle of PQR passes through O.

Comment. The proposer observes that (as can be seen from the above solutions) the point under discussion can also be identified as the second common point of the circumcircles of triangles BCP and DAP.

G5. Let ABC be an acute-angled triangle with $AB \neq AC$, let H be its orthocentre and M the midpoint of BC. Points D on AB and E on AC are such that $AE = AD$ and D, H, E are collinear. Prove that HM is orthogonal to the common chord of the circumcircles of triangles ABC and ADE.

Solution. Let O and O_1 be the circumcentres of the triangles ABC and ADE, respectively. Since the radical axis of two circles is perpendicular to their line of centres, we have to prove that OO_1 is parallel to HM.

Consider the diameter AP of the circumcircle of ABC and let B_1 and C_1 be the feet of the altitudes from B and C in the triangle ABC. Since $AB \perp BP$ and $AC \perp CP$, it follows that $HC \parallel BP$ and $HB \parallel CP$. Thus $BPCH$ is a parallelogram; as a consequence, HM cuts the circle at P.



The triangle ADE is isosceles, so its circumcentre O_1 lies on the bisector of the angle BAC. We shall prove that the intersection Q of AO_1 with HP is the symmetric of A with respect to O_1. The rays AH and AO are isogonal conjugates, so the line AQ bisects $\angle HAP$. Then the bisector theorem in the triangle AHP yields

$$\frac{QH}{QP} = \frac{AH}{AP}.$$

Because ADE is an isosceles triangle, an easy angle computation shows that HD bisects $\angle C_1HB$. Hence the bisector theorem again gives

$$\frac{DC_1}{DB} = \frac{HC_1}{HB}.$$

Applying once more the fact that AH and AP are isogonal lines, we see that the right triangles C_1HA and CPA are similar, so

$$\frac{AH}{AP} = \frac{C_1H}{CP} = \frac{C_1H}{BH};$$

the last equality holds because $BPCH$ is a parallelogram, so that $PC = BH$.

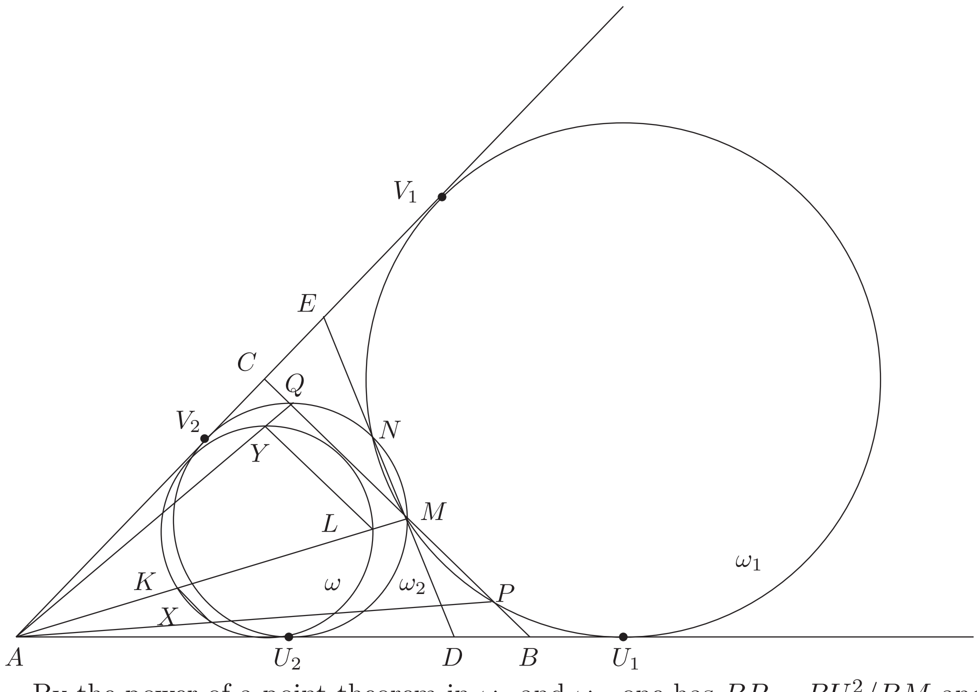
Summarizing, we conclude that

$$\frac{DC_1}{DB} = \frac{QH}{QP},$$

that is, $QD \parallel HC_1$. In the same way we obtain $QE \parallel HB_1$. As a consequence, AQ is a diameter of the circumcircle of triangle ADE, implying that O_1 is the midpoint of AQ. Thus $OO_1 \parallel PQ$; that is, OO_1 is parallel to HM.

G6. The median AM of a triangle ABC intersects its incircle ω at K and L. The lines through K and L parallel to BC intersect ω again at X and Y. The lines AX and AY intersect BC at P and Q. Prove that $BP = CQ$.

First solution. Without loss of generality, one can assume the notation in the figure. Let ω_1 be the image of ω under the homothety with centre A and ratio AM/AK. This homothety takes K to M and hence X to P, because $KX \parallel BC$. So ω_1 is a circle through M and P inscribed in $\angle BAC$. Denote its points of tangency with AB and AC by U_1 and V_1, respectively. Analogously, let ω_2 be the image of ω under the homothety with center A and ratio AM/AL. Then ω_2 is a circle through M and Q also inscribed in $\angle BAC$. Let it touch AB and AC at U_2 and V_2, respectively. Then $U_1U_2 = V_1V_2$, as U_1U_2 and V_1V_2 are the common external tangents of ω_1 and ω_2.



By the power-of-a-point theorem in ω_1 and ω_2, one has $BP = BU_1^2/BM$ and $CQ = CV_2^2/CM$. Since $BM = CM$, it suffices to show that $BU_1 = CV_2$.

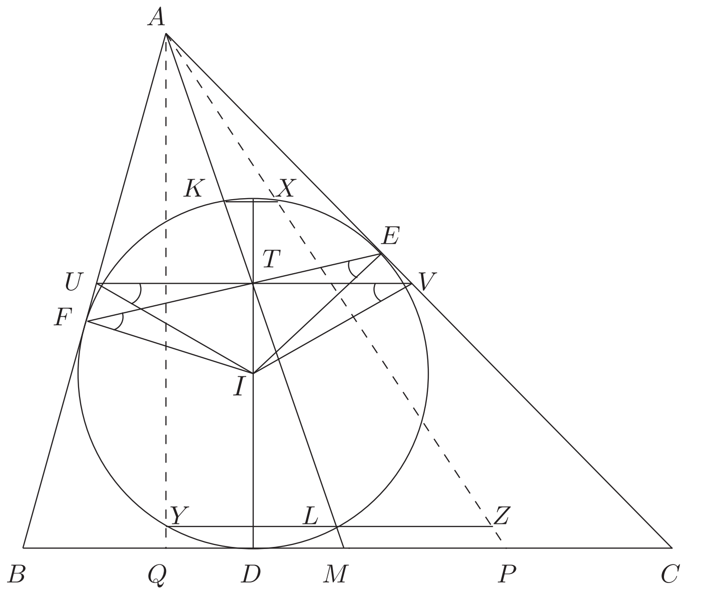
Consider the second common point N of ω_1 and ω_2 (M and N may coincide, in which case the "line MN" is the common tangent). Let the line MN meet AB and AC at D and E, respectively. Clearly D is the midpoint of U_1U_2 because $DU_1^2 = DM \cdot DN = DU_2^2$ by the power-of-a-point theorem again. Likewise, E is the midpoint of V_1V_2. Note that B and C are on different sides of DE, which reduces the problem to proving that $BD = CE$.

Since DE is perpendicular to the line of centres of ω_1 and ω_2, we have $\angle ADM = \angle AEM$. Then the law of sines for triangles BDM and CEM gives

$$BD = \frac{BM \sin \angle BMD}{\sin \angle BDM} = \frac{BM \sin \angle BMD}{\sin \angle ADM}, \qquad CE = \frac{CM \sin \angle CME}{\sin \angle AEM}.$$

Because $BM = CM$ and $\angle BMD = \angle CME$, the conclusion follows.

Second solution. Let ω touch BC, CA and AB at D, E and F, respectively, and let I be the incentre of triangle ABC. The key step of this solution is the observation that the lines AM, EF and DI are concurrent.



Indeed, suppose that EF and DI meet at T. Let the parallel through T to BC meet AB and AC at U and V, respectively. One has $IT \perp UV$, and since $IE \perp AC$, it follows that the points I, T, V and E are concyclic. Moreover, V and E lie on the same side of the line IT, so that $\angle IVT = \angle IET$. By symmetry, $\angle IUT = \angle IFT$. But $\angle IET = \angle IFT$, hence UVI is an isosceles triangle with altitude IT to its base UV. So T is the midpoint of UV, implying that AT meets BC at its midpoint M.

Now observe that EF is the polar of A with respect to ω, therefore

$$\frac{AK}{AL} = \frac{TK}{TL}.$$

Furthermore, let LY meet AP at Z. Then

$$\frac{KX}{LZ} = \frac{AK}{AL}.$$

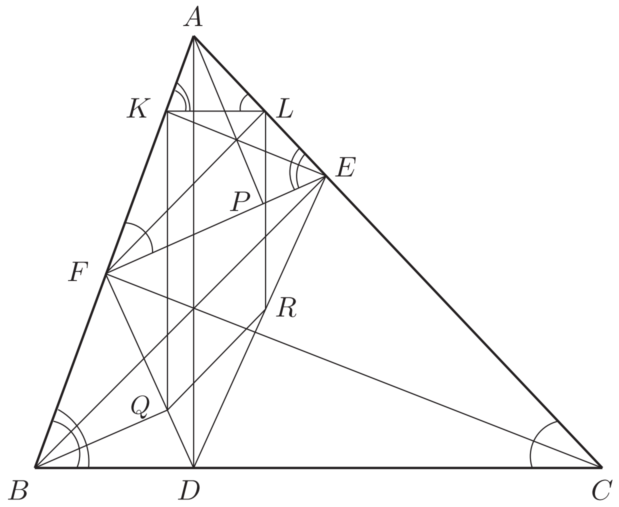
The line IT is the common perpendicular bisector of KX and LY. As we have shown, T lies on AM, i. e. on KL. Hence

$$\frac{KX}{LY} = \frac{TK}{TL}.$$

The last three relations show that L is the midpoint of YZ, so M is the midpoint of PQ.

G7. In an acute triangle ABC, let D, E, F, P, Q, R be the feet of perpendiculars from A, B, C, A, B, C to BC, CA, AB, EF, FD, DE, respectively. Prove that $p(ABC)p(PQR) \geq p(DEF)^2$, where $p(T)$ denotes the perimeter of the triangle T.

First solution. The points D, E and F are interior to the sides of triangle ABC which is acute-angled. It is widely known that the triangles ABC and AEF are similar. Equivalently, the lines BC and EF are antiparallel with respect to the sides of $\angle A$. Similar conclusions hold true for the pairs of lines CA, FD and AB, DE. This is a general property related to the feet of the altitudes in every triangle. In particular, it follows that P, Q and R are interior to the respective sides of triangle DEF.



Let K and L be the feet of the perpendiculars from E and F to AB and AC, respectively. By the remark above, KL and EF are antiparallel with respect to the sides of the same $\angle A$. Therefore $\angle AKL = \angle AEF = \angle ABC$, meaning that $KL \parallel BC$.

Now, EK and BQ are respective altitudes in the similar triangles AEF and DBF, so they divide the opposite sides in the same ratio:

$$\frac{AK}{KF} = \frac{DQ}{QF}.$$

This implies $KQ \parallel AD$. By symmetry, $LR \parallel AD$. Since KL is parallel to BC, it is perpendicular to AD. It follows that $QR \geq KL$.

From the similar triangles AKL, AEF, ABC we obtain

$$\frac{KL}{EF} = \frac{AK}{AE} = \cos\angle A = \frac{AE}{AB} = \frac{EF}{BC}.$$

Hence $QR \geq EF^2/BC$. Likewise, $PQ \geq DE^2/AB$ and $RP \geq FD^2/CA$.

Therefore it suffices to show that

$$(AB+BC+CA)\left(\frac{DE^2}{AB}+\frac{EF^2}{BC}+\frac{FD^2}{CA}\right) \geq (DE+EF+FA)^2,$$

which is a direct consequence of the Cauchy-Schwarz inequality.

Second solution. Let $\alpha = \angle A$, $\beta = \angle B$, $\gamma = \angle C$. There is no loss of generality in assuming that triangle ABC has circumradius 1. The triangles AEF and ABC are similar in ratio $\cos\alpha$, so $EF = BC\cos\alpha = \sin 2\alpha$. By symmetry, $FD = \sin 2\beta$, $DE = \sin 2\gamma$. Next, since $\angle BDF = \angle CDE = \alpha$, it follows that $DQ = BD\cos\alpha = AB\cos\beta\cos\alpha = 2\cos\alpha\cos\beta\sin\gamma$.

Similarly, $DR = 2\cos\alpha\sin\beta\cos\gamma$. Now the law of cosines for triangle DQR gives after short manipulation

$$QR = \sin 2\alpha\sqrt{1-\sin 2\beta\sin 2\gamma}.$$

Likewise, $RP = \sin 2\beta\sqrt{1-\sin 2\gamma\sin 2\alpha}$, $PQ = \sin 2\gamma\sqrt{1-\sin 2\alpha\sin 2\beta}$.

Therefore the given inequality is equivalent to

$$2\sum\sin\alpha\sum\sin 2\alpha\sqrt{1-\sin 2\beta\sin 2\gamma} \geq \left(\sum\sin 2\alpha\right)^2,$$

where Σ means a cyclic sum over α, β, γ, the angles of an acute triangle. In view of this, all trigonometric functions below are positive. To eliminate the square roots, observe that

$$1-\sin 2\beta\sin 2\gamma = \sin^2(\beta-\gamma)+\cos^2\alpha \geq \cos^2\alpha.$$

Hence it suffices to establish $2\sum\sin\alpha\sum\sin 2\alpha\cos\alpha \geq (\sum\sin 2\alpha)^2$. This is yet another immediate consequence of the Cauchy-Schwarz inequality:

$$\sum 2\sin\alpha\sum\sin 2\alpha\cos\alpha \geq \left(\sum\sqrt{2\sin\alpha}\sqrt{\sin 2\alpha\cos\alpha}\right)^2 = \left(\sum\sin 2\alpha\right)^2.$$

Third solution. A stronger conclusion is true, namely:

$$\frac{p(ABC)}{p(DEF)} \geq 2 \geq \frac{p(DEF)}{p(PQR)}.$$

The left inequality is a known fact, so we consider only the right one.

It is immediate that the points A, B and C are the excentres of triangle DEF. Therefore P, Q and R are the tangency points of the excircles of this triangle with its sides. For the sake of clarity, let us adopt the notation $a = EF$, $b = FD$, $c = DE$, $\alpha = \angle D$, $\beta = \angle E$, $\gamma = \angle F$ now for the sides and angles of triangle DEF. Also, let $s = (a+b+c)/2$. Then $ER = FQ = s - a$, $FP = DR = s - b$, $DQ = EP = s - c$.

Now we regard the line DE as an axis by choosing the direction from D to E as the positive direction. The signed length of a line segment UV on this axis will be denoted by $\overline{UV}$. Let X and Y be the orthogonal projections onto DE of P and Q, respectively. On one hand, $\overline{DE} = \overline{DY} + \overline{YX} + \overline{XE}$. On the other hand,

$$\overline{DY} = DQ\cos\alpha, \qquad \overline{XE} = EP\cos\beta.$$

Observe that these inequalities hold true in all cases, regardless of whether or not α and β are acute. Finally, it is clear that $\overline{YX} \le PQ$. In conclusion,

$$DE = (s-c)(\cos\alpha + \cos\beta) + \overline{YX} \le (s-c)(\cos\alpha + \cos\beta) + PQ.$$

By symmetry,

$$EF \le (s-a)(\cos\beta + \cos\gamma) + QR, \qquad FD \le (s-b)(\cos\gamma + \cos\alpha) + RP.$$

Adding up yields $p(DEF) \le \sum(s-c)(\cos\alpha + \cos\beta) + p(PQR)$, where again Σ denotes a cyclic sum over α, β, γ. This sum is equal to $a\cos\alpha + b\cos\beta + c\cos\gamma$, since $(s-b) + (s-c) = a$, $(s-c) + (s-a) = b$, $(s-a) + (s-b) = c$.

Now it suffices to show that $a\cos\alpha + b\cos\beta + c\cos\gamma \le (1/2)p(DEF)$. Suppose that $a \le b \le c$; then $\cos\alpha \ge \cos\beta \ge \cos\gamma$, so one can apply Chebyshev's inequality to the triples (a, b, c) and $(\cos\alpha, \cos\beta, \cos\gamma)$. This gives

$$a\cos\alpha + b\cos\beta + c\cos\gamma \le \frac{1}{3}(a+b+c)(\cos\alpha + \cos\beta + \cos\gamma).$$

But $\cos\alpha + \cos\beta + \cos\gamma \le 3/2$ for every triangle, and the result follows.

Comment. This last solution shows that the proposed inequality splits into two independent ones, which can be expressed in words:

> In every triangle, the perimeter of its orthic triangle is not greater than half the perimeter of the triangle itself, and the perimeter of its Nagel triangle is not smaller than half the perimeter of the triangle itself.

Whereas the first of these inequalities is indeed a very well-known fact, this seems not to be the case with the second one.

Number Theory

N1. Determine all positive integers relatively prime to all terms of the infinite sequence $a_n = 2^n + 3^n + 6^n - 1$ $(n = 1, 2, 3, \dots)$.

Solution. We claim that 1 is the only such number. This amounts to showing that every prime p is a divisor of a certain a_n. This is true for $p = 2$ and $p = 3$ as $a_2 = 48$.

Fix a prime $p > 3$. All congruences that follow are considered modulo p. By Fermat's little theorem, one has $2^{p-1} \equiv 1$, $3^{p-1} \equiv 1$, $6^{p-1} \equiv 1$. Then the evident congruence $3 + 2 + 1 \equiv 6$ can be written as

$$3 \cdot 2^{p-1} + 2 \cdot 3^{p-1} + 6^{p-1} \equiv 6, \quad \text{or} \quad 6 \cdot 2^{p-2} + 6 \cdot 3^{p-2} + 6 \cdot 6^{p-2} \equiv 6.$$

Simplifying by 6 shows that $a_{p-2} = 2^{p-2} + 3^{p-2} + 6^{p-2} - 1$ is divisible by p, and the proof is complete.

N2. Let $a_1, a_2, \ldots$ be a sequence of integers with infinitely many positive and infinitely many negative terms. Suppose that for every positive integer M the numbers $a_1, a_2, \ldots, a_M$ leave different remainders upon division by M. Prove that every integer occurs exactly once in the sequence $a_1, a_2, \ldots$.

Solution. The hypothesis of the problem can be reformulated by saying that for every positive integer M the numbers $a_1, a_2, \ldots, a_M$ form a complete system of residue classes modulo M. Note that if $i < j$ then $a_i \neq a_j$, otherwise the set $\{a_1, \ldots, a_j\}$ would contain at most $j-1$ distinct residues modulo j. Furthermore, if $i < j \leq n$, then $|a_i - a_j| \leq n-1$, for if $m = |a_i - a_j| \geq n$, then the set $\{a_1, \ldots, a_m\}$ would contain two numbers congruent modulo m, which is impossible.

Given any $n \geq 1$, let $i(n)$, $j(n)$ be the indices such that $a_{i(n)}$, $a_{j(n)}$ are respectively the smallest and the largest number among $a_1, \ldots, a_n$. The above arguments show that $|a_{i(n)} - a_{j(n)}| = n-1$, therefore the set $\{a_1, \ldots, a_n\}$ consists of all integers between $a_{i(n)}$ and $a_{j(n)}$.

Now let x be an arbitrary integer. Since $a_k < 0$ for infinitely many k and the terms of the sequence are distinct, we conclude that there exists i such that $a_i < x$. By a similar argument, there exists j such that $x < a_j$. Hence, if $n > \max\{i, j\}$, we conclude that every number between a_i and a_j (x in particular) is in $\{a_1, \ldots, a_n\}$.

Comment. Proving that for every M the set $\{a_1, \ldots, a_M\}$ is a block of consecutive integers can be also achieved by induction.

N3. Let a, b, c, d, e and f be positive integers. Suppose that the sum $S = a+b+c+d+e+f$ divides both $abc+def$ and $ab+bc+ca-de-ef-fd$. Prove that S is composite.

Solution. By hypothesis, all coefficients of the quadratic polynomial

$$\begin{aligned} f(x) &= (x+a)(x+b)(x+c) - (x-d)(x-e)(x-f) \\ &= Sx^2 + (ab+bc+ca-de-ef-fd)x + (abc+def) \end{aligned}$$

are multiples of S. Evaluating f at d we get that $f(d) = (a+d)(b+d)(c+d)$ is a multiple of S. This readily implies that S is composite because each of $a+d$, $b+d$ and $c+d$ is less than S.

N4. Find all positive integers $n > 1$ for which there exists a unique integer a with $0 < a \leq n!$ such that $a^n + 1$ is divisible by $n!$.

Solution. The answer is "n is prime."

If $n = 2$, the only solution is $a = 1$. If $n > 2$ is even, then a^n is a square, therefore $a^n + 1$ is congruent to 1 or 2 modulo 4, while $n!$ is divisible by 4. So there is no appropriate a in this case.

From now on, n is odd. Assume that $n = p$ is a prime and that $p! \mid a^p + 1$ for some a, $0 < a \leq p!$. By Fermat's little theorem, $a^p + 1 \equiv a + 1 \pmod{p}$. So, if p does not divide $a + 1$, then $a^{p-1} + \cdots + a + 1 = (a^p + 1)/(a + 1) \equiv 1 \pmod{p}$, which is a contradiction. Thus, $p \mid a + 1$.

We shall show that $(a^p + 1)/(a + 1)$ has no prime divisors $q < p$. This will be enough to deduce the uniqueness of a. Indeed, the relation

$$(p-1)! \mid (a+1)\left(\frac{a^p+1}{a+1}\right)$$

forces $(p-1)! \mid a+1$. Combined with $p \mid a+1$, this leads to $p! \mid a+1$, and hence showing $a = p! - 1$.

Suppose on the contrary that $q \mid (a^p + 1)/(a + 1)$, where $q < p$ is prime. Note that q is odd. We get $a^p \equiv -1 \pmod{q}$, therefore $a^{2p} \equiv 1 \pmod{q}$. Clearly, q is coprime to a, so $a^{q-1} \equiv 1 \pmod{q}$. Writing $d = \gcd(q-1, 2p)$, we obtain $a^d \equiv 1 \pmod{q}$. Since $q < p$, we have $d = 2$. Hence, $a \equiv \pm 1 \pmod{q}$. The case $a \equiv 1 \pmod{q}$ gives $(a^p + 1)/(a + 1) \equiv 1 \pmod{q}$, which is impossible. The case $a \equiv -1 \pmod{q}$ gives

$$\begin{aligned} \frac{a^p+1}{a+1} &\equiv a^{p-1} - a^{p-2} + \cdots + 1 \\ &\equiv (-1)^{p-1} - (-1)^{p-2} + \cdots + 1 \equiv p \pmod{q}, \end{aligned}$$

leading to $q \mid p$ which is not possible as $q < p$. So, we see that primes fulfill the conditions under discussion.

It remains to deal with the case when n is odd and composite. Let $p < n$ be the least prime divisor of n. Let p^α be the highest power of p which divides $n!$. Since $2p < p^2 \leq n$, we have $n! = 1 \ldots p \ldots (2p) \ldots$, so $\alpha \geq 2$. Write $m = n!/p^\alpha$, and take any integer a satisfying

$$a \equiv -1 \left(\bmod\ p^{\alpha-1} m\right). \tag{1}$$

Write $a = -1 + p^{\alpha-1}k$. Then

$$a^p = (-1 + p^{\alpha-1}k)^p = -1 + p^\alpha k + p^\alpha \sum_{j=2}^{p} (-1)^{p-j} \binom{p}{j} p^{j(\alpha-1)} k^j = -1 + p^\alpha M,$$

where M is an integer because $j(\alpha - 1) \geq \alpha$ for all $j \geq 2$ and $\alpha \geq 2$. Thus p^α divides $a^p + 1$, and hence also $a^n + 1$, because $p \mid n$ and n is odd. Furthermore, m too is a divisor of $a + 1$, and hence of $a^n + 1$. Since m is coprime to p, $(a^n + 1)/n!$ is an integer for all a satisfying congruence (1). Since it is clear that there are $p > 2$ integers in the interval $[1, n!]$ satisfying (1), we conclude that composite values of n do not satisfy the condition given in the problem.

Comment. The fact that no prime divisor of $(a^p + 1)/(a + 1)$ is smaller than p is not a mere curiosity. More is true and can be deduced easily from the above proof, namely that if q is a prime factor of the above number, then either $q = p$ (and this happens if and only if $p \mid a + 1$) or $q \equiv 1 \pmod{p}$.

N5. Denote by $d(n)$ the number of divisors of the positive integer n. A positive integer n is called *highly divisible* if $d(n) > d(m)$ for all positive integers $m < n$. Two highly divisible integers m and n with $m < n$ are called consecutive if there exists no highly divisible integer s satisfying $m < s < n$.

(a) Show that there are only finitely many pairs of consecutive highly divisible integers of the form (a, b) with $a \mid b$.

(b) Show that for every prime number p there exist infinitely many positive highly divisible integers r such that pr is also highly divisible.

Solution. This problem requires an analysis of the structure of the highly divisible integers. Recall that if n has prime factorization

$$n = \prod_{p^{\alpha_p(n)} || n} p^{\alpha_p(n)},$$

where p stands for a prime, then $d(n) = \prod_{p^{\alpha_p}||n}(\alpha_p(n) + 1)$.

Let us start by noting that since $d(n)$ takes arbitrarily large values (think of $d(m!)$, for example, for arbitrary large m's), there exist infinitely many highly divisible integers. Furthermore, it is easy to see that if n is highly divisible and

$$n = 2^{\alpha_2(n)} 3^{\alpha_3(n)} \dots p^{\alpha_p(n)},$$

then $\alpha_2(n) \geq \dots \geq \alpha_p(n)$. Thus, if $q < p$ are primes and $p \mid n$, then $q \mid n$.

We show that for every prime p all but finitely many highly divisible integers are multiples of p. This is obviously so for $p = 2$. Assume that this were not so, that p is the rth prime ($r > 1$), and that n is one of the infinitely many highly divisible integers whose largest prime factor is less than p. For such an n, $(\alpha_2(n) + 1)^{r-1} \geq d(n)$, therefore $\alpha_2(n)$ takes arbitrarily large values. Let n be such that $2^{\alpha_2(n)-1} > p^2$ and look at $m = np/2^{\lfloor \alpha_2(n)/2 \rfloor}$. Clearly, $m < n$, while

$$d(m) = 2d(n)\frac{\alpha_2(n) - \lfloor \alpha_2(n)/2 \rfloor + 1}{\alpha_2(n) + 1} > d(n)$$

contradicting the fact that n is highly divisible.

We now show a stronger property, namely that for any prime p and constant κ, there are only finitely many highly divisible positive integers n such that $\alpha_p(n) \leq \kappa$. Indeed, assume that this were not so. Let κ be a constant such that $\alpha_p(n) \leq \kappa$ for infinitely many highly divisible n. Let q be a large prime satisfying $q > p^{2\kappa+1}$. All but finitely many such positive integers n are multiples of q. Look at the number $m = p^{\alpha_p(n)\alpha_q(n)+\alpha_p(n)+\alpha_q(n)} n/q^{\alpha_q(n)}$. An immediate calculation shows that $d(m) = d(n)$, therefore $m > n$. Thus,

$$p^{2\alpha_p(n)\alpha_q(n)+\alpha_q(n)} \geq p^{\alpha_p(n)\alpha_q(n)+\alpha_q(n)+\alpha_p(n)} > q^{\alpha_q(n)},$$

giving $p^{2\alpha_p(n)+1} > q > p^{2\kappa+1}$, and we get a contradiction with the fact that $\alpha_p(n) \le \kappa$.

We are now ready to prove both (a) and (b). For (a), let n be highly divisible and such that $\alpha_3(n) \ge 8$. All but finitely many highly divisible integers n have this property. Now $8n/9$ is an integer and $8n/9 < n$, therefore $d(8n/9) < d(n)$. This implies

$$(\alpha_2(n)+4)(\alpha_3(n)-1) < (\alpha_2(n)+1)(\alpha_3(n)+1),$$

which is equivalent to

$$3\alpha_3(n) - 5 < 2\alpha_2(n). \tag{1}$$

Assume now that $n \mid m$ are consecutive and highly divisible. Since already $d(2n) > d(n)$, we get that there must be a highly divisible integer in $(n, 2n]$. Thus $m = 2n$, leading to $d(3n/2) \le d(n)$ (or else there must be a highly divisible number between n and $3n/2$). This gives

$$\alpha_2(n)(\alpha_3(n)+2) \le (\alpha_2(n)+1)(\alpha_3(n)+1),$$

which is equivalent to

$$\alpha_2(n) \le \alpha_3(n) + 1,$$

which together with $\alpha_3(n) \ge 8$ contradicts inequality (1). This proves (a).

For part (b), let k be any positive integer and look at the smallest highly divisible positive integer n such that $\alpha_p(n) \ge k$. All but finitely many highly divisible integers n satisfy this last inequality. We claim that n/p is also highly divisible. If this were not so, then there would exist a highly divisible positive integer $m < n/p$ with $d(m) \ge d(n/p)$. Note that, by assumption, $\alpha_p(m) < \alpha_p(n)$. Then,

$$d(mp) = d(m)\frac{\alpha_p(m)+2}{\alpha_p(m)+1} \ge d(n/p)\frac{\alpha_p(n)+1}{\alpha_p(n)} = d(n),$$

where for the above inequality we used the fact that the function $(x+1)/x$ is decreasing. However, $mp < n$, so the above inequality contradicts the fact that n is highly divisible. This contradiction shows that n/p is highly divisible, and since k can be taken to be arbitrarily large, we get infinitely many examples of highly divisible integers n such that n/p is also highly divisible.

Comment. The notion of a highly divisible integer first appeared in a paper of Ramanujan in 1915. Eric Weinstein's *World of Mathematics* has one web page mentioning some properties of these numbers (called highly composite) and giving some bibliographical references, while Ross Honsberger's *Mathematical*

Gems (Third Edition) has a chapter dedicated to them. In spite of all these references, the properties of these numbers mentioned in the above sources have little relevance for the problem at hand and we believe that if given to the exam, the students who have seen these numbers before will not have any significant advantage over the ones who encounter them for the first time.

N6. Let a and b be positive integers such that $a^n + n$ divides $b^n + n$ for every positive integer n. Show that $a = b$.

Solution. Assume that $b \neq a$. Taking $n = 1$ shows that $a + 1$ divides $b + 1$, so that $b \geq a$. Let $p > b$ be a prime and let n be a positive integer such that

$$n \equiv 1 \pmod{p-1} \quad \text{and} \quad n \equiv -a \pmod{p}.$$

Such an n exists by the Chinese remainder theorem. (Without the Chinese remainder theorem, one could notice that $n = (a+1)(p-1)+1$ has this property.)

By Fermat's little theorem, $a^n = a(a^{p-1} \cdots a^{p-1}) \equiv a \pmod{p}$, and therefore $a^n + n \equiv 0 \pmod{p}$. So p divides the number $a^n + n$, hence also $b^n + n$. However, by Fermat's little theorem again, we have analogously $b^n + n \equiv b - a \pmod{p}$. We are therefore led to the conclusion $p \mid b - a$, which is a contradiction.

Comment. The first thing coming to mind is to show that a and b share the same prime divisors. This is easily established by using Fermat's little theorem or Wilson's theorem. However, we know of no solution which uses this fact in any meaningful way.

For the conclusion to remain true, it is not sufficient that $a^n + n \mid b^n + n$ holds for infinitely many n. Indeed, take $a = 1$ and any $b > 1$. The given divisibility relation holds for all positive integers n of the form $p - 1$, where $p > b$ is a prime, but $a \neq b$.

N7. Let $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$, where $a_0, \ldots, a_n$ are integers, $a_n > 0$, $n \geq 2$. Prove that there exists a positive integer m such that $P(m!)$ is a composite number.

Solution. We may assume that $a_0 = \pm 1$, otherwise the conclusion is immediate. Observe that if $p > k \geq 1$ and p is a prime then

$$(p-k)! \equiv (-1)^k \left((k-1)!\right)^{-1} \pmod{p}, \tag{1}$$

where t^{-1} denotes the multiplicative inverse (mod p) of t. Indeed, this is proved by writing

$$(p-1)! = (p-k)![p-(k-1)][p-(k-2)]\cdots(p-1),$$

reducing modulo p and using Wilson's theorem. With (1) in mind, we see that it might be worth looking at the rational numbers

$$P\left(\frac{(-1)^k}{(k-1)!}\right) = \frac{(-1)^{kn}}{((k-1)!)^n} Q((-1)^k(k-1)!),$$

where $Q(x) = a_n + a_{n-1}x + \cdots + a_0 x^n$.

If $k - 1 > a_n^2$, then $a_n \mid (k-1)!$ and $(k-1)!/a_n = 1 \cdot 2 \cdots (a_n^2/a_n) \cdots (k-1)$ is divisible by all primes $\leq k-1$. Hence, for such k we have $Q((k-1)!) = a_n b_k$, where $b_k = 1 + a_{n-1}(k-1)!/a_n + \cdots$ has no prime factors $\leq k-1$. Clearly, $Q(x)$ is not a constant polynomial, because its leading term is $a_0 = \pm 1$. Therefore $|Q((k-1)!)|$ becomes arbitrarily large when k is large, and so does $|b_k|$. In particular, $|b_k| > 1$ if k is large enough.

Take such an even k and choose any prime factor p of b_k. The above argument, combined with (1), shows that $p > k$ and that $P((p-k)!) \equiv 0 \pmod{p}$.

In order to complete the proof, we only need to ensure that k can be chosen so that $|P((p-k)!)| > p$. We do not know p, but we know that $p \geq k$. Our best bet is to take k such that the first possible prime following k is "far away" from it; i. e., $p-k$ is large. For this, we may choose $k = m!$, where $m = q-1 > 2$ and q is a prime. Then $m!$ is composite, $m!+1$ is also composite (because $m!+1 > m+1 = q$ and $m!+1$ is a multiple of q by Wilson's theorem), and $m! + \ell$ is also composite for all $\ell = 2, \ldots, m$. So, $p = m! + m + t$ for some $t \geq 1$, therefore $p - k = m + t$. For large m,

$$P((p-k)!) = P((m+t)!) > \frac{(m+t)!}{2},$$

because $a_n > 0$. So it suffices to observe that

$$\frac{(m+t)!}{2} > m! + m + t,$$

which is obviously true for m large enough and $t \geq 1$.

47th INTERNATIONAL MATHEMATICAL OLYMPIAD
SLOVENIA 2006

Shortlisted problems with solutions

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Algebra

A1. A sequence of real numbers $a_0, a_1, a_2, \dots$ is defined by the formula

$$a_{i+1} = \lfloor a_i \rfloor \cdot \langle a_i \rangle \qquad \text{for} \quad i \geq 0;$$

here a_0 is an arbitrary real number, $\lfloor a_i \rfloor$ denotes the greatest integer not exceeding a_i, and $\langle a_i \rangle = a_i - \lfloor a_i \rfloor$. Prove that $a_i = a_{i+2}$ for i sufficiently large.

(Estonia)

Solution. First note that if $a_0 \geq 0$, then all $a_i \geq 0$. For $a_i \geq 1$ we have (in view of $\langle a_i \rangle < 1$ and $\lfloor a_i \rfloor > 0$)

$$\lfloor a_{i+1} \rfloor \leq a_{i+1} = \lfloor a_i \rfloor \cdot \langle a_i \rangle < \lfloor a_i \rfloor;$$

the sequence $\lfloor a_i \rfloor$ is strictly decreasing as long as its terms are in $[1, \infty)$. Eventually there appears a number from the interval $[0, 1)$ and all subsequent terms are 0.

Now pass to the more interesting situation where $a_0 < 0$; then all $a_i \leq 0$. Suppose the sequence never hits 0. Then we have $\lfloor a_i \rfloor \leq -1$ for all i, and so

$$1 + \lfloor a_{i+1} \rfloor > a_{i+1} = \lfloor a_i \rfloor \cdot \langle a_i \rangle > \lfloor a_i \rfloor;$$

this means that the sequence $\lfloor a_i \rfloor$ is nondecreasing. And since all its terms are integers from $(-\infty, -1]$, this sequence must be constant from some term on:

$$\lfloor a_i \rfloor = c \qquad \text{for} \quad i \geq i_0\,; \qquad c \;\; \text{a negative integer.}$$

The defining formula becomes

$$a_{i+1} = c \cdot \langle a_i \rangle = c(a_i - c) = ca_i - c^2.$$

Consider the sequence

$$b_i = a_i - \frac{c^2}{c-1}\,. \tag{1}$$

It satisfies the recursion rule

$$b_{i+1} = a_{i+1} - \frac{c^2}{c-1} = ca_i - c^2 - \frac{c^2}{c-1} = cb_i,$$

implying

$$b_i = c^{i-i_0} b_{i_0} \qquad \text{for} \quad i \geq i_0. \tag{2}$$

Since all the numbers a_i (for $i \geq i_0$) lie in $[c, c+1)$, the sequence (b_i) is bounded. The equation (2) can be satisfied only if either $b_{i_0} = 0$ or $|c| = 1$, i.e., $c = -1$.

In the first case, $b_i = 0$ for all $i \geq i_0$, so that

$$a_i = \frac{c^2}{c-1} \qquad \text{for} \quad i \geq i_0.$$

In the second case, $c = -1$, equations (1) and (2) say that

$$a_i = -\frac{1}{2} + (-1)^{i-i_0} b_{i_0} = \begin{cases} a_{i_0} & \text{for} \;\; i = i_0, i_0+2, i_0+4, \ldots\,, \\ 1 - a_{i_0} & \text{for} \;\; i = i_0+1, i_0+3, i_0+5, \ldots\,. \end{cases}$$

Summarising, we see that (from some point on) the sequence (a_i) either is constant or takes alternately two values from the interval $(-1, 0)$. The result follows.

Comment. There is nothing mysterious in introducing the sequence (b_i). The sequence (a_i) arises by iterating the function $x \mapsto cx - c^2$ whose unique fixed point is $c^2/(c-1)$.

A2. The sequence of real numbers $a_0, a_1, a_2, \ldots$ is defined recursively by

$$a_0 = -1, \qquad \sum_{k=0}^{n} \frac{a_{n-k}}{k+1} = 0 \quad \text{for} \quad n \geq 1.$$

Show that $a_n > 0$ for $n \geq 1$.

(Poland)

Solution. The proof goes by induction. For $n = 1$ the formula yields $a_1 = 1/2$. Take $n \geq 1$, assume $a_1, \ldots, a_n > 0$ and write the recurrence formula for n and $n+1$, respectively as

$$\sum_{k=0}^{n} \frac{a_k}{n-k+1} = 0 \qquad \text{and} \qquad \sum_{k=0}^{n+1} \frac{a_k}{n-k+2} = 0.$$

Subtraction yields

$$\begin{aligned} 0 = (n+2) \sum_{k=0}^{n+1} \frac{a_k}{n-k+2} - (n+1) \sum_{k=0}^{n} \frac{a_k}{n-k+1} \\ = (n+2) a_{n+1} + \sum_{k=0}^{n} \left(\frac{n+2}{n-k+2} - \frac{n+1}{n-k+1} \right) a_k. \end{aligned}$$

The coefficient of a_0 vanishes, so

$$a_{n+1} = \frac{1}{n+2} \sum_{k=1}^{n} \left(\frac{n+1}{n-k+1} - \frac{n+2}{n-k+2} \right) a_k = \frac{1}{n+2} \sum_{k=1}^{n} \frac{k}{(n-k+1)(n-k+2)} a_k.$$

The coefficients of $a_1, , \ldots, a_n$ are all positive. Therefore, $a_1, \ldots, a_n > 0$ implies $a_{n+1} > 0$.

Comment. Students familiar with the technique of generating functions will immediately recognise $\sum a_n x^n$ as the power series expansion of $x / \ln(1-x)$ (with value -1 at 0). But this can be a trap; attempts along these lines lead to unpleasant differential equations and integrals hard to handle. Using only tools from real analysis (e.g. computing the coefficients from the derivatives) seems very difficult.

On the other hand, the coefficients can be approached applying complex contour integrals and some other techniques from complex analysis and an attractive formula can be obtained for the coefficients:

$$a_n = \int_1^{\infty} \frac{\mathrm{d}x}{x^n \left(\pi^2 + \log^2(x-1) \right)} \qquad (n \geq 1)$$

which is evidently positive.

A3. The sequence $c_0, c_1, \ldots, c_n, \ldots$ is defined by $c_0 = 1$, $c_1 = 0$ and $c_{n+2} = c_{n+1} + c_n$ for $n \geq 0$. Consider the set S of ordered pairs (x, y) for which there is a finite set J of positive integers such that $x = \sum_{j \in J} c_j$, $y = \sum_{j \in J} c_{j-1}$. Prove that there exist real numbers α, β and m, M with the following property: An ordered pair of nonnegative integers (x, y) satisfies the inequality

$$m < \alpha x + \beta y < M$$

if and only if $(x, y) \in S$.

N. B. A sum over the elements of the empty set is assumed to be 0.

(Russia)

Solution. Let $\varphi = (1+\sqrt{5})/2$ and $\psi = (1-\sqrt{5})/2$ be the roots of the quadratic equation $t^2 - t - 1 = 0$. So $\varphi\psi = -1$, $\varphi + \psi = 1$ and $1 + \psi = \psi^2$. An easy induction shows that the general term c_n of the given sequence satisfies

$$c_n = \frac{\varphi^{n-1} - \psi^{n-1}}{\varphi - \psi} \qquad \text{for } n \geq 0.$$

Suppose that the numbers α and β have the stated property, for appropriately chosen m and M. Since $(c_n, c_{n-1}) \in S$ for each n, the expression

$$\alpha c_n + \beta c_{n-1} = \frac{\alpha}{\sqrt{5}}\left(\varphi^{n-1} - \psi^{n-1}\right) + \frac{\beta}{\sqrt{5}}\left(\varphi^{n-2} - \psi^{n-2}\right) = \frac{1}{\sqrt{5}}\left[(\alpha\varphi + \beta)\varphi^{n-2} - (\alpha\psi + \beta)\psi^{n-2}\right]$$

is bounded as n grows to infinity. Because $\varphi > 1$ and $-1 < \psi < 0$, this implies $\alpha\varphi + \beta = 0$.

To satisfy $\alpha\varphi + \beta = 0$, one can set for instance $\alpha = \psi$, $\beta = 1$. We now find the required m and M for this choice of α and β.

Note first that the above displayed equation gives $c_n\psi + c_{n-1} = \psi^{n-1}$, $n \geq 1$. In the sequel, we denote the pairs in S by (a_J, b_J), where J is a finite subset of the set $\mathbb{N}$ of positive integers and $a_J = \sum_{j \in J} c_j$, $b_J = \sum_{j \in J} c_{j-1}$. Since $\psi a_J + b_J = \sum_{j \in J}(c_j\psi + c_{j-1})$, we obtain

$$\psi a_J + b_J = \sum_{j \in J} \psi^{j-1} \qquad \text{for each } (a_J, b_J) \in S. \tag{1}$$

On the other hand, in view of $-1 < \psi < 0$,

$$-1 = \frac{\psi}{1-\psi^2} = \sum_{j=0}^{\infty} \psi^{2j+1} < \sum_{j \in J} \psi^{j-1} < \sum_{j=0}^{\infty} \psi^{2j} = \frac{1}{1-\psi^2} = 1 - \psi = \varphi.$$

Therefore, according to (1),

$$-1 < \psi a_J + b_J < \varphi \qquad \text{for each } (a_J, b_J) \in S.$$

Thus $m = -1$ and $M = \varphi$ is an appropriate choice.

Conversely, we prove that if an ordered pair of nonnegative integers (x, y) satisfies the inequality $-1 < \psi x + y < \varphi$ then $(x, y) \in S$.

Lemma. Let x, y be nonnegative integers such that $-1 < \psi x + y < \varphi$. Then there exists a subset J of $\mathbb{N}$ such that

$$\psi x + y = \sum_{j \in J} \psi^{j-1} \tag{2}$$

Proof. For $x = y = 0$ it suffices to choose the empty subset of $\mathbb{N}$ as J, so let at least one of x, y be nonzero. There exist representations of $\psi x + y$ of the form

$$\psi x + y = \psi^{i_1} + \cdots + \psi^{i_k}$$

where $i_1 \le \cdots \le i_k$ is a sequence of nonnegative integers, not necessarily distinct. For instance, we can take x summands $\psi^1 = \psi$ and y summands $\psi^0 = 1$. Consider all such representations of minimum length k and focus on the ones for which i_1 has the minimum possible value j_1. Among them, consider the representations where i_2 has the minimum possible value j_2. Upon choosing $j_3, \ldots, j_k$ analogously, we obtain a sequence $j_1 \le \cdots \le j_k$ which clearly satisfies $\psi x + y = \sum_{r=1}^{k} \psi^{j_r}$. To prove the lemma, it suffices to show that $j_1, \ldots, j_k$ are pairwise distinct.

Suppose on the contrary that $j_r = j_{r+1}$ for some $r = 1, \ldots, k-1$. Let us consider the case $j_r \ge 2$ first. Observing that $2\psi^2 = 1 + \psi^3$, we replace j_r and j_{r+1} by $j_r - 2$ and $j_r + 1$, respectively. Since

$$\psi^{j_r} + \psi^{j_{r+1}} = 2\psi^{j_r} = \psi^{j_r - 2}(1 + \psi^3) = \psi^{j_r - 2} + \psi^{j_r + 1},$$

the new sequence also represents $\psi x + y$ as needed, and the value of i_r in it contradicts the minimum choice of j_r.

Let $j_r = j_{r+1} = 0$. Then the sum $\psi x + y = \sum_{r=1}^{k} \psi^{j_r}$ contains at least two summands equal to $\psi^0 = 1$. On the other hand $j_s \ne 1$ for all s, because the equality $1 + \psi = \psi^2$ implies that a representation of minimum length cannot contain consecutive i_r's. It follows that

$$\psi x + y = \sum_{r=1}^{k} \psi^{j_r} > 2 + \psi^3 + \psi^5 + \psi^7 + \cdots = 2 - \psi^2 = \varphi,$$

contradicting the condition of the lemma.

Let $j_r = j_{r+1} = 1$; then $\sum_{r=1}^{k} \psi^{j_r}$ contains at least two summands equal to $\psi^1 = \psi$. Like in the case $j_r = j_{r+1} = 0$, we also infer that $j_s \ne 0$ and $j_s \ne 2$ for all s. Therefore

$$\psi x + y = \sum_{r=1}^{k} \psi^{j_r} < 2\psi + \psi^4 + \psi^6 + \psi^8 + \cdots = 2\psi - \psi^3 = -1,$$

which is a contradiction again. The conclusion follows. $\square$

Now let the ordered pair (x, y) satisfy $-1 < \psi x + y < \varphi$; hence the lemma applies to (x, y). Let $J \subset \mathbb{N}$ be such that (2) holds. Comparing (1) and (2), we conclude that $\psi x + y = \psi a_J + b_J$. Now, x, y, a_J and b_J are integers, and ψ is irrational. So the last equality implies $x = a_J$ and $y = b_J$. This shows that the numbers $\alpha = \psi$, $\beta = 1$, $m = -1$, $M = \varphi$ meet the requirements.

Comment. We present another way to prove the lemma, constructing the set J inductively. For $x = y = 0$, choose $J = \emptyset$. We induct on $n = 3x + 2y$. Suppose that an appropriate set J exists when $3x + 2y < n$. Now assume $3x + 2y = n > 0$. The current set J should be

$$\text{either} \quad 1 \le j_1 < j_2 < \cdots < j_k \quad \text{or} \quad j_1 = 0,\ 1 \le j_2 < \cdots < j_k.$$

These sets fulfil the condition if

$$\frac{\psi x + y}{\psi} = \psi^{i_1 - 1} + \cdots + \psi^{i_k - 1} \quad \text{or} \quad \frac{\psi x + y - 1}{\psi} = \psi^{i_2 - 1} + \cdots + \psi^{i_k - 1},$$

respectively; therefore it suffices to find an appropriate set for $\frac{\psi x+y}{\psi}$ or $\frac{\psi x+y-1}{\psi}$, respectively.

Consider $\frac{\psi x+y}{\psi}$. Knowing that

$$\frac{\psi x+y}{\psi} = x + (\psi - 1)y = \psi y + (x - y),$$

let $x' = y$, $y' = x - y$ and test the induction hypothesis on these numbers. We require $\frac{\psi x+y}{\psi} \in (-1, \varphi)$ which is equivalent to

$$\psi x + y \in (\varphi \cdot \psi, (-1) \cdot \psi) = (-1, -\psi). \tag{3}$$

Relation (3) implies $y' = x - y \geq -\psi x - y > \psi > -1$; therefore $x', y' \geq 0$. Moreover, we have $3x' + 2y' = 2x + y \leq \frac{2}{3}n$; therefore, if (3) holds then the induction applies: the numbers x', y' are represented in the form as needed, hence x, y also.

Now consider $\frac{\psi x+y-1}{\psi}$. Since

$$\frac{\psi x+y-1}{\psi} = x + (\psi - 1)(y - 1) = \psi(y - 1) + (x - y + 1),$$

we set $x' = y - 1$ and $y' = x - y + 1$. Again we require that $\frac{\psi x+y-1}{\psi} \in (-1, \varphi)$, i.e.

$$\psi x + y \in (\varphi \cdot \psi + 1, (-1) \cdot \psi + 1) = (0, \varphi). \tag{4}$$

If (4) holds then $y - 1 \geq \psi x + y - 1 > -1$ and $x - y + 1 \geq -\psi x - y + 1 > -\varphi + 1 > -1$, therefore $x', y' \geq 0$. Moreover, $3x' + 2y' = 2x + y - 1 < \frac{2}{3}n$ and the induction works.

Finally, $(-1, -\psi) \cup (0, \varphi) = (-1, \varphi)$ so at least one of (3) and (4) holds and the induction step is justified.

A4. Prove the inequality

$$\sum_{i<j} \frac{a_i a_j}{a_i + a_j} \leq \frac{n}{2(a_1 + a_2 + \cdots + a_n)} \sum_{i<j} a_i a_j$$

for positive real numbers $a_1, a_2, \ldots, a_n$.

(Serbia)

Solution 1. Let $S = \sum_i a_i$. Denote by L and R the expressions on the left and right hand side of the proposed inequality. We transform L and R using the identity

$$\sum_{i<j} (a_i + a_j) = (n-1) \sum_i a_i. \tag{1}$$

And thus:

$$L = \sum_{i<j} \frac{a_i a_j}{a_i + a_j} = \sum_{i<j} \frac{1}{4}\Big(a_i + a_j - \frac{(a_i - a_j)^2}{a_i + a_j}\Big) = \frac{n-1}{4} \cdot S - \frac{1}{4} \sum_{i<j} \frac{(a_i - a_j)^2}{a_i + a_j}. \tag{2}$$

To represent R we express the sum $\sum\limits_{i<j} a_i a_j$ in two ways; in the second transformation identity (1) will be applied to the squares of the numbers a_i:

$$\sum_{i<j} a_i a_j = \frac{1}{2}\Big(S^2 - \sum_i a_i^2\Big);$$

$$\sum_{i<j} a_i a_j = \frac{1}{2} \sum_{i<j} \Big(a_i^2 + a_j^2 - (a_i - a_j)^2\Big) = \frac{n-1}{2} \cdot \sum_i a_i^2 - \frac{1}{2} \sum_{i<j} (a_i - a_j)^2.$$

Multiplying the first of these equalities by $n-1$ and adding the second one we obtain

$$n \sum_{i<j} a_i a_j = \frac{n-1}{2} \cdot S^2 - \frac{1}{2} \sum_{i<j} (a_i - a_j)^2.$$

Hence

$$R = \frac{n}{2S} \sum_{i<j} a_i a_j = \frac{n-1}{4} \cdot S - \frac{1}{4} \sum_{i<j} \frac{(a_i - a_j)^2}{S}. \tag{3}$$

Now compare (2) and (3). Since $S \geq a_i + a_j$ for any $i < j$, the claim $L \geq R$ results.

Solution 2. Let $S = a_1 + a_2 + \cdots + a_n$. For any $i \neq j$,

$$4\frac{a_i a_j}{a_i + a_j} = a_i + a_j - \frac{(a_i - a_j)^2}{a_i + a_j} \leq a_i + a_j - \frac{(a_i - a_j)^2}{a_1 + a_2 + \cdots + a_n} = \frac{\sum\limits_{k \neq i} a_i a_k + \sum\limits_{k \neq j} a_j a_k + 2a_i a_j}{S}.$$

The statement is obtained by summing up these inequalities for all pairs i, j:

$$\sum_{i<j} \frac{a_i a_j}{a_i + a_j} = \frac{1}{2} \sum_i \sum_{j \neq i} \frac{a_i a_j}{a_i + a_j} \leq \frac{1}{8S} \sum_i \sum_{j \neq i} \left(\sum_{k \neq i} a_i a_k + \sum_{k \neq j} a_j a_k + 2a_i a_j \right)$$

$$= \frac{1}{8S} \left(\sum_k \sum_{i \neq k} \sum_{j \neq i} a_i a_k + \sum_k \sum_{j \neq k} \sum_{i \neq j} a_j a_k + \sum_i \sum_{j \neq i} 2a_i a_j \right)$$

$$= \frac{1}{8S} \left(\sum_k \sum_{i \neq k} (n-1) a_i a_k + \sum_k \sum_{j \neq k} (n-1) a_j a_k + \sum_i \sum_{j \neq i} 2a_i a_j \right)$$

$$= \frac{n}{4S} \sum_i \sum_{j \neq i} a_i a_j = \frac{n}{2S} \sum_{i<j} a_i a_j.$$

Comment. Here is an outline of another possible approach. Examine the function $R - L$ subject to constraints $\sum_i a_i = S$, $\sum_{i<j} a_i a_j = U$ for fixed constants $S, U > 0$ (which can jointly occur as values of these symmetric forms). Suppose that among the numbers a_i there are some three, say a_k, a_l, a_m such that $a_k < a_l \leq a_m$. Then it is possible to decrease the value of $R - L$ by perturbing this triple so that in the new triple a'_k, a'_l, a'_m one has $a'_k = a'_l \leq a'_m$, without touching the remaining a_is and without changing the values of S and U; this requires some skill in algebraic manipulations. It follows that the constrained minimum can be only attained for $n - 1$ of the a_is equal and a single one possibly greater. In this case, $R - L \geq 0$ holds almost trivially.

A5. Let a, b, c be the sides of a triangle. Prove that

$$\frac{\sqrt{b+c-a}}{\sqrt{b}+\sqrt{c}-\sqrt{a}}+\frac{\sqrt{c+a-b}}{\sqrt{c}+\sqrt{a}-\sqrt{b}}+\frac{\sqrt{a+b-c}}{\sqrt{a}+\sqrt{b}-\sqrt{c}} \leq 3.$$

(Korea)

Solution 1. Note first that the denominators are all positive, e.g. $\sqrt{a}+\sqrt{b}>\sqrt{a+b}>\sqrt{c}$.

Let $x=\sqrt{b}+\sqrt{c}-\sqrt{a}$, $y=\sqrt{c}+\sqrt{a}-\sqrt{b}$ and $z=\sqrt{a}+\sqrt{b}-\sqrt{c}$. Then

$$b+c-a=\left(\frac{z+x}{2}\right)^2+\left(\frac{x+y}{2}\right)^2-\left(\frac{y+z}{2}\right)^2=\frac{x^2+xy+xz-yz}{2}=x^2-\frac{1}{2}(x-y)(x-z)$$

and

$$\frac{\sqrt{b+c-a}}{\sqrt{b}+\sqrt{c}-\sqrt{a}}=\sqrt{1-\frac{(x-y)(x-z)}{2x^2}} \leq 1-\frac{(x-y)(x-z)}{4x^2},$$

applying $\sqrt{1+2u} \leq 1+u$ in the last step. Similarly we obtain

$$\frac{\sqrt{c+a-b}}{\sqrt{c}+\sqrt{a}-\sqrt{b}} \leq 1-\frac{(z-x)(z-y)}{4z^2} \quad \text{and} \quad \frac{\sqrt{a+b-c}}{\sqrt{a}+\sqrt{b}-\sqrt{c}} \leq 1-\frac{(y-z)(y-x)}{4y^2}.$$

Substituting these quantities into the statement, it is sufficient to prove that

$$\frac{(x-y)(x-z)}{x^2}+\frac{(y-z)(y-x)}{y^2}+\frac{(z-x)(z-y)}{z^2} \geq 0. \tag{1}$$

By symmetry we can assume $x \leq y \leq z$. Then

$$\frac{(x-y)(x-z)}{x^2}=\frac{(y-x)(z-x)}{x^2} \geq \frac{(y-x)(z-y)}{y^2}=-\frac{(y-z)(y-x)}{y^2},$$

$$\frac{(z-x)(z-y)}{z^2} \geq 0$$

and (1) follows.

Comment 1. Inequality (1) is a special case of the well-known inequality

$$x^t(x-y)(x-z)+y^t(y-z)(y-x)+z^t(z-x)(z-y) \geq 0$$

which holds for all positive numbers x, y, z and real t; in our case $t=-2$. Case $t>0$ is called Schur's inequality. More generally, if $x \leq y \leq z$ are real numbers and p, q, r are nonnegative numbers such that $q \leq p$ or $q \leq r$ then

$$p(x-y)(x-z)+q(y-z)(y-x)+r(z-x)(z-y) \geq 0.$$

Comment 2. One might also start using Cauchy–Schwarz' inequality (or the root mean square vs. arithmetic mean inequality) to the effect that

$$\left(\sum \frac{\sqrt{b+c-a}}{\sqrt{b}+\sqrt{c}-\sqrt{a}}\right)^2 \le 3 \cdot \sum \frac{b+c-a}{\left(\sqrt{b}+\sqrt{c}-\sqrt{a}\,\right)^2}\,, \tag{2}$$

in cyclic sum notation. There are several ways to prove that the right-hand side of (2) never exceeds 9 (and this is just what we need). One of them is to introduce new variables x, y, z, as in Solution 1, which upon some manipulation brings the problem again to inequality (1).

Alternatively, the claim that right-hand side of (2) is not greater than 9 can be expressed in terms of the symmetric forms $\sigma_1 = \sum x$, $\sigma_2 = \sum xy$, $\sigma_3 = xyz$ equivalently as

$$4\sigma_1\sigma_2\sigma_3 \le \sigma_2^3 + 9\sigma_3^2, \tag{3}$$

which is a known inequality. A yet different method to deal with the right-hand expression in (2) is to consider $\sqrt{a}, \sqrt{b}, \sqrt{c}$ as sides of a triangle. Through standard trigonometric formulas the problem comes down to showing that

$$p^2 \le 4R^2 + 4Rr + 3r^2, \tag{4}$$

p, R and r standing for the semiperimeter, the circumradius and the inradius of that triangle. Again, (4) is another known inequality. Note that the inequalities (1), (3), (4) are equivalent statements about the same mathematical situation.

Solution 2. Due to the symmetry of variables, it can be assumed that $a \ge b \ge c$. We claim that

$$\frac{\sqrt{a+b-c}}{\sqrt{a}+\sqrt{b}-\sqrt{c}} \le 1 \quad \text{and} \quad \frac{\sqrt{b+c-a}}{\sqrt{b}+\sqrt{c}-\sqrt{a}} + \frac{\sqrt{c+a-b}}{\sqrt{c}+\sqrt{a}-\sqrt{b}} \le 2.$$

The first inequality follows from

$$\sqrt{a+b-c} - \sqrt{a} = \frac{(a+b-c)-a}{\sqrt{a+b-c}+\sqrt{a}} \le \frac{b-c}{\sqrt{b}+\sqrt{c}} = \sqrt{b} - \sqrt{c}.$$

For proving the second inequality, let $p = \sqrt{a}+\sqrt{b}$ and $q = \sqrt{a}-\sqrt{b}$. Then $a-b = pq$ and the inequality becomes

$$\frac{\sqrt{c-pq}}{\sqrt{c}-q} + \frac{\sqrt{c+pq}}{\sqrt{c}+q} \le 2.$$

From $a \ge b \ge c$ we have $p \ge 2\sqrt{c}$. Applying the Cauchy-Schwarz inequality,

$$\left(\frac{\sqrt{c-pq}}{\sqrt{c}-q} + \frac{\sqrt{c+pq}}{\sqrt{c}+q}\right)^2 \le \left(\frac{c-pq}{\sqrt{c}-q} + \frac{c+pq}{\sqrt{c}+q}\right)\left(\frac{1}{\sqrt{c}-q} + \frac{1}{\sqrt{c}+q}\right)$$

$$= \frac{2(c\sqrt{c}-pq^2)}{c-q^2} \cdot \frac{2\sqrt{c}}{c-q^2} = 4 \cdot \frac{c^2 - \sqrt{c}pq^2}{(c-q^2)^2} \le 4 \cdot \frac{c^2 - 2cq^2}{(c-q^2)^2} \le 4.$$

A6. Determine the smallest number M such that the inequality

$$\left|ab(a^2-b^2)+bc(b^2-c^2)+ca(c^2-a^2)\right| \le M\left(a^2+b^2+c^2\right)^2$$

holds for all real numbers a, b, c.

(Ireland)

Solution. We first consider the cubic polynomial

$$P(t) = tb(t^2-b^2)+bc(b^2-c^2)+ct(c^2-t^2).$$

It is easy to check that $P(b)=P(c)=P(-b-c)=0$, and therefore

$$P(t)=(b-c)(t-b)(t-c)(t+b+c),$$

since the cubic coefficient is $b-c$. The left-hand side of the proposed inequality can therefore be written in the form

$$|ab(a^2-b^2)+bc(b^2-c^2)+ca(c^2-a^2)| = |P(a)| = |(b-c)(a-b)(a-c)(a+b+c)|.$$

The problem comes down to finding the smallest number M that satisfies the inequality

$$|(b-c)(a-b)(a-c)(a+b+c)| \le M\cdot(a^2+b^2+c^2)^2. \tag{1}$$

Note that this expression is symmetric, and we can therefore assume $a \le b \le c$ without loss of generality. With this assumption,

$$|(a-b)(b-c)| = (b-a)(c-b) \le \left(\frac{(b-a)+(c-b)}{2}\right)^2 = \frac{(c-a)^2}{4}, \tag{2}$$

with equality if and only if $b-a=c-b$, i.e. $2b=a+c$. Also

$$\left(\frac{(c-b)+(b-a)}{2}\right)^2 \le \frac{(c-b)^2+(b-a)^2}{2},$$

or equivalently,

$$3(c-a)^2 \le 2\cdot[(b-a)^2+(c-b)^2+(c-a)^2], \tag{3}$$

again with equality only for $2b=a+c$. From (2) and (3) we get

$$\begin{aligned}
& |(b-c)(a-b)(a-c)(a+b+c)| \\
\le\ & \frac{1}{4}\cdot|(c-a)^3(a+b+c)| \\
=\ & \frac{1}{4}\cdot\sqrt{(c-a)^6(a+b+c)^2} \\
\le\ & \frac{1}{4}\cdot\sqrt{\left(\frac{2\cdot[(b-a)^2+(c-b)^2+(c-a)^2]}{3}\right)^3\cdot(a+b+c)^2} \\
=\ & \frac{\sqrt{2}}{2}\cdot\left(\sqrt[4]{\left(\frac{(b-a)^2+(c-b)^2+(c-a)^2}{3}\right)^3\cdot(a+b+c)^2}\right)^2.
\end{aligned}$$

By the weighted AM-GM inequality this estimate continues as follows:

$$\begin{aligned}
& |(b-c)(a-b)(a-c)(a+b+c)| \\
\leq\ & \frac{\sqrt{2}}{2} \cdot \left(\frac{(b-a)^2+(c-b)^2+(c-a)^2+(a+b+c)^2}{4}\right)^2 \\
=\ & \frac{9\sqrt{2}}{32} \cdot (a^2+b^2+c^2)^2.
\end{aligned}$$

We see that the inequality (1) is satisfied for $M = \frac{9}{32}\sqrt{2}$, with equality if and only if $2b = a+c$ and

$$\frac{(b-a)^2+(c-b)^2+(c-a)^2}{3} = (a+b+c)^2.$$

Plugging $b = (a+c)/2$ into the last equation, we bring it to the equivalent form

$$2(c-a)^2 = 9(a+c)^2.$$

The conditions for equality can now be restated as

$$2b = a+c \qquad \text{and} \qquad (c-a)^2 = 18b^2.$$

Setting $b = 1$ yields $a = 1 - \frac{3}{2}\sqrt{2}$ and $c = 1 + \frac{3}{2}\sqrt{2}$. We see that $M = \frac{9}{32}\sqrt{2}$ is indeed the smallest constant satisfying the inequality, with equality for any triple (a, b, c) proportional to $\left(1 - \frac{3}{2}\sqrt{2}, 1, 1 + \frac{3}{2}\sqrt{2}\right)$, up to permutation.

Comment. With the notation $x = b-a$, $y = c-b$, $z = a-c$, $s = a+b+c$ and $r^2 = a^2+b^2+c^2$, the inequality (1) becomes just $|sxyz| \leq Mr^4$ (with suitable constraints on s and r). The original asymmetric inequality turns into a standard symmetric one; from this point on the solution can be completed in many ways. One can e.g. use the fact that, for fixed values of $\sum x$ and $\sum x^2$, the product xyz is a maximum/minimum only if some of x, y, z are equal, thus reducing one degree of freedom, etc.

As observed by the proposer, a specific attraction of the problem is that the maximum is attained at a point (a, b, c) with all coordinates distinct.

Combinatorics

C1. We have $n \geq 2$ lamps $L_1, \ldots, L_n$ in a row, each of them being either *on* or *off*. Every second we simultaneously modify the state of each lamp as follows:
— if the lamp L_i and its neighbours (only one neighbour for $i = 1$ or $i = n$, two neighbours for other i) are in the same state, then L_i is switched off;
— otherwise, L_i is switched on.
Initially all the lamps are off except the leftmost one which is on.

(a) Prove that there are infinitely many integers n for which all the lamps will eventually be off.

(b) Prove that there are infinitely many integers n for which the lamps will never be all off.

(France)

Solution. (a) Experiments with small n lead to the guess that every n of the form 2^k should be good. This is indeed the case, and more precisely: let A_k be the $2^k \times 2^k$ matrix whose rows represent the evolution of the system, with entries $0, 1$ (for *off* and *on* respectively). The top row shows the initial state $[1, 0, 0, \ldots, 0]$; the bottom row shows the state after $2^k - 1$ steps. The claim is that:

$$\text{\textit{The bottom row of } } A_k \text{ \textit{is} } \ [1, 1, 1, \ldots, 1].$$

This will of course suffice because one more move then produces $[0, 0, 0, \ldots, 0]$, as required.

The proof is by induction on k. The base $k = 1$ is obvious. Assume the claim to be true for a $k \geq 1$ and write the matrix A_{k+1} in the block form $\begin{pmatrix} A_k & O_k \\ B_k & C_k \end{pmatrix}$ with four $2^k \times 2^k$ matrices. After m steps, the last 1 in a row is at position $m + 1$. Therefore O_k is the zero matrix. According to the induction hypothesis, the bottom row of $[A_k \ O_k]$ is $[1, \ldots, 1, 0, \ldots, 0]$, with 2^k ones and 2^k zeros. The next row is thus

$$[\underbrace{0, \ldots, 0}_{2^k - 1}, 1, 1, \underbrace{0, \ldots, 0}_{2^k - 1}]$$

It is symmetric about its midpoint, and this symmetry is preserved in all subsequent rows because the procedure described in the problem statement is left/right symmetric. Thus B_k is the mirror image of C_k. In particular, the rightmost column of B_k is identical with the leftmost column of C_k.

Imagine the matrix C_k in isolation from the rest of A_{k+1}. Suppose it is subject to evolution as defined in the problem: the first (leftmost) term in a row depends only on the two first terms in the preceding row, according as they are equal or not. Now embed C_k again in A_k. The 'leftmost' terms in the rows of C_k now have neighbours on their left side—but these neighbours are their exact copies. Consequently the actual evolution within C_k is the same, whether or not C_k is considered as a piece of A_{k+1} or in isolation. And since the top row of C_k is $[1, 0, \ldots, 0]$, it follows that C_k is identical with A_k.

The bottom row of A_k is $[1, 1, \ldots, 1]$; the same is the bottom row of C_k, hence also of B_k, which mirrors C_k. So the bottom row of A_{k+1} consists of ones only and the induction is complete.

(b) There are many ways to produce an infinite sequence of those n for which the state $[0, 0, \ldots, 0]$ will never be achieved. As an example, consider $n = 2^k + 1$ (for $k \geq 1$). The evolution of the system can be represented by a matrix $\mathcal{A}$ of width $2^k + 1$ with infinitely many rows. The top 2^k rows form the matrix A_k discussed above, with one column of zeros attached at its right.

In the next row we then have the vector $[0, 0, \ldots, 0, 1, 1]$. But this is just the second row of $\mathcal{A}$ reversed. Subsequent rows will be mirror copies of the foregoing ones, starting from the second one. So the configuration $[1, 1, 0, \ldots, 0, 0]$, i.e. the second row of $\mathcal{A}$, will reappear. Further rows will periodically repeat this pattern and there will be no row of zeros.

C2. A diagonal of a regular 2006-gon is called *odd* if its endpoints divide the boundary into two parts, each composed of an odd number of sides. Sides are also regarded as odd diagonals.

Suppose the 2006-gon has been dissected into triangles by 2003 nonintersecting diagonals. Find the maximum possible number of isosceles triangles with two odd sides.

(Serbia)

Solution 1. Call an isosceles triangle *odd* if it has two odd sides. Suppose we are given a dissection as in the problem statement. A triangle in the dissection which is odd and isosceles will be called *iso-odd* for brevity.

Lemma. Let AB be one of dissecting diagonals and let $\mathcal{L}$ be the shorter part of the boundary of the 2006-gon with endpoints A, B. Suppose that $\mathcal{L}$ consists of n segments. Then the number of iso-odd triangles with vertices on $\mathcal{L}$ does not exceed $n/2$.

Proof. This is obvious for $n = 2$. Take n with $2 < n \leq 1003$ and assume the claim to be true for every $\mathcal{L}$ of length less than n. Let now $\mathcal{L}$ (endpoints A, B) consist of n segments. Let PQ be the longest diagonal which is a side of an iso-odd triangle PQS with all vertices on $\mathcal{L}$ (if there is no such triangle, there is nothing to prove). Every triangle whose vertices lie on $\mathcal{L}$ is obtuse or right-angled; thus S is the summit of PQS. We may assume that the five points A, P, S, Q, B lie on $\mathcal{L}$ in this order and partition $\mathcal{L}$ into four pieces $\mathcal{L}_{AP}$, $\mathcal{L}_{PS}$, $\mathcal{L}_{SQ}$, $\mathcal{L}_{QB}$ (the outer ones possibly reducing to a point).

By the definition of PQ, an iso-odd triangle cannot have vertices on both $\mathcal{L}_{AP}$ and $\mathcal{L}_{QB}$. Therefore every iso-odd triangle within $\mathcal{L}$ has all its vertices on just one of the four pieces. Applying to each of these pieces the induction hypothesis and adding the four inequalities we get that the number of iso-odd triangles within $\mathcal{L}$ other than PQS does not exceed $n/2$. And since each of $\mathcal{L}_{PS}$, $\mathcal{L}_{SQ}$ consists of an odd number of sides, the inequalities for these two pieces are actually strict, leaving a $1/2 + 1/2$ in excess. Hence the triangle PSQ is also covered by the estimate $n/2$. This concludes the induction step and proves the lemma. $\square$

The remaining part of the solution in fact repeats the argument from the above proof. Consider the longest dissecting diagonal XY. Let $\mathcal{L}_{XY}$ be the shorter of the two parts of the boundary with endpoints X, Y and let XYZ be the triangle in the dissection with vertex Z not on $\mathcal{L}_{XY}$. Notice that XYZ is acute or right-angled, otherwise one of the segments XZ, YZ would be longer than XY. Denoting by $\mathcal{L}_{XZ}$, $\mathcal{L}_{YZ}$ the two pieces defined by Z and applying the lemma to each of $\mathcal{L}_{XY}$, $\mathcal{L}_{XZ}$, $\mathcal{L}_{YZ}$ we infer that there are no more than 2006/2 iso-odd triangles in all, unless XYZ is one of them. But in that case XZ and YZ are odd diagonals and the corresponding inequalities are strict. This shows that also in this case the total number of iso-odd triangles in the dissection, including XYZ, is not greater than 1003.

This bound can be achieved. For this to happen, it just suffices to select a vertex of the 2006-gon and draw a broken line joining every second vertex, starting from the selected one. Since 2006 is even, the line closes. This already gives us the required 1003 iso-odd triangles. Then we can complete the triangulation in an arbitrary fashion.

Solution 2. Let the terms *odd triangle* and *iso-odd triangle* have the same meaning as in the first solution.

Let ABC be an iso-odd triangle, with AB and BC odd sides. This means that there are an odd number of sides of the 2006-gon between A and B and also between B and C. We say that these sides *belong* to the iso-odd triangle ABC.

At least one side in each of these groups does not belong to any other iso-odd triangle. This is so because any odd triangle whose vertices are among the points between A and B has two sides of equal length and therefore has an even number of sides belonging to it in total. Eliminating all sides belonging to any other iso-odd triangle in this area must therefore leave one side that belongs to no other iso-odd triangle. Let us *assign* these two sides (one in each group) to the triangle ABC.

To each iso-odd triangle we have thus assigned a pair of sides, with no two triangles sharing an assigned side. It follows that at most 1003 iso-odd triangles can appear in the dissection.

This value can be attained, as shows the example from the first solution.

C3. Let S be a finite set of points in the plane such that no three of them are on a line. For each convex polygon P whose vertices are in S, let $a(P)$ be the number of vertices of P, and let $b(P)$ be the number of points of S which are outside P. Prove that for every real number x

$$\sum_P x^{a(P)}(1-x)^{b(P)} = 1,$$

where the sum is taken over all convex polygons with vertices in S.

NB. A line segment, a point and the empty set are considered as convex polygons of 2, 1 and 0 vertices, respectively.

(Colombia)

Solution 1. For each convex polygon P whose vertices are in S, let $c(P)$ be the number of points of S which are inside P, so that $a(P)+b(P)+c(P)=n$, the total number of points in S. Denoting $1-x$ by y,

$$\sum_P x^{a(P)}y^{b(P)} = \sum_P x^{a(P)}y^{b(P)}(x+y)^{c(P)} = \sum_P \sum_{i=0}^{c(P)} \binom{c(P)}{i} x^{a(P)+i}y^{b(P)+c(P)-i}.$$

View this expression as a homogeneous polynomial of degree n in two independent variables x, y. In the expanded form, it is the sum of terms $x^r y^{n-r}$ $(0 \le r \le n)$ multiplied by some nonnegative integer coefficients.

For a fixed r, the coefficient of $x^r y^{n-r}$ represents the number of ways of choosing a convex polygon P and then choosing some of the points of S inside P so that the number of vertices of P and the number of chosen points inside P jointly add up to r.

This corresponds to just choosing an r-element subset of S. The correspondence is bijective because every set T of points from S splits in exactly one way into the union of two disjoint subsets, of which the first is the set of vertices of a convex polygon — namely, the convex hull of T — and the second consists of some points inside that polygon.

So the coefficient of $x^r y^{n-r}$ equals $\binom{n}{r}$. The desired result follows:

$$\sum_P x^{a(P)}y^{b(P)} = \sum_{r=0}^{n} \binom{n}{r} x^r y^{n-r} = (x+y)^n = 1.$$

Solution 2. Apply induction on the number n of points. The case $n = 0$ is trivial. Let $n > 0$ and assume the statement for less than n points. Take a set S of n points.

Let C be the set of vertices of the convex hull of S, let $m = |C|$.

Let $X \subset C$ be an arbitrary nonempty set. For any convex polygon P with vertices in the set $S \setminus X$, we have $b(P)$ points of S outside P. Excluding the points of X — all outside P — the set $S \setminus X$ contains exactly $b(P) - |X|$ of them. Writing $1 - x = y$, by the induction hypothesis

$$\sum_{P \subset S \setminus X} x^{a(P)} y^{b(P)-|X|} = 1$$

(where $P \subset S \setminus X$ means that the vertices of P belong to the set $S \setminus X$). Therefore

$$\sum_{P \subset S \setminus X} x^{a(P)} y^{b(P)} = y^{|X|}.$$

All convex polygons appear at least once, except the convex hull C itself. The convex hull adds x^m. We can use the inclusion-exclusion principle to compute the sum of the other terms:

$$\sum_{P \neq C} x^{a(P)} y^{b(P)} = \sum_{k=1}^{m} (-1)^{k-1} \sum_{|X|=k} \sum_{P \subset S \setminus X} x^{a(P)} y^{b(P)} = \sum_{k=1}^{m} (-1)^{k-1} \sum_{|X|=k} y^k$$

$$= \sum_{k=1}^{m} (-1)^{k-1} \binom{m}{k} y^k = -\big((1-y)^m - 1\big) = 1 - x^m$$

and then

$$\sum_{P} x^{a(P)} y^{b(P)} = \sum_{P=C} + \sum_{P \neq C} = x^m + (1 - x^m) = 1.$$

C4. A cake has the form of an $n \times n$ square composed of n^2 unit squares. Strawberries lie on some of the unit squares so that each row or column contains exactly one strawberry; call this arrangement $\mathcal{A}$.

Let $\mathcal{B}$ be another such arrangement. Suppose that every grid rectangle with one vertex at the top left corner of the cake contains no fewer strawberries of arrangement $\mathcal{B}$ than of arrangement $\mathcal{A}$. Prove that arrangement $\mathcal{B}$ can be obtained from $\mathcal{A}$ by performing a number of *switches*, defined as follows:

A *switch* consists in selecting a grid rectangle with only two strawberries, situated at its top right corner and bottom left corner, and moving these two strawberries to the other two corners of that rectangle.

(Taiwan)

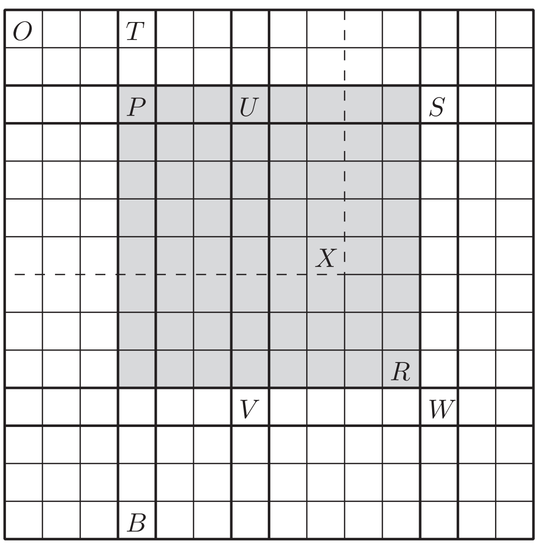
Solution. We use capital letters to denote unit squares; O is the top left corner square. For any two squares X and Y let $[XY]$ be the smallest grid rectangle containing these two squares. Strawberries lie on some squares in arrangement $\mathcal{A}$. Put a plum on each square of the target configuration $\mathcal{B}$. For a square X denote by $a(X)$ and $b(X)$ respectively the number of strawberries and the number of plums in $[OX]$. By hypothesis $a(X) \leq b(X)$ for each X, with strict inequality for some X (otherwise the two arrangements coincide and there is nothing to prove).

The idea is to show that by a legitimate switch one can obtain an arrangement $\mathcal{A}'$ such that

$$a(X) \leq a'(X) \leq b(X) \quad \text{for each } X; \qquad \sum_X a(X) < \sum_X a'(X) \tag{1}$$

(with $a'(X)$ defined analogously to $a(X)$; the sums range over all unit squares X). This will be enough because the same reasoning then applies to $\mathcal{A}'$, giving rise to a new arrangement $\mathcal{A}''$, and so on (induction). Since $\sum a(X) < \sum a'(X) < \sum a''(X) < \ldots$ and all these sums do not exceed $\sum b(X)$, we eventually obtain a sum with all summands equal to the respective $b(X)$s; all strawberries will meet with plums.

Consider the uppermost row in which the plum and the strawberry lie on different squares P and S (respectively); clearly P must be situated left to S. In the column passing through P, let T be the top square and B the bottom square. The strawberry in that column lies below the plum (because there is no plum in that column above P, and the positions of strawberries and plums coincide everywhere above the row of P). Hence there is at least one strawberry in the region $[BS]$ below $[PS]$. Let V be the position of the uppermost strawberry in that region.



Denote by W the square at the intersection of the row through V with the column through S and let R be the square vertex-adjacent to W up-left. We claim that

$$a(X) < b(X) \qquad \text{for all} \quad X \in [PR]. \tag{2}$$

This is so because if $X \in [PR]$ then the portion of $[OX]$ left to column $[TB]$ contains at least as many plums as strawberries (the hypothesis of the problem); in the portion above the row through P and S we have perfect balance; and in the remaining portion, i.e. rectangle $[PX]$ we have a plum on square P and no strawberry at all.

Now we are able to perform the required switch. Let U be the square at the intersection of the row through P with the column through V (some of P, U, R can coincide). We move strawberries from squares S and V to squares U and W. Then

$$a'(X) = a(X) + 1 \quad \text{for} \quad X \in [UR]; \qquad a'(X) = a(X) \quad \text{for other } X.$$

And since the rectangle $[UR]$ is contained in $[PR]$, we still have $a'(X) \leq b(X)$ for all S, in view of (2); conditions (1) are satisfied and the proof is complete.

C5. An (n, k)-tournament is a contest with n players held in k rounds such that:

(i) Each player plays in each round, and every two players meet at most once.

(ii) If player A meets player B in round i, player C meets player D in round i, and player A meets player C in round j, then player B meets player D in round j.

Determine all pairs (n, k) for which there exists an (n, k)-tournament.

(Argentina)

Solution. For each k, denote by t_k the unique integer such that $2^{t_k-1} < k+1 \leq 2^{t_k}$. We show that an (n, k)-tournament exists if and only if 2^{t_k} divides n.

First we prove that if $n = 2^t$ for some t then there is an (n, k)-tournament for all $k \leq 2^t - 1$. Let S be the set of $0-1$ sequences with length t. We label the 2^t players with the elements of S in an arbitrary fashion (which is possible as there are exactly 2^t sequences in S). Players are identified with their labels in the construction below. If $\alpha, \beta \in S$, let $\alpha + \beta \in S$ be the result of the modulo 2 term-by-term addition of α and β (with rules $0+0=0$, $0+1=1+0=1$, $1+1=0$; there is no carryover). For each $i = 1, \ldots, 2^t - 1$ let $\omega(i) \in S$ be the sequence of base 2 digits of i, completed with leading zeros if necessary to achieve length t.

Now define a tournament with $n = 2^t$ players in $k \leq 2^t - 1$ rounds as follows: For all $i = 1, \ldots, k$, let player α meet player $\alpha + \omega(i)$ in round i. The tournament is well-defined as $\alpha + \omega(i) \in S$ and $\alpha + \omega(i) = \beta + \omega(i)$ implies $\alpha = \beta$; also $[\alpha + \omega(i)] + \omega(i) = \alpha$ for each $\alpha \in S$ (meaning that player $\alpha + \omega(i)$ meets player α in round i, as needed). Each player plays in each round. Next, every two players meet at most once (exactly once if $k = 2^t - 1$), since $\omega(i) \neq \omega(j)$ if $i \neq j$. Thus condition (i) holds true, and condition (ii) is also easy to check.

Let player α meet player β in round i, player γ meet player δ in round i, and player α meet player γ in round j. Then $\beta = \alpha + \omega(i)$, $\delta = \gamma + \omega(i)$ and $\gamma = \alpha + \omega(j)$. By definition, β will play in round j with

$$\beta + \omega(j) = [\alpha + \omega(i)] + \omega(j) = [\alpha + \omega(j)] + \omega(i) = \gamma + \omega(i) = \delta,$$

as required by (ii).

So there exists an (n, k)-tournament for pairs (n, k) such that $n = 2^t$ and $k \leq 2^t - 1$. The same conclusion is straightforward for n of the form $n = 2^t s$ and $k \leq 2^t - 1$. Indeed, consider s different $(2^t, k)$-tournaments $T_1, \ldots, T_s$, no two of them having players in common. Their union can be regarded as a $(2^t s, k)$-tournament T where each round is the union of the respective rounds in $T_1, \ldots, T_s$.

In summary, the condition that 2^{t_k} divides n is sufficient for an (n, k)-tournament to exist. We prove that it is also necessary.

Consider an arbitrary (n, k)-tournament. Represent each player by a point and after each round, join by an edge every two players who played in this round. Thus to a round $i = 1, \ldots, k$ there corresponds a graph G_i. We say that player Q is an *i-neighbour* of player P if there is a path of edges in G_i from P to Q; in other words, if there are players $P = X_1, X_2, \ldots, X_m = Q$ such that player X_j meets player X_{j+1} in one of the first i rounds, $j = 1, 2 \ldots, m-1$. The set of i-neighbours of a player will be called its *i-component*. Clearly two i-components are either disjoint or coincide.

Hence after each round i the set of players is partitioned into pairwise disjoint i-components. So, to achieve our goal, it suffices to show that all k-components have size divisible by 2^{t_k}.

To this end, let us see how the i-component Γ of a player A changes after round $i+1$. Suppose that A meets player B with i-component Δ in round $i+1$ (components Γ and Δ are not necessarily distinct). We claim that then *in round* $i{+}1$ *each player from* Γ *meets a player from* Δ*, and vice versa.*

Indeed, let C be any player in Γ, and let C meet D in round $i{+}1$. Since C is an i-neighbour of A, there is a sequence of players $A = X_1, X_2, \ldots, X_m = C$ such that X_j meets X_{j+1} in one of the first i rounds, $j = 1, 2 \ldots, m{-}1$. Let X_j meet Y_j in round $i{+}1$, for $j = 1, 2 \ldots, m$; in particular $Y_1 = B$ and $Y_m = D$. Players Y_j exists in view of condition (i). Suppose that X_j and X_{j+1} met in round r, where $r \leq i$. Then condition (ii) implies that and Y_j and Y_{j+1} met in round r, too. Hence $B = Y_1, Y_2, \ldots, Y_m = D$ is a path in G_i from B to D. This is to say, D is in the i-component Δ of B, as claimed. By symmetry, each player from Δ meets a player from Γ in round $i{+}1$. It follows in particular that Γ and Δ have the same cardinality.

It is straightforward now that the $(i{+}1)$-component of A is $\Gamma \cup \Delta$, the union of two sets with the same size. Since Γ and Δ are either disjoint or coincide, we have either $|\Gamma \cup \Delta| = 2|\Gamma|$ or $|\Gamma \cup \Delta| = |\Gamma|$; as usual, $|\cdots|$ denotes the cardinality of a finite set.

Let $\Gamma_1, \ldots, \Gamma_k$ be the consecutive components of a given player A. We obtained that either $|\Gamma_{i+1}| = 2|\Gamma_i|$ or $|\Gamma_{i+1}| = |\Gamma_i|$ for $i = 1, \ldots, k{-}1$. Because $|\Gamma_1| = 2$, each $|\Gamma_i|$ is a power of 2, $i = 1, \ldots, k{-}1$. In particular $|\Gamma_k| = 2^u$ for some u.

On the other hand, player A has played with k different opponents by (i). All of them belong to Γ_k, therefore $|\Gamma_k| \geq k{+}1$.

Thus $2^u \geq k{+}1$, and since t_k is the least integer satisfying $2^{t_k} \geq k{+}1$, we conclude that $u \geq t_k$. So the size of each k-component is divisible by 2^{t_k}, which completes the argument.

C6. A *holey triangle* is an upward equilateral triangle of side length n with n upward unit triangular holes cut out. A *diamond* is a $60^\circ - 120^\circ$ unit rhombus. Prove that a holey triangle T can be tiled with diamonds if and only if the following condition holds: Every upward equilateral triangle of side length k in T contains at most k holes, for $1 \leq k \leq n$.

(Colombia)

Solution. Let T be a holey triangle. The unit triangles in it will be called *cells*. We say simply "triangle" instead of "upward equilateral triangle" and "size" instead of "side length."

The necessity will be proven first. Assume that a holey triangle T can be tiled with diamonds and consider such a tiling. Let T' be a triangle of size k in T containing h holes. Focus on the diamonds which cover (one or two) cells in T'. Let them form a figure R. The boundary of T' consists of upward cells, so R is a triangle of size k with h upward holes cut out and possibly some downward cells sticking out. Hence there are exactly $(k^2+k)/2 - h$ upward cells in R, and at least $(k^2-k)/2$ downward cells (not counting those sticking out). On the other hand each diamond covers one upward and one downward cell, which implies $(k^2+k)/2 - h \geq (k^2-k)/2$. It follows that $h \leq k$, as needed.

We pass on to the sufficiency. For brevity, let us say that a set of holes in a given triangle T is *spread out* if every triangle of size k in T contains at most k holes. For any set S of spread out holes, a triangle of size k will be called *full of S* if it contains exactly k holes of S. The proof is based on the following observation.

Lemma. Let S be a set of spread out holes in T. Suppose that two triangles T' and T'' are full of S, and that they touch or intersect. Let $T'+T''$ denote the smallest triangle in T containing them. Then $T'+T''$ is also full of S.

Proof. Let triangles T', T'', $T' \cap T''$ and $T'+T''$ have sizes a, b, c and d, and let them contain a, b, x and y holes of S, respectively. (Note that $T' \cap T''$ could be a point, in which case $c=0$.) Since S is spread out, we have $x \leq c$ and $y \leq d$. The geometric configuration of triangles clearly satisfies $a+b=c+d$. Furthermore, $a+b \leq x+y$, since $a+b$ counts twice the holes in $T' \cap T''$. These conclusions imply $x=c$ and $y=d$, as we wished to show. $\square$

Now let T_n be a holey triangle of size n, and let the set H of its holes be spread out. We show by induction on n that T_n can be tiled with diamonds. The base $n=1$ is trivial. Suppose that $n \geq 2$ and that the claim holds for holey triangles of size less than n.

Denote by B the bottom row of T_n and by T' the triangle formed by its top $n-1$ rows. There is at least one hole in B as T' contains at most $n-1$ holes. If this hole is only one, there is a unique way to tile B with diamonds. Also, T' contains exactly $n-1$ holes, making it a holey triangle of size $n-1$, and these holes are spread out. Hence it remains to apply the induction hypothesis.

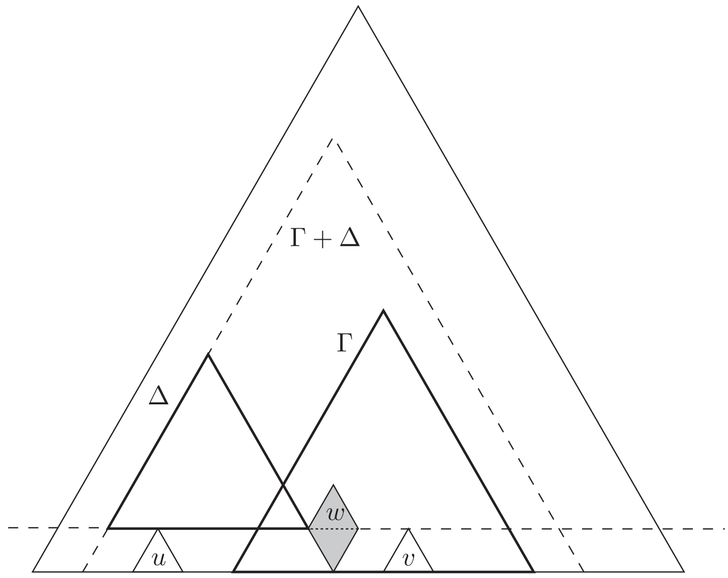
So suppose that there are $m \geq 2$ holes in B and label them $a_1, \ldots, a_m$ from left to right. Let ℓ be the line separating B from T'. For each $i = 1, \ldots, m-1$, pick an upward cell b_i between a_i and a_{i+1}, with base on ℓ. Place a diamond to cover b_i and its lower neighbour, a downward cell in B. The remaining part of B can be tiled uniquely with diamonds. Remove from T_n row B and the cells $b_1, \ldots, b_{m-1}$ to obtain a holey triangle T_{n-1} of size $n-1$. The conclusion will follow by induction if the choice of $b_1, \ldots, b_{m-1}$ guarantees that the following condition is satisfied: If the holes $a_1, \ldots, a_{m-1}$ are replaced by $b_1, \ldots, b_{m-1}$ then the new set of holes is spread out again.

We show that such a choice is possible. The cells $b_1, \ldots, b_{m-1}$ can be defined one at a time in this order, making sure that the above condition holds at each step. Thus it suffices to prove that there is an appropriate choice for b_1, and we set $a_1 = u$, $a_2 = v$ for clarity.

Let Δ be the triangle of maximum size which is full of H, contains the top vertex of the hole u, and has base on line ℓ. Call Δ the *associate of* u. Observe that Δ does not touch v. Indeed, if Δ has size r then it contains r holes of T_n. Extending its slanted sides downwards produces a triangle Δ' of size $r+1$ containing at least one more hole, namely u. Since there are at most $r+1$ holes in Δ', it cannot contain v. Consequently, Δ does not contain the top vertex of v.

Let w be the upward cell with base on ℓ which is to the right of Δ and shares a common vertex with it. The observation above shows that w is to the left of v. Note that w is not a hole, or else Δ could be extended to a larger triangle full of H.

We prove that if the hole u is replaced by w then the new set of holes is spread out again. To verify this, we only need to check that if a triangle Γ in T_n contains w but not u then Γ is not full of H. Suppose to the contrary that Γ is full of H. Consider the minimum triangle $\Gamma+\Delta$ containing Γ and the associate Δ of u. Clearly $\Gamma+\Delta$ is larger than Δ, because Γ contains w but Δ does not. Next, $\Gamma+\Delta$ is full of $H \setminus \{u\}$ by the lemma, since Γ and Δ have a common point and neither of them contains u.



If Γ is above line ℓ then so is $\Gamma+\Delta$, which contradicts the maximum choice of Δ. If Γ contains cells from row B, observe that $\Gamma+\Delta$ contains u. Let s be the size of $\Gamma+\Delta$. Being full of $H \setminus \{u\}$, $\Gamma+\Delta$ contains s holes other than u. But it also contains u, contradicting the assumption that H is spread out.

The claim follows, showing that $b_1 = w$ is an appropriate choice for $a_1 = u$ and $a_2 = v$. As explained above, this is enough to complete the induction.

C7. Consider a convex polyhedron without parallel edges and without an edge parallel to any face other than the two faces adjacent to it.

Call a pair of points of the polyhedron *antipodal* if there exist two parallel planes passing through these points and such that the polyhedron is contained between these planes.

Let A be the number of antipodal pairs of vertices, and let B be the number of antipodal pairs of midpoints of edges. Determine the difference $A-B$ in terms of the numbers of vertices, edges and faces.

(Japan)

Solution 1. Denote the polyhedron by Γ; let its vertices, edges and faces be $V_1, V_2, \ldots, V_n$, $E_1, E_2, \ldots, E_m$ and $F_1, F_2, \ldots, F_\ell$, respectively. Denote by Q_i the midpoint of edge E_i.

Let S be the unit sphere, the set of all unit vectors in three-dimensional space. Map the boundary elements of Γ to some objects on S as follows.

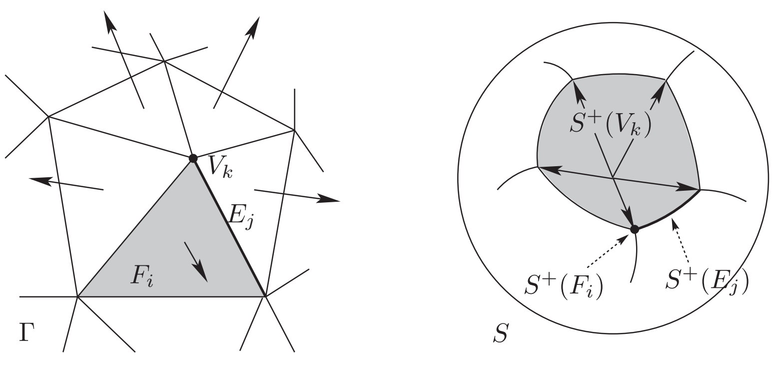
For a face F_i, let $S^+(F_i)$ and $S^-(F_i)$ be the unit normal vectors of face F_i, pointing outwards from Γ and inwards to Γ, respectively. These points are diametrically opposite.

For an edge E_j, with neighbouring faces F_{i_1} and F_{i_2}, take all support planes of Γ (planes which have a common point with Γ but do not intersect it) containing edge E_j, and let $S^+(E_j)$ be the set of their outward normal vectors. The set $S^+(E_j)$ is an arc of a great circle on S. Arc $S^+(E_j)$ is perpendicular to edge E_j and it connects points $S^+(F_{i_1})$ and $S^+(F_{i_2})$.

Define also the set of inward normal vectors $S^-(E_i)$ which is the reflection of $S^+(E_i)$ across the origin.

For a vertex V_k, which is the common endpoint of edges $E_{j_1}, \ldots, E_{j_h}$ and shared by faces $F_{i_1}, \ldots, F_{i_h}$, take all support planes of Γ through point V_k and let $S^+(V_k)$ be the set of their outward normal vectors. This is a region on S, a spherical polygon with vertices $S^+(F_{i_1}), \ldots, S^+(F_{i_h})$ bounded by arcs $S^+(E_{j_1}), \ldots, S^+(E_{j_h})$. Let $S^-(V_k)$ be the reflection of $S^+(V_k)$, the set of inward normal vectors.

Note that region $S^+(V_k)$ is convex in the sense that it is the intersection of several half spheres.



Now translate the conditions on Γ to the language of these objects.

(a) Polyhedron Γ has no parallel edges — the great circles of arcs $S^+(E_i)$ and $S^-(E_j)$ are different for all $i \neq j$.

(b) If an edge E_i does not belong to a face F_j then they are not parallel — the great circle which contains arcs $S^+(E_i)$ and $S^-(E_i)$ does not pass through points $S^+(F_j)$ and $S^-(F_j)$.

(c) Polyhedron Γ has no parallel faces — points $S^+(F_i)$ and $S^-(F_j)$ are pairwise distinct.

The regions $S^+(V_k)$, arcs $S^+(E_j)$ and points $S^+(F_i)$ provide a decomposition of the surface of the sphere. Regions $S^-(V_k)$, arcs $S^-(E_j)$ and points $S^-(F_i)$ provide the reflection of this decomposition. These decompositions are closely related to the problem.

Lemma 1. For any $1 \le i, j \le n$, regions $S^-(V_i)$ and $S^+(V_j)$ overlap if and only if vertices V_i and V_j are antipodal.

Lemma 2. For any $1 \le i, j \le m$, arcs $S^-(E_i)$ and $S^+(E_j)$ intersect if and only if the midpoints Q_i and Q_j of edges E_i and E_j are antipodal.

Proof of lemma 1. First note that by properties (a,b,c) above, the two regions cannot share only a single point or an arc. They are either disjoint or they overlap.

Assume that the two regions have a common interior point u. Let P_1 and P_2 be two parallel support planes of Γ through points V_i and V_j, respectively, with normal vector u. By the definition of regions $S^-(V_i)$ and $S^+(V_j)$, u is the inward normal vector of P_1 and the outward normal vector of P_2. Therefore polyhedron Γ lies between the two planes; vertices V_i and V_j are antipodal.

To prove the opposite direction, assume that V_i and V_j are antipodal. Then there exist two parallel support planes P_1 and P_2 through V_i and V_j, respectively, such that Γ is between them. Let u be the inward normal vector of P_1; then u is the outward normal vector of P_2, therefore $u \in S^-(V_i) \cap S^+(V_j)$. The two regions have a common point, so they overlap. □

Proof of lemma 2. Again, by properties (a,b) above, the endpoints of arc $S^-(E_i)$ cannot belong to $S^+(E_j)$ and vice versa. The two arcs are either disjoint or intersecting.

Assume that arcs $S^-(E_i)$ and $S^+(E_j)$ intersect at point u. Let P_1 and P_2 be the two support planes through edges E_i and E_j, respectively, with normal vector u. By the definition of arcs $S^-(E_i)$ and $S^+(E_j)$, vector u points inwards from P_1 and outwards from P_2. Therefore Γ is between the planes. Since planes P_1 and P_2 pass through Q_i and Q_j, these points are antipodal.

For the opposite direction, assume that points Q_i and Q_j are antipodal. Let P_1 and P_2 be two support planes through these points, respectively. An edge cannot intersect a support plane, therefore E_i and E_j lie in the planes P_1 and P_2, respectively. Let u be the inward normal vector of P_1, which is also the outward normal vector of P_2. Then $u \in S^-(E_i) \cap S^+(E_j)$. So the two arcs are not disjoint; they therefore intersect. □

Now create a new decomposition of sphere S. Draw all arcs $S^+(E_i)$ and $S^-(E_j)$ on sphere S and put a knot at each point where two arcs meet. We have ℓ knots at points $S^+(F_i)$ and another ℓ knots at points $S^-(F_i)$, corresponding to the faces of Γ; by property (c) they are different. We also have some pairs $1 \le i, j \le m$ where arcs $S^-(E_i)$ and $S^+(E_j)$ intersect. By Lemma 2, each antipodal pair (Q_i, Q_j) gives rise to two such intersections; hence, the number of all intersections is $2B$ and we have $2\ell + 2B$ knots in all.

Each intersection knot splits two arcs, increasing the number of arcs by 2. Since we started with $2m$ arcs, corresponding the edges of Γ, the number of the resulting curve segments is $2m + 4B$.

The network of these curve segments divides the sphere into some "new" regions. Each new region is the intersection of some overlapping sets $S^-(V_i)$ and $S^+(V_j)$. Due to the convexity, the intersection of two overlapping regions is convex and thus contiguous. By Lemma 1, each pair of overlapping regions corresponds to an antipodal vertex pair and each antipodal vertex pair gives rise to two different overlaps, which are symmetric with respect to the origin. So the number of new regions is $2A$.

The result now follows from Euler's polyhedron theorem. We have $n + l = m + 2$ and

$$(2\ell + 2B) + 2A = (2m + 4B) + 2,$$

therefore

$$A - B = m - \ell + 1 = n - 1.$$

Therefore $A - B$ is by one less than the number of vertices of Γ.

Solution 2. Use the same notations for the polyhedron and its vertices, edges and faces as in Solution 1. We regard points as vectors starting from the origin. Polyhedron Γ is regarded as a closed convex set, including its interior. In some cases the edges and faces of Γ are also regarded as sets of points. The symbol ∂ denotes the boundary of the certain set; e.g. $\partial\Gamma$ is the surface of Γ.

Let $\Delta = \Gamma - \Gamma = \{U - V : U, V \in \Gamma\}$ be the set of vectors between arbitrary points of Γ. Then Δ, being the sum of two bounded convex sets, is also a bounded convex set and, by construction, it is also centrally symmetric with respect to the origin. We will prove that Δ is also a polyhedron and express the numbers of its faces, edges and vertices in terms n, m, ℓ, A and B.

Lemma 1. For points $U, V \in \Gamma$, point $W = U - V$ is a boundary point of Δ if and only if U and V are antipodal. Moreover, for each boundary point $W \in \partial\Delta$ there exists exactly one pair of points $U, V \in \Gamma$ such that $W = U - V$.

Proof. Assume first that U and V are antipodal points of Γ. Let parallel support planes P_1 and P_2 pass through them such that Γ is in between. Consider plane $P = P_1 - U = P_2 - V$. This plane separates the interiors of $\Gamma - U$ and $\Gamma - V$. After reflecting one of the sets, e.g. $\Gamma - V$, the sets $\Gamma - U$ and $-\Gamma + V$ lie in the same half space bounded by P. Then $(\Gamma - U) + (-\Gamma + V) = \Delta - W$ lies in that half space, so $0 \in P$ is a boundary point of the set $\Delta - W$. Translating by W we obtain that W is a boundary point of Δ.

To prove the opposite direction, let $W = U - V$ be a boundary point of Δ, and let $\Psi = (\Gamma - U) \cap (\Gamma - V)$. We claim that $\Psi = \{0\}$. Clearly Ψ is a bounded convex set and $0 \in \Psi$. For any two points $X, Y \in \Psi$, we have $U + X, V + Y \in \Gamma$ and $W + (X - Y) = (U + X) - (V + Y) \in \Delta$. Since W is a boundary point of Δ, the vector $X - Y$ cannot have the same direction as W. This implies that the interior of Ψ is empty. Now suppose that Ψ contains a line segment S. Then $S + U$ and $S + V$ are subsets of some faces or edges of Γ and these faces/edges are parallel to S. In all cases, we find two faces, two edges, or a face and an edge which are parallel, contradicting the conditions of the problem. Therefore, $\Psi = \{0\}$ indeed.

Since $\Psi = (\Gamma - U) \cap (\Gamma - V)$ consists of a single point, the interiors of bodies $\Gamma - U$ and $\Gamma - V$ are disjoint and there exists a plane P which separates them. Let u be the normal vector of P pointing into that half space bounded by P which contains $\Gamma - U$. Consider the planes $P + U$ and $P + V$; they are support planes of Γ, passing through U and V, respectively. From plane $P + U$, the vector u points into that half space which contains Γ. From plane $P + V$, vector u points into the opposite half space containing Γ. Therefore, we found two proper support through points U and V such that Γ is in between.

For the uniqueness part, assume that there exist points $U_1, V_1 \in \Gamma$ such that $U_1 - V_1 = U - V$. The points $U_1 - U$ and $V_1 - V$ lie in the sets $\Gamma - U$ and $\Gamma - V$ separated by P. Since $U_1 - U = V_1 - V$, this can happen only if both are in P; but the only such point is 0. Therefore, $U_1 - V_1 = U - V$ implies $U_1 = U$ and $V_1 = V$. The lemma is complete. $\square$

Lemma 2. Let U and V be two antipodal points and assume that plane P, passing through 0, separates the interiors of $\Gamma - U$ and $\Gamma - V$. Let $\Psi_1 = (\Gamma - U) \cap P$ and $\Psi_2 = (\Gamma - V) \cap P$. Then $\Delta \cap (P + U - V) = \Psi_1 - \Psi_2 + U - V$.

Proof. The sets $\Gamma - U$ and $-\Gamma + V$ lie in the same closed half space bounded by P. Therefore, for any points $X \in (\Gamma - U)$ and $Y \in (-\Gamma + V)$, we have $X + Y \in P$ if and only if $X, Y \in P$. Then

$$(\Delta - (U - V)) \cap P = \big((\Gamma - U) + (-\Gamma + V)\big) \cap P = \big((\Gamma - U) \cap P\big) + \big((-\Gamma + V) \cap P\big) = \Psi_1 - \Psi_2.$$

Now a translation by $(U - V)$ completes the lemma. $\square$

Now classify the boundary points $W = U - V$ of Δ, according to the types of points U and V. In all cases we choose a plane P through 0 which separates the interiors of $\Gamma - U$ and $\Gamma - V$. We will use the notation $\Psi_1 = (\Gamma - U) \cap P$ and $\Psi_2 = (\Gamma - V) \cap P$ as well.

Case 1: Both U and V are vertices of Γ. Bodies $\Gamma - U$ and $\Gamma - V$ have a common vertex which is 0. Choose plane P in such a way that $\Psi_1 = \Psi_2 = \{0\}$. Then Lemma 2 yields $\Delta \cap (P + W) = \{W\}$. Therefore $P + W$ is a support plane of Δ such that they have only one common point so no line segment exists on $\partial\Delta$ which would contain W in its interior.

Since this case occurs for antipodal vertex pairs and each pair is counted twice, the number of such boundary points on Δ is $2A$.

Case 2: Point U is an interior point of an edge E_i and V is a vertex of Γ. Choose plane P such that $\Psi_1 = E_i - U$ and $\Psi_2 = \{0\}$. By Lemma 2, $\Delta \cap (P + W) = E_i - V$. Hence there exists a line segment in $\partial\Delta$ which contains W in its interior, but there is no planar region in $\partial\Delta$ with the same property.

We obtain a similar result if V belongs to an edge of Γ and U is a vertex.

Case 3: Points U and V are interior points of edges E_i and E_j, respectively. Let P be the plane of $E_i - U$ and $E_j - V$. Then $\Psi_1 = E_i - U$, $\Psi_2 = E_j - V$ and $\Delta \cap (P + W) = E_i - E_j$. Therefore point W belongs to a parallelogram face on $\partial\Delta$.

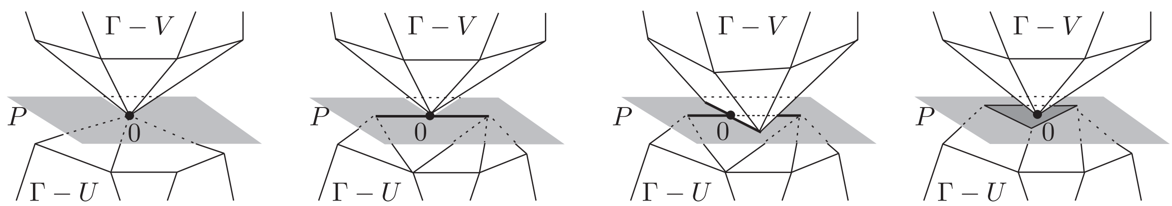
The centre of the parallelogram is $Q_i - Q_j$, the vector between the midpoints. Therefore an edge pair (E_i, E_j) occurs if and only if Q_i and Q_j are antipodal which happens $2B$ times.

Case 4: Point U lies in the interior of a face F_i and V is a vertex of Γ. The only choice for P is the plane of $F_i - U$. Then we have $\Psi_1 = F_i - U$, $\Psi_2 = \{0\}$ and $\Delta \cap (P + W) = F_i - V$. This is a planar face of $\partial\Delta$ which is congruent to F_i.

For each face F_i, the only possible vertex V is the farthest one from the plane of F_i.

If U is a vertex and V belongs to face F_i then we obtain the same way that W belongs to a face $-F_i + U$ which is also congruent to F_i. Therefore, each face of Γ has two copies on $\partial\Delta$, a translated and a reflected copy.

Case 5: Point U belongs to a face F_i of Γ and point V belongs to an edge or a face G. In this case objects F_i and G must be parallel which is not allowed.



Now all points in $\partial\Delta$ belong to some planar polygons (cases 3 and 4), finitely many line segments (case 2) and points (case 1). Therefore Δ is indeed a polyhedron. Now compute the numbers of its vertices, edges and faces.

The vertices are obtained in case 1, their number is $2A$.

Faces are obtained in cases 3 and 4. Case 3 generates $2B$ parallelogram faces. Case 4 generates 2ℓ faces.

We compute the number of edges of Δ from the degrees (number of sides) of faces of Γ. Let d_i be the the degree of face F_i. The sum of degrees is twice as much as the number of edges, so $d_1+d_2+\ldots+d_l = 2m$. The sum of degrees of faces of Δ is $2B\cdot 4+2(d_1+d_2+\cdots+d_l) = 8B+4m$, so the number of edges on Δ is $4B + 2m$.

Applying Euler's polyhedron theorem on Γ and Δ, we have $n+l = m+2$ and $2A+(2B+2\ell) = (4B + 2m) + 2$. Then the conclusion follows:

$$A - B = m - \ell + 1 = n - 1.$$

Geometry

G1. Let ABC be a triangle with incentre I. A point P in the interior of the triangle satisfies

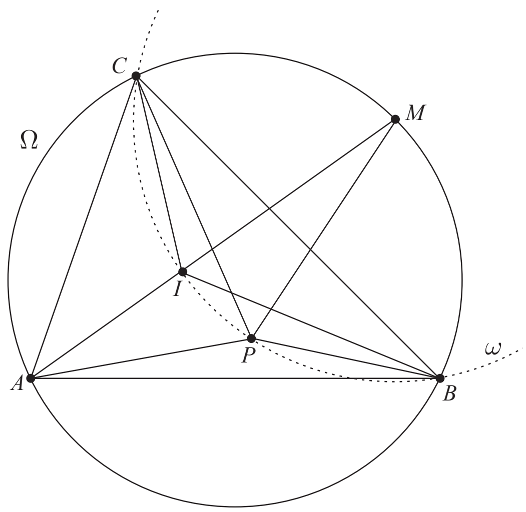
$$\angle PBA + \angle PCA = \angle PBC + \angle PCB.$$

Show that $AP \geq AI$ and that equality holds if and only if P coincides with I.

(Korea)

Solution. Let $\angle A = \alpha$, $\angle B = \beta$, $\angle C = \gamma$. Since $\angle PBA + \angle PCA + \angle PBC + \angle PCB = \beta + \gamma$, the condition from the problem statement is equivalent to $\angle PBC + \angle PCB = (\beta + \gamma)/2$, i. e. $\angle BPC = 90^\circ + \alpha/2$.

On the other hand $\angle BIC = 180^\circ - (\beta + \gamma)/2 = 90^\circ + \alpha/2$. Hence $\angle BPC = \angle BIC$, and since P and I are on the same side of BC, the points B, C, I and P are concyclic. In other words, P lies on the circumcircle ω of triangle BCI.



Let Ω be the circumcircle of triangle ABC. It is a well-known fact that the centre of ω is the midpoint M of the arc BC of Ω. This is also the point where the angle bisector AI intersects Ω.

From triangle APM we have

$$AP + PM \geq AM = AI + IM = AI + PM.$$

Therefore $AP \geq AI$. Equality holds if and only if P lies on the line segment AI, which occurs if and only if $P = I$.

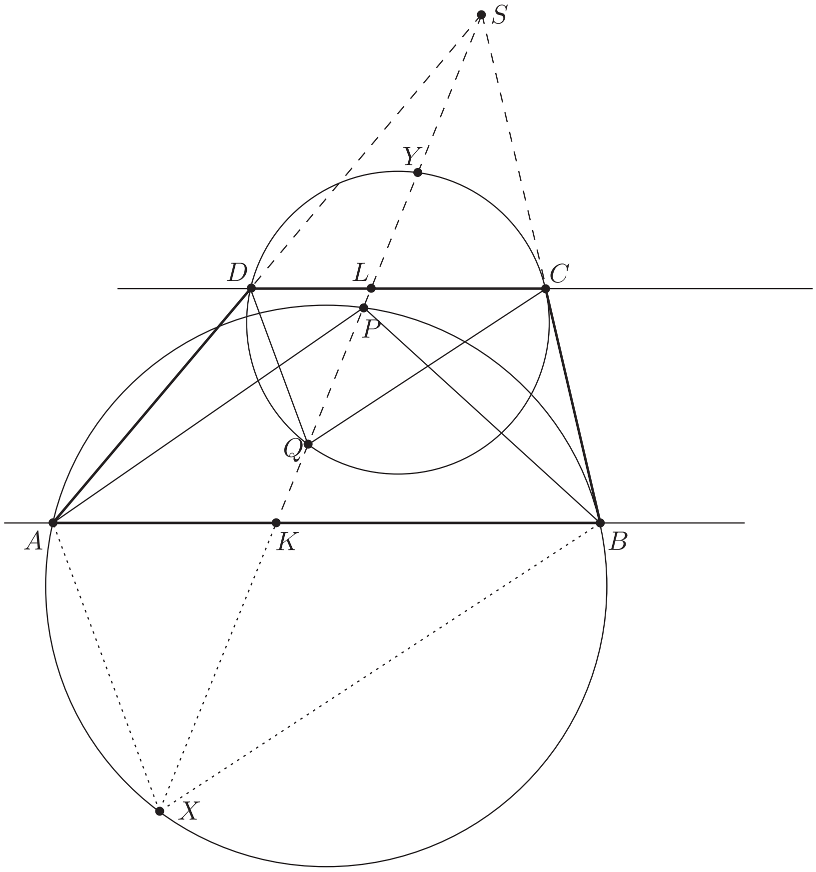
G2. Let $ABCD$ be a trapezoid with parallel sides $AB > CD$. Points K and L lie on the line segments AB and CD, respectively, so that $AK/KB = DL/LC$. Suppose that there are points P and Q on the line segment KL satisfying

$$\angle APB = \angle BCD \qquad \text{and} \qquad \angle CQD = \angle ABC.$$

Prove that the points P, Q, B and C are concyclic.

(Ukraine)

Solution 1. Because $AB \parallel CD$, the relation $AK/KB = DL/LC$ readily implies that the lines AD, BC and KL have a common point S.



Consider the second intersection points X and Y of the line SK with the circles (ABP) and (CDQ), respectively. Since $APBX$ is a cyclic quadrilateral and $AB \parallel CD$, one has

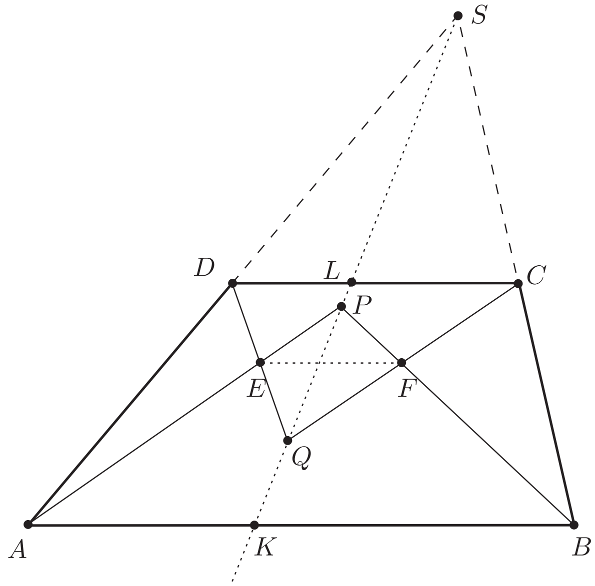
$$\angle AXB = 180^\circ - \angle APB = 180^\circ - \angle BCD = \angle ABC.$$

This shows that BC is tangent to the circle (ABP) at B. Likewise, BC is tangent to the circle (CDQ) at C. Therefore $SP \cdot SX = SB^2$ and $SQ \cdot SY = SC^2$.

Let h be the homothety with centre S and ratio SC/SB. Since $h(B) = C$, the above conclusion about tangency implies that h takes circle (ABP) to circle (CDQ). Also, h takes AB to CD, and it easily follows that $h(P) = Y$, $h(X) = Q$, yielding $SP/SY = SB/SC = SX/SQ$.

Equalities $SP \cdot SX = SB^2$ and $SQ/SX = SC/SB$ imply $SP \cdot SQ = SB \cdot SC$, which is equivalent to P, Q, B and C being concyclic.

Solution 2. The case where $P = Q$ is trivial. Thus assume that P and Q are two distinct points. As in the first solution, notice that the lines AD, BC and KL concur at a point S.



Let the lines AP and DQ meet at E, and let BP and CQ meet at F. Then $\angle EPF = \angle BCD$ and $\angle FQE = \angle ABC$ by the condition of the problem. Since the angles BCD and ABC add up to $180°$, it follows that $PEQF$ is a cyclic quadrilateral.

Applying Menelaus' theorem, first to triangle ASP and line DQ and then to triangle BSP and line CQ, we have

$$\frac{AD}{DS} \cdot \frac{SQ}{QP} \cdot \frac{PE}{EA} = 1 \quad \text{and} \quad \frac{BC}{CS} \cdot \frac{SQ}{QP} \cdot \frac{PF}{FB} = 1.$$

The first factors in these equations are equal, as $AB \| CD$. Thus the last factors are also equal, which implies that EF is parallel to AB and CD. Using this and the cyclicity of $PEQF$, we obtain

$$\angle BCD = \angle BCF + \angle FCD = \angle BCQ + \angle EFQ = \angle BCQ + \angle EPQ.$$

On the other hand,

$$\angle BCD = \angle APB = \angle EPF = \angle EPQ + \angle QPF,$$

and consequently $\angle BCQ = \angle QPF$. The latter angle either coincides with $\angle QPB$ or is supplementary to $\angle QPB$, depending on whether Q lies between K and P or not. In either case it follows that P, Q, B and C are concyclic.

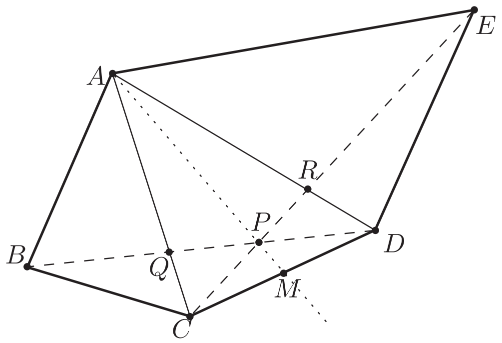
G3. Let $ABCDE$ be a convex pentagon such that

$$\angle BAC = \angle CAD = \angle DAE \qquad \text{and} \qquad \angle ABC = \angle ACD = \angle ADE.$$

The diagonals BD and CE meet at P. Prove that the line AP bisects the side CD.

(USA)

Solution. Let the diagonals AC and BD meet at Q, the diagonals AD and CE meet at R, and let the ray AP meet the side CD at M. We want to prove that $CM = MD$ holds.



The idea is to show that Q and R divide AC and AD in the same ratio, or more precisely

$$\frac{AQ}{QC} = \frac{AR}{RD} \tag{1}$$

(which is equivalent to $QR \| CD$). The given angle equalities imply that the triangles ABC, ACD and ADE are similar. We therefore have

$$\frac{AB}{AC} = \frac{AC}{AD} = \frac{AD}{AE}.$$

Since $\angle BAD = \angle BAC + \angle CAD = \angle CAD + \angle DAE = \angle CAE$, it follows from $AB/AC = AD/AE$ that the triangles ABD and ACE are also similar. Their angle bisectors in A are AQ and AR, respectively, so that

$$\frac{AB}{AC} = \frac{AQ}{AR}.$$

Because $AB/AC = AC/AD$, we obtain $AQ/AR = AC/AD$, which is equivalent to (1). Now Ceva's theorem for the triangle ACD yields

$$\frac{AQ}{QC} \cdot \frac{CM}{MD} \cdot \frac{DR}{RA} = 1.$$

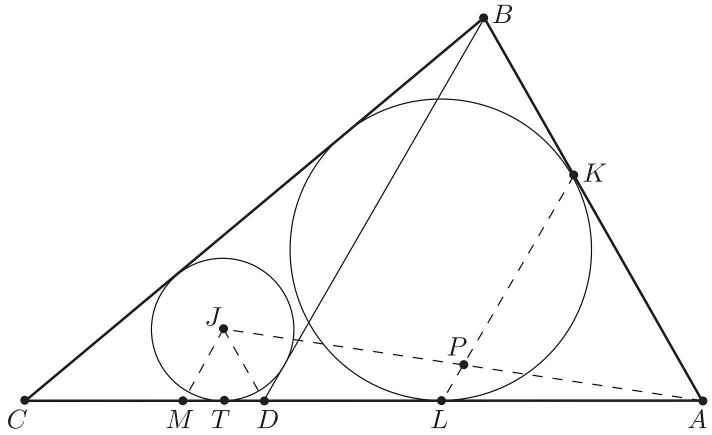
In view of (1), this reduces to $CM = MD$, which completes the proof.

Comment. Relation (1) immediately follows from the fact that quadrilaterals $ABCD$ and $ACDE$ are similar.

G4. A point D is chosen on the side AC of a triangle ABC with $\angle C < \angle A < 90^\circ$ in such a way that $BD = BA$. The incircle of ABC is tangent to AB and AC at points K and L, respectively. Let J be the incentre of triangle BCD. Prove that the line KL intersects the line segment AJ at its midpoint.

(Russia)

Solution. Denote by P be the common point of AJ and KL. Let the parallel to KL through J meet AC at M. Then P is the midpoint of AJ if and only if $AM = 2 \cdot AL$, which we are about to show.



Denoting $\angle BAC = 2\alpha$, the equalities $BA = BD$ and $AK = AL$ imply $\angle ADB = 2\alpha$ and $\angle ALK = 90^\circ - \alpha$. Since DJ bisects $\angle BDC$, we obtain $\angle CDJ = \frac{1}{2} \cdot (180^\circ - \angle ADB) = 90^\circ - \alpha$. Also $\angle DMJ = \angle ALK = 90^\circ - \alpha$ since $JM \| KL$. It follows that $JD = JM$.

Let the incircle of triangle BCD touch its side CD at T. Then $JT \perp CD$, meaning that JT is the altitude to the base DM of the isosceles triangle DMJ. It now follows that $DT = MT$, and we have

$$DM = 2 \cdot DT = BD + CD - BC.$$

Therefore

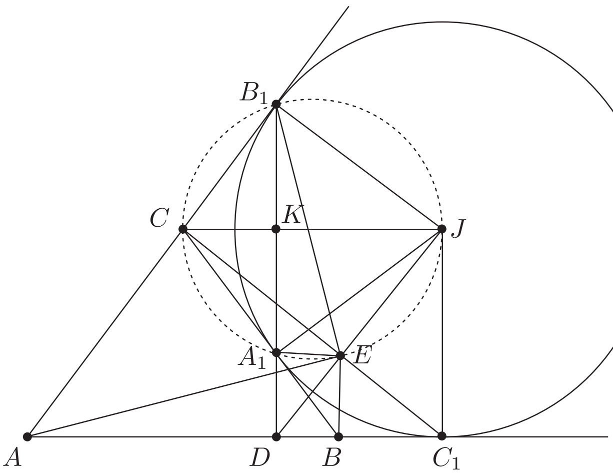
$$\begin{aligned} AM &= AD + (BD + CD - BC) \\ &= AD + AB + DC - BC \\ &= AC + AB - BC \\ &= 2 \cdot AL, \end{aligned}$$

which completes the proof.

G5. In triangle ABC, let J be the centre of the excircle tangent to side BC at A_1 and to the extensions of sides AC and AB at B_1 and C_1, respectively. Suppose that the lines A_1B_1 and AB are perpendicular and intersect at D. Let E be the foot of the perpendicular from C_1 to line DJ. Determine the angles $\angle BEA_1$ and $\angle AEB_1$.

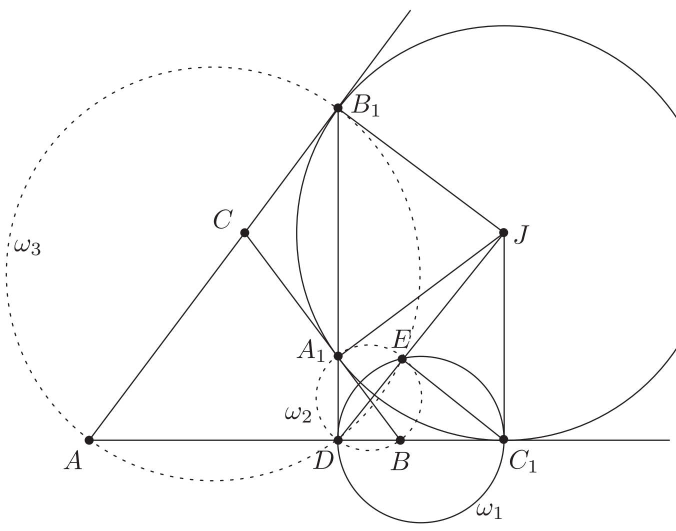
(Greece)

Solution 1. Let K be the intersection point of lines JC and A_1B_1. Obviously $JC \perp A_1B_1$ and since $A_1B_1 \perp AB$, the lines JK and C_1D are parallel and equal. From the right triangle B_1CJ we obtain $JC_1^2 = JB_1^2 = JC \cdot JK = JC \cdot C_1D$ from which we infer that $DC_1/C_1J = C_1J/JC$ and the right triangles DC_1J and C_1JC are similar. Hence $\angle C_1DJ = \angle JC_1C$, which implies that the lines DJ and C_1C are perpendicular, i.e. the points C_1, E, C are collinear.



Since $\angle CA_1J = \angle CB_1J = \angle CEJ = 90°$, points A_1, B_1 and E lie on the circle of diameter CJ. Then $\angle DBA_1 = \angle A_1CJ = \angle DEA_1$, which implies that quadrilateral BEA_1D is cyclic; therefore $\angle A_1EB = 90°$.

Quadrilateral $ADEB_1$ is also cyclic because $\angle EB_1A = \angle EJC = \angle EDC_1$, therefore we obtain $\angle AEB_1 = \angle ADB = 90°$.



Solution 2. Consider the circles ω_1, ω_2 and ω_3 of diameters C_1D, A_1B and AB_1, respectively. Line segments JC_1, JB_1 and JA_1 are tangents to those circles and, due to the right angle at D, ω_2 and ω_3 pass through point D. Since $\angle C_1ED$ is a right angle, point E lies on circle ω_1, therefore

$$JC_1^2 = JD \cdot JE.$$

Since $JA_1 = JB_1 = JC_1$ are all radii of the excircle, we also have

$$JA_1^2 = JD \cdot JE \qquad \text{and} \qquad JB_1^2 = JD \cdot JE.$$

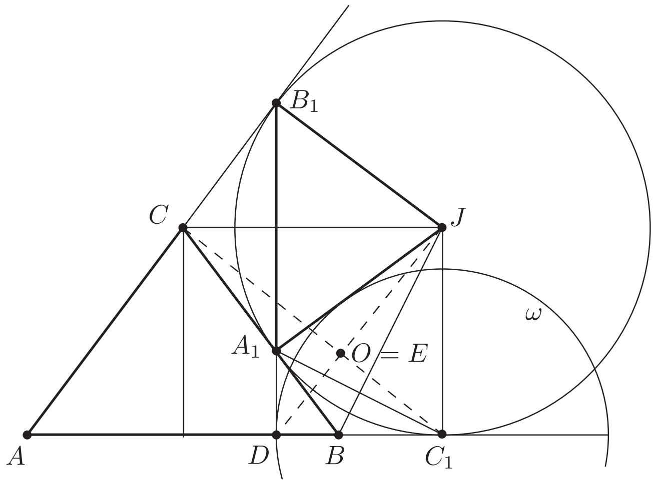
These equalities show that E lies on circles ω_2 and ω_3 as well, so $\angle BEA_1 = \angle AEB_1 = 90°$.

Solution 3. First note that A_1B_1 is perpendicular to the external angle bisector CJ of $\angle BCA$ and parallel to the internal angle bisector of that angle. Therefore, A_1B_1 is perpendicular to AB if and only if triangle ABC is isosceles, $AC = BC$. In that case the external bisector CJ is parallel to AB.

Triangles ABC and B_1A_1J are similar, as their corresponding sides are perpendicular. In particular, we have $\angle DA_1J = \angle C_1BA_1$; moreover, from cyclic deltoid JA_1BC_1,

$$\angle C_1A_1J = \angle C_1BJ = \frac{1}{2}\angle C_1BA_1 = \frac{1}{2}\angle DA_1J.$$

Therefore, A_1C_1 bisects angle $\angle DA_1J$.



In triangle B_1A_1J, line JC_1 is the external bisector at vertex J. The point C_1 is the intersection of two external angle bisectors (at A_1 and J) so C_1 is the centre of the excircle ω, tangent to side A_1J, and to the extension of B_1A_1 at point D.

Now consider the similarity transform φ which moves B_1 to A, A_1 to B and J to C. This similarity can be decomposed into a rotation by 90° around a certain point O and a homothety from the same centre. This similarity moves point C_1 (the centre of excircle ω) to J and moves D (the point of tangency) to C_1.

Since the rotation angle is 90°, we have $\angle XO\varphi(X) = 90°$ for an arbitrary point $X \neq O$. For $X = D$ and $X = C_1$ we obtain $\angle DOC_1 = \angle C_1OJ = 90°$. Therefore O lies on line segment DJ and C_1O is perpendicular to DJ. This means that $O = E$.

For $X = A_1$ and $X = B_1$ we obtain $\angle A_1OB = \angle B_1OA = 90°$, i.e.

$$\angle BEA_1 = \angle AEB_1 = 90°.$$

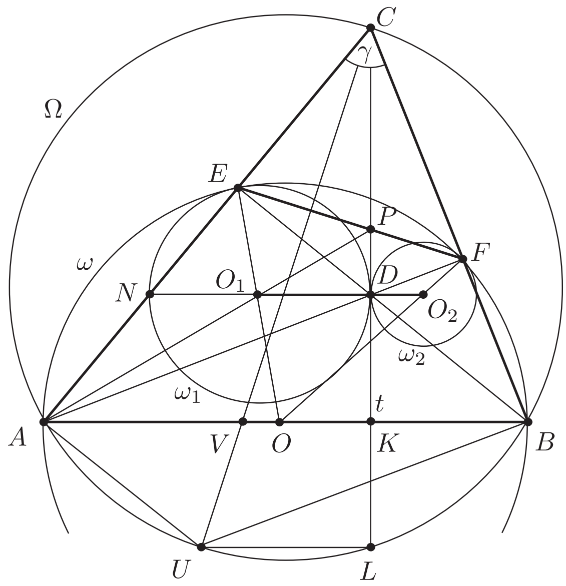
Comment. Choosing $X = J$, it also follows that $\angle JEC = 90°$ which proves that lines DJ and CC_1 intersect at point E. However, this is true more generally, without the assumption that A_1B_1 and AB are perpendicular, because points C and D are conjugates with respect to the excircle. The last observation could replace the first paragraph of Solution 1.

G6. Circles ω_1 and ω_2 with centres O_1 and O_2 are externally tangent at point D and internally tangent to a circle ω at points E and F, respectively. Line t is the common tangent of ω_1 and ω_2 at D. Let AB be the diameter of ω perpendicular to t, so that A, E and O_1 are on the same side of t. Prove that lines AO_1, BO_2, EF and t are concurrent.

(Brasil)

Solution 1. Point E is the centre of a homothety h which takes circle ω_1 to circle ω. The radii O_1D and OB of these circles are parallel as both are perpendicular to line t. Also, O_1D and OB are on the same side of line EO, hence h takes O_1D to OB. Consequently, points E, D and B are collinear. Likewise, points F, D and A are collinear as well.

Let lines AE and BF intersect at C. Since AF and BE are altitudes in triangle ABC, their common point D is the orthocentre of this triangle. So CD is perpendicular to AB, implying that C lies on line t. Note that triangle ABC is acute-angled. We mention the well-known fact that triangles FEC and ABC are similar in ratio $\cos\gamma$, where $\gamma = \angle ACB$. In addition, points C, E, D and F lie on the circle with diameter CD.



Let P be the common point of lines EF and t. We are going to prove that P lies on line AO_1. Denote by N the second common point of circle ω_1 and AC; this is the point of ω_1 diametrically opposite to D. By Menelaus' theorem for triangle DCN, points A, O_1 and P are collinear if and only if

$$\frac{CA}{AN} \cdot \frac{NO_1}{O_1D} \cdot \frac{DP}{PC} = 1.$$

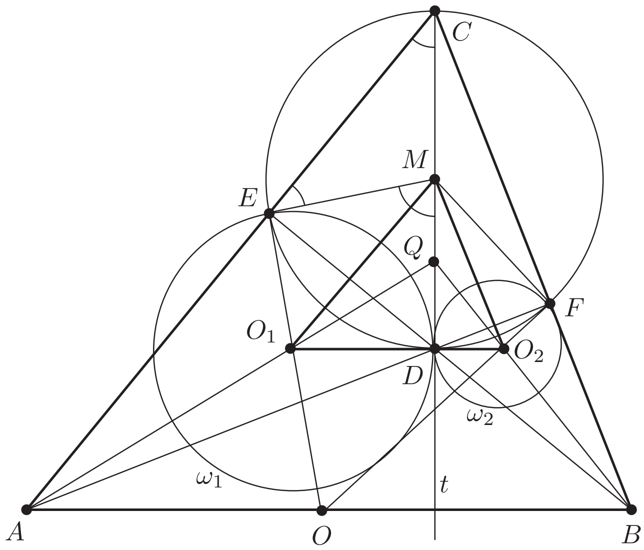
Because $NO_1 = O_1D$, this reduces to $CA/AN = CP/PD$. Let line t meet AB at K. Then $CA/AN = CK/KD$, so it suffices to show that

$$\frac{CP}{PD} = \frac{CK}{KD}. \tag{1}$$

To verify (1), consider the circumcircle Ω of triangle ABC. Draw its diameter CU through C, and let CU meet AB at V. Extend CK to meet Ω at L. Since AB is parallel to UL, we have $\angle ACU = \angle BCL$. On the other hand $\angle EFC = \angle BAC$, $\angle FEC = \angle ABC$ and $EF/AB = \cos\gamma$, as stated above. So reflection in the bisector of $\angle ACB$ followed by a homothety with centre C and ratio $1/\cos\gamma$ takes triangle FEC to triangle ABC. Consequently, this transformation takes CD to CU, which implies $CP/PD = CV/VU$. Next, we have $KL = KD$, because D is the orthocentre of triangle ABC. Hence $CK/KD = CK/KL$. Finally, $CV/VU = CK/KL$ because AB is parallel to UL. Relation (1) follows, proving that P lies on line AO_1. By symmetry, P also lies on line AO_2 which completes the solution.

Solution 2. We proceed as in the first solution to define a triangle ABC with orthocentre D, in which AF and BE are altitudes.

Denote by M the midpoint of CD. The quadrilateral $CEDF$ is inscribed in a circle with centre M, hence $MC = ME = MD = MF$.



Consider triangles ABC and O_1O_2M. Lines O_1O_2 and AB are parallel, both of them being perpendicular to line t. Next, MO_1 is the line of centres of circles (CEF) and ω_1 whose common chord is DE. Hence MO_1 bisects $\angle DME$ which is the external angle at M in the isosceles triangle CEM. It follows that $\angle DMO_1 = \angle DCA$, so that MO_1 is parallel to AC. Likewise, MO_2 is parallel to BC.

Thus the respective sides of triangles ABC and O_1O_2M are parallel; in addition, these triangles are not congruent. Hence there is a homothety taking ABC to O_1O_2M. The lines AO_1, BO_2 and $CM = t$ are concurrent at the centre Q of this homothety.

Finally, apply Pappus' theorem to the triples of collinear points A, O, B and O_2, D, O_1. The theorem implies that the points $AD \cap OO_2 = F$, $AO_1 \cap BO_2 = Q$ and $OO_1 \cap BD = E$ are collinear. In other words, line EF passes through the common point Q of AO_1, BO_2 and t.

Comment. Relation (1) from Solution 1 expresses the well-known fact that points P and K are harmonic conjugates with respect to points C and D. It is also easy to justify it by direct computation. Denoting $\angle CAB = \alpha$, $\angle ABC = \beta$, it is straightforward to obtain $CP/PD = CK/KD = \tan\alpha\tan\beta$.

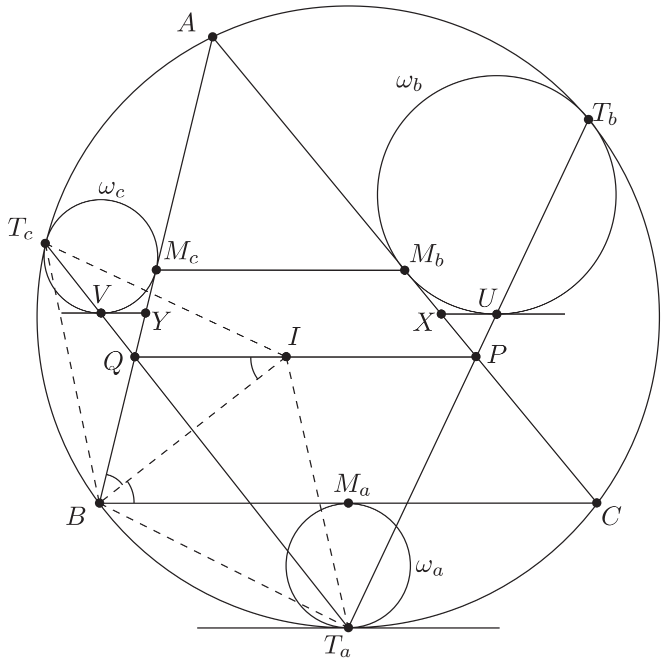
G7. In a triangle ABC, let M_a, M_b, M_c be respectively the midpoints of the sides BC, CA, AB and T_a, T_b, T_c be the midpoints of the arcs BC, CA, AB of the circumcircle of ABC, not containing the opposite vertices. For $i \in \{a, b, c\}$, let ω_i be the circle with M_iT_i as diameter. Let p_i be the common external tangent to ω_j, ω_k ($\{i, j, k\} = \{a, b, c\}$) such that ω_i lies on the opposite side of p_i than ω_j, ω_k do. Prove that the lines p_a, p_b, p_c form a triangle similar to ABC and find the ratio of similitude.

(Slovakia)

Solution. Let T_aT_b intersect circle ω_b at T_b and U, and let T_aT_c intersect circle ω_c at T_c and V. Further, let UX be the tangent to ω_b at U, with X on AC, and let VY be the tangent to ω_c at V, with Y on AB. The homothety with centre T_b and ratio T_bT_a/T_bU maps the circle ω_b onto the circumcircle of ABC and the line UX onto the line tangent to the circumcircle at T_a, which is parallel to BC; thus $UX \| BC$. The same is true of VY, so that $UX \| BC \| VY$.

Let T_aT_b cut AC at P and let T_aT_c cut AB at Q. The point X lies on the hypotenuse PM_b of the right triangle PUM_b and is equidistant from U and M_b. So X is the midpoint of M_bP. Similarly Y is the midpoint of M_cQ.

Denote the incentre of triangle ABC as usual by I. It is a known fact that $T_aI = T_aB$ and $T_cI = T_cB$. Therefore the points B and I are symmetric across T_aT_c, and consequently $\angle QIB = \angle QBI = \angle IBC$. This implies that BC is parallel to the line IQ, and likewise, to IP. In other words, PQ is the line parallel to BC passing through I.



Clearly $M_bM_c \| BC$. So PM_bM_cQ is a trapezoid and the segment XY connects the midpoints of its nonparallel sides; hence $XY \| BC$. This combined with the previously established relations $UX \| BC \| VY$ shows that all the four points U, X, Y, V lie on a line which is the common tangent to circles ω_b, ω_c. Since it leaves these two circles on one side and the circle ω_a on the other, this line is just the line p_a from the problem statement.

Line p_a runs midway between I and M_bM_c. Analogous conclusions hold for the lines p_b and p_c. So these three lines form a triangle homothetic from centre I to triangle $M_aM_bM_c$ in ratio $1/2$, hence similar to ABC in ratio $1/4$.

G8. Let $ABCD$ be a convex quadrilateral. A circle passing through the points A and D and a circle passing through the points B and C are externally tangent at a point P inside the quadrilateral. Suppose that

$$\angle PAB + \angle PDC \leq 90^\circ \qquad \text{and} \qquad \angle PBA + \angle PCD \leq 90^\circ.$$

Prove that $AB + CD \geq BC + AD$.

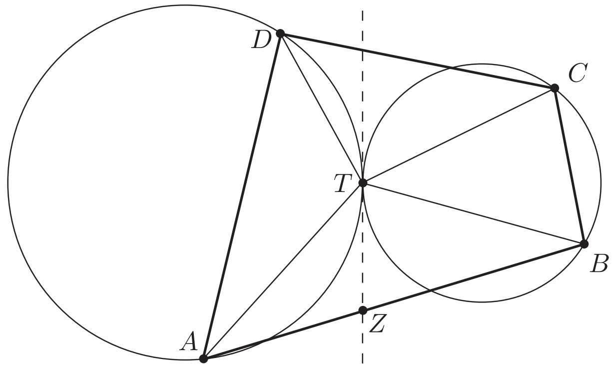
(Poland)

Solution. We start with a preliminary observation. Let T be a point inside the quadrilateral $ABCD$. Then:

Circles (BCT) and (DAT) are tangent at T
if and only if $\quad \angle ADT + \angle BCT = \angle ATB.$ (1)

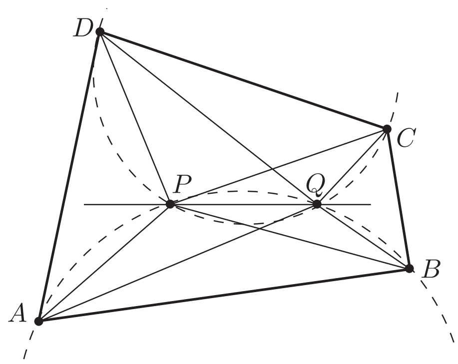
Indeed, if the two circles touch each other then their common tangent at T intersects the segment AB at a point Z, and so $\angle ADT = \angle ATZ$, $\angle BCT = \angle BTZ$, by the tangent-chord theorem. Thus $\angle ADT + \angle BCT = \angle ATZ + \angle BTZ = \angle ATB$.

And conversely, if $\angle ADT + \angle BCT = \angle ATB$ then one can draw from T a ray TZ with Z on AB so that $\angle ADT = \angle ATZ$, $\angle BCT = \angle BTZ$. The first of these equalities implies that TZ is tangent to the circle (DAT); by the second equality, TZ is tangent to the circle (BCT), so the two circles are tangent at T.



So the equivalence (1) is settled. It will be used later on. Now pass to the actual solution. Its key idea is to introduce the circumcircles of triangles ABP and CDP and to consider their second intersection Q (assume for the moment that they indeed meet at two distinct points P and Q).

Since the point A lies outside the circle (BCP), we have $\angle BCP + \angle BAP < 180^\circ$. Therefore the point C lies outside the circle (ABP). Analogously, D also lies outside that circle. It follows that P and Q lie on the same arc CD of the circle (BCP).



By symmetry, P and Q lie on the same arc AB of the circle (ABP). Thus the point Q lies either inside the angle BPC or inside the angle APD. Without loss of generality assume that Q lies inside the angle BPC. Then

$$\angle AQD = \angle PQA + \angle PQD = \angle PBA + \angle PCD \leq 90^\circ, \tag{2}$$

by the condition of the problem.

In the cyclic quadrilaterals $APQB$ and $DPQC$, the angles at vertices A and D are acute. So their angles at Q are obtuse. This implies that Q lies not only inside the angle BPC but in fact inside the triangle BPC, hence also inside the quadrilateral $ABCD$.

Now an argument similar to that used in deriving (2) shows that

$$\angle BQC = \angle PAB + \angle PDC \leq 90^\circ. \tag{3}$$

Moreover, since $\angle PCQ = \angle PDQ$, we get

$$\angle ADQ + \angle BCQ = \angle ADP + \angle PDQ + \angle BCP - \angle PCQ = \angle ADP + \angle BCP.$$

The last sum is equal to $\angle APB$, according to the observation (1) applied to $T = P$. And because $\angle APB = \angle AQB$, we obtain

$$\angle ADQ + \angle BCQ = \angle AQB.$$

Applying now (1) to $T = Q$ we conclude that the circles (BCQ) and (DAQ) are externally tangent at Q. (We have assumed $P \neq Q$; but if $P = Q$ then the last conclusion holds trivially.)

Finally consider the halfdiscs with diameters BC and DA constructed inwardly to the quadrilateral $ABCD$. They have centres at M and N, the midpoints of BC and DA respectively. In view of (2) and (3), these two halfdiscs lie entirely inside the circles (BQC) and (AQD); and since these circles are tangent, the two halfdiscs cannot overlap. Hence $MN \geq \frac{1}{2}BC + \frac{1}{2}DA$.

On the other hand, since $\overrightarrow{MN} = \frac{1}{2}(\overrightarrow{BA} + \overrightarrow{CD})$, we have $MN \leq \frac{1}{2}(AB + CD)$. Thus indeed $AB + CD \geq BC + DA$, as claimed.

G9. Points A_1, B_1, C_1 are chosen on the sides BC, CA, AB of a triangle ABC, respectively. The circumcircles of triangles AB_1C_1, BC_1A_1, CA_1B_1 intersect the circumcircle of triangle ABC again at points A_2, B_2, C_2, respectively ($A_2 \neq A$, $B_2 \neq B$, $C_2 \neq C$). Points A_3, B_3, C_3 are symmetric to A_1, B_1, C_1 with respect to the midpoints of the sides BC, CA, AB respectively. Prove that the triangles $A_2B_2C_2$ and $A_3B_3C_3$ are similar.

(Russia)

Solution. We will work with oriented angles between lines. For two straight lines ℓ, m in the plane, $\angle(\ell, m)$ denotes the angle of counterclockwise rotation which transforms line ℓ into a line parallel to m (the choice of the rotation centre is irrelevant). This is a signed quantity; values differing by a multiple of π are identified, so that

$$\angle(\ell, m) = -\angle(m, \ell), \qquad \angle(\ell, m) + \angle(m, n) = \angle(\ell, n).$$

If ℓ is the line through points K, L and m is the line through M, N, one writes $\angle(KL, MN)$ for $\angle(\ell, m)$; the characters K, L are freely interchangeable; and so are M, N.

The counterpart of the classical theorem about cyclic quadrilaterals is the following: *If K, L, M, N are four noncollinear points in the plane then*

$$K, L, M, N \text{ are concyclic if and only if } \angle(KM, LM) = \angle(KN, LN). \tag{1}$$

Passing to the solution proper, we first show that the three circles (AB_1C_1), (BC_1A_1), (CA_1B_1) have a common point. So, let (AB_1C_1) and (BC_1A_1) intersect at the points C_1 and P. Then by (1)

$$\angle(PA_1, CA_1) = \angle(PA_1, BA_1) = \angle(PC_1, BC_1)$$

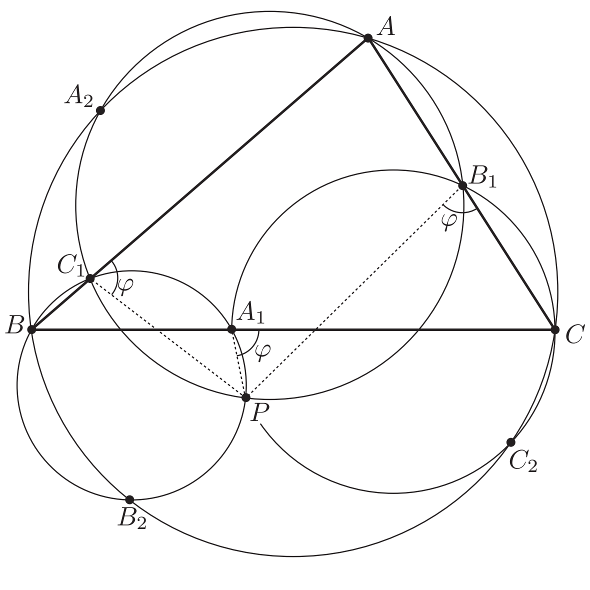
$$= \angle(PC_1, AC_1) = \angle(PB_1, AB_1) = \angle(PB_1, CB_1).$$

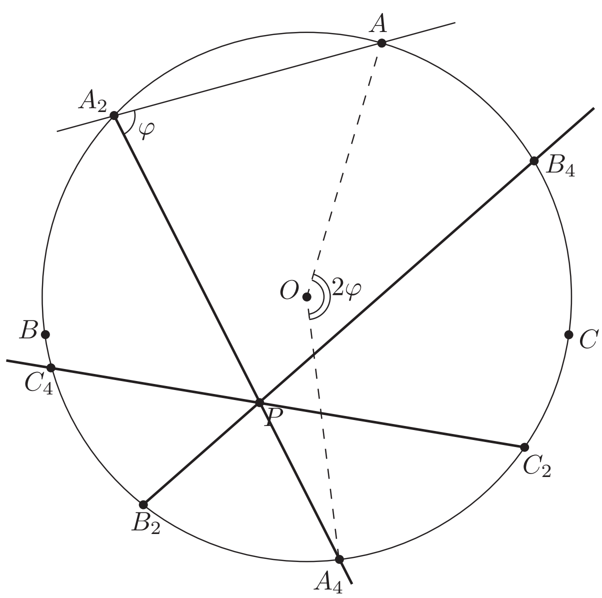
Denote this angle by φ.

The equality between the outer terms shows, again by (1), that the points A_1, B_1, P, C are concyclic. Thus P is the common point of the three mentioned circles.

From now on the basic property (1) will be used without explicit reference. We have

$$\varphi = \angle(PA_1, BC) = \angle(PB_1, CA) = \angle(PC_1, AB). \tag{2}$$





Let lines A_2P, B_2P, C_2P meet the circle (ABC) again at A_4, B_4, C_4, respectively. As

$$\angle(A_4A_2, AA_2) = \angle(PA_2, AA_2) = \angle(PC_1, AC_1) = \angle(PC_1, AB) = \varphi,$$

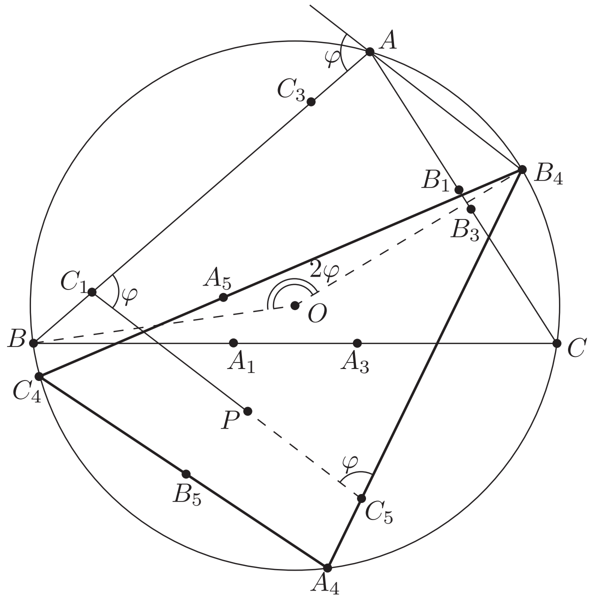
we see that line A_2A is the image of line A_2A_4 under rotation about A_2 by the angle φ. Hence the point A is the image of A_4 under rotation by 2φ about O, the centre of (ABC). The same rotation sends B_4 to B and C_4 to C. Triangle ABC is the image of $A_4B_4C_4$ in this map. Thus

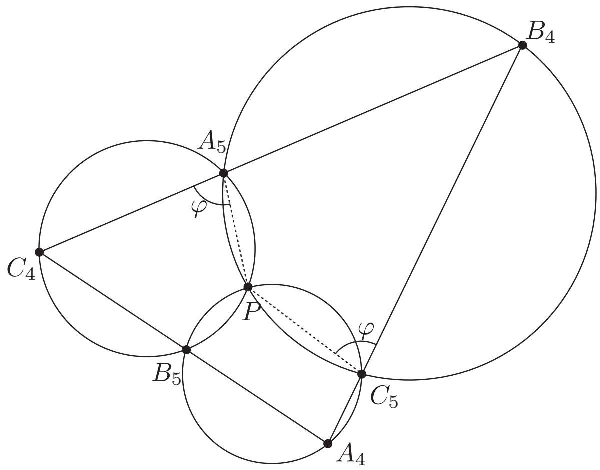
$$\angle(A_4B_4, AB) = \angle(B_4C_4, BC) = \angle(C_4A_4, CA) = 2\varphi. \tag{3}$$

Since the rotation by 2φ about O takes B_4 to B, we have $\angle(AB_4, AB) = \varphi$. Hence by (2)

$$\angle(AB_4, PC_1) = \angle(AB_4, AB) + \angle(AB, PC_1) = \varphi + (-\varphi) = 0,$$

which means that $AB_4 \| PC_1$.





Let C_5 be the intersection of lines PC_1 and A_4B_4; define A_5, B_5 analogously. So $AB_4 \| C_1C_5$ and, by (3) and (2),

$$\angle(A_4B_4, PC_1) = \angle(A_4B_4, AB) + \angle(AB, PC_1) = 2\varphi + (-\varphi) = \varphi; \tag{4}$$

i.e., $\angle(B_4C_5, C_5C_1) = \varphi$. This combined with $\angle(C_5C_1, C_1A) = \angle(PC_1, AB) = \varphi$ (see (2)) proves that the quadrilateral $AB_4C_5C_1$ is an isosceles trapezoid with $AC_1 = B_4C_5$.

Interchanging the roles of A and B we infer that also $BC_1 = A_4C_5$. And since $AC_1 + BC_1 = AB = A_4B_4$, it follows that the point C_5 lies on the line segment A_4B_4 and partitions it into segments A_4C_5, B_4C_5 of lengths BC_1 $(= AC_3)$ and AC_1 $(= BC_3)$. In other words, the rotation which maps triangle $A_4B_4C_4$ onto ABC carries C_5 onto C_3. Likewise, it sends A_5 to A_3 and B_5 to B_3. So the triangles $A_3B_3C_3$ and $A_5B_5C_5$ are congruent. It now suffices to show that the latter is similar to $A_2B_2C_2$.

Lines B_4C_5 and PC_5 coincide respectively with A_4B_4 and PC_1. Thus by (4)

$$\angle(B_4C_5, PC_5) = \varphi.$$

Analogously (by cyclic shift) $\varphi = \angle(C_4A_5, PA_5)$, which rewrites as

$$\varphi = \angle(B_4A_5, PA_5).$$

These relations imply that the points P, B_4, C_5, A_5 are concyclic. Analogously, P, C_4, A_5, B_5 and P, A_4, B_5, C_5 are concyclic quadruples. Therefore

$$\angle(A_5B_5, C_5B_5) = \angle(A_5B_5, PB_5) + \angle(PB_5, C_5B_5) = \angle(A_5C_4, PC_4) + \angle(PA_4, C_5A_4). \qquad (5)$$

On the other hand, since the points $A_2, B_2, C_2, A_4, B_4, C_4$ all lie on the circle (ABC), we have

$$\angle(A_2B_2, C_2B_2) = \angle(A_2B_2, B_4B_2) + \angle(B_4B_2, C_2B_2) = \angle(A_2A_4, B_4A_4) + \angle(B_4C_4, C_2C_4). \quad (6)$$

But the lines A_2A_4, B_4A_4, B_4C_4, C_2C_4 coincide respectively with PA_4, C_5A_4, A_5C_4, PC_4. So the sums on the right-hand sides of (5) and (6) are equal, leading to equality between their left-hand sides: $\angle(A_5B_5, C_5B_5) = \angle(A_2B_2, C_2B_2)$. Hence (by cyclic shift, once more) also $\angle(B_5C_5, A_5C_5) = \angle(B_2C_2, A_2C_2)$ and $\angle(C_5A_5, B_5A_5) = \angle(C_2A_2, B_2A_2)$. This means that the triangles $A_5B_5C_5$ and $A_2B_2C_2$ have their corresponding angles equal, and consequently they are similar.

Comment 1. This is the way in which the proof has been presented by the proposer. Trying to work it out in the language of classical geometry, so as to avoid oriented angles, one is led to difficulties due to the fact that the reasoning becomes heavily case-dependent. Disposition of relevant points can vary in many respects. Angles which are equal in one case become supplementary in another. Although it seems not hard to translate all formulas from the shapes they have in one situation to the one they have in another, the real trouble is to identify all cases possible and rigorously verify that the key conclusions retain validity in each case.

The use of oriented angles is a very efficient method to omit this trouble. It seems to be the most appropriate environment in which the solution can be elaborated.

Comment 2. Actually, the fact that the circles (AB_1C_1), (BC_1A_1) and (CA_1B_1) have a common point does not require a proof; it is known as Miquel's theorem.

G10. To each side a of a convex polygon we assign the maximum area of a triangle contained in the polygon and having a as one of its sides. Show that the sum of the areas assigned to all sides of the polygon is not less than twice the area of the polygon.

(Serbia)

Solution 1.

Lemma. Every convex $(2n)$-gon, of area S, has a side and a vertex that jointly span a triangle of area not less than S/n.

Proof. By *main diagonals* of the $(2n)$-gon we shall mean those which partition the $(2n)$-gon into two polygons with equally many sides. For any side b of the $(2n)$-gon denote by Δ_b the triangle ABP where A, B are the endpoints of b and P is the intersection point of the main diagonals AA', BB'. We claim that the union of triangles Δ_b, taken over all sides, covers the whole polygon.

To show this, choose any side AB and consider the main diagonal AA' as a directed segment. Let X be any point in the polygon, not on any main diagonal. For definiteness, let X lie on the left side of the ray AA'. Consider the sequence of main diagonals AA', BB', CC', $\ldots$, where $A, B, C, \ldots$ are consecutive vertices, situated right to AA'.

The n-th item in this sequence is the diagonal $A'A$ (i.e. AA' reversed), having X on its right side. So there are two successive vertices K, L in the sequence $A, B, C, \ldots$ before A' such that X still lies to the left of KK' but to the right of LL'. And this means that X is in the triangle $\Delta_{\ell'}$, $\ell' = K'L'$. Analogous reasoning applies to points X on the right of AA' (points lying on main diagonals can be safely ignored). Thus indeed the triangles Δ_b jointly cover the whole polygon.

The sum of their areas is no less than S. So we can find two opposite sides, say $b = AB$ and $b' = A'B'$ (with AA', BB' main diagonals) such that $[\Delta_b] + [\Delta_{b'}] \geq S/n$, where $[\cdots]$ stands for the area of a region. Let AA', BB' intersect at P; assume without loss of generality that $PB \geq PB'$. Then

$$[ABA'] = [ABP] + [PBA'] \geq [ABP] + [PA'B'] = [\Delta_b] + [\Delta_{b'}] \geq S/n,$$

proving the lemma. □

Now, let $\mathcal{P}$ be any convex polygon, of area S, with m sides $a_1, \ldots, a_m$. Let S_i be the area of the greatest triangle in $\mathcal{P}$ with side a_i. Suppose, contrary to the assertion, that

$$\sum_{i=1}^{m} \frac{S_i}{S} < 2.$$

Then there exist rational numbers $q_1, \ldots, q_m$ such that $\sum q_i = 2$ and $q_i > S_i/S$ for each i.

Let n be a common denominator of the m fractions $q_1, \ldots, q_m$. Write $q_i = k_i/n$; so $\sum k_i = 2n$. Partition each side a_i of $\mathcal{P}$ into k_i equal segments, creating a convex $(2n)$-gon of area S (with some angles of size $180°$), to which we apply the lemma. Accordingly, this refined polygon has a side b and a vertex H spanning a triangle T of area $[T] \geq S/n$. If b is a piece of a side a_i of $\mathcal{P}$, then the triangle W with base a_i and summit H has area

$$[W] = k_i \cdot [T] \geq k_i \cdot S/n = q_i \cdot S > S_i,$$

in contradiction with the definition of S_i. This ends the proof.

Solution 2. As in the first solution, we allow again angles of size 180° at some vertices of the convex polygons considered.

To each convex n-gon $\mathcal{P} = A_1A_2\dots A_n$ we assign a centrally symmetric convex $(2n)$-gon $\mathcal{Q}$ with side vectors $\pm\overrightarrow{A_iA_{i+1}}$, $1 \le i \le n$. The construction is as follows. Attach the $2n$ vectors $\pm\overrightarrow{A_iA_{i+1}}$ at a common origin and label them $\overrightarrow{\mathbf{b}_1}, \overrightarrow{\mathbf{b}_2}, \dots, \overrightarrow{\mathbf{b}_{2n}}$ in counterclockwise direction; the choice of the first vector $\overrightarrow{\mathbf{b}_1}$ is irrelevant. The order of labelling is well-defined if $\mathcal{P}$ has neither parallel sides nor angles equal to 180°. Otherwise several collinear vectors with the same direction are labelled consecutively $\overrightarrow{\mathbf{b}_j}, \overrightarrow{\mathbf{b}_{j+1}}, \dots, \overrightarrow{\mathbf{b}_{j+r}}$. One can assume that in such cases the respective opposite vectors occur in the order $-\overrightarrow{\mathbf{b}_j}, -\overrightarrow{\mathbf{b}_{j+1}}, \dots, -\overrightarrow{\mathbf{b}_{j+r}}$, ensuring that $\overrightarrow{\mathbf{b}_{j+n}} = -\overrightarrow{\mathbf{b}_j}$ for $j = 1, \dots, 2n$. Indices are taken cyclically here and in similar situations below.

Choose points $B_1, B_2, \dots, B_{2n}$ satisfying $\overrightarrow{B_jB_{j+1}} = \overrightarrow{\mathbf{b}_j}$ for $j = 1, \dots, 2n$. The polygonal line $\mathcal{Q} = B_1B_2\dots B_{2n}$ is closed, since $\sum_{j=1}^{2n} \overrightarrow{\mathbf{b}_j} = \overrightarrow{0}$. Moreover, $\mathcal{Q}$ is a convex $(2n)$-gon due to the arrangement of the vectors $\overrightarrow{\mathbf{b}_j}$, possibly with 180°-angles. The side vectors of $\mathcal{Q}$ are $\pm\overrightarrow{A_iA_{i+1}}$, $1 \le i \le n$. So in particular $\mathcal{Q}$ is centrally symmetric, because it contains as side vectors $\overrightarrow{A_iA_{i+1}}$ and $-\overrightarrow{A_iA_{i+1}}$ for each $i = 1, \dots, n$. Note that B_jB_{j+1} and $B_{j+n}B_{j+n+1}$ are opposite sides of $\mathcal{Q}$, $1 \le j \le n$. We call $\mathcal{Q}$ the *associate* of $\mathcal{P}$.

Let S_i be the maximum area of a triangle with side A_iA_{i+1} in $\mathcal{P}$, $1 \le i \le n$. We prove that

$$[B_1B_2\dots B_{2n}] = 2\sum_{i=1}^{n} S_i \tag{1}$$

and

$$[B_1B_2\dots B_{2n}] \ge 4\,[A_1A_2\dots A_n]\,. \tag{2}$$

It is clear that (1) and (2) imply the conclusion of the original problem.

Lemma. For a side A_iA_{i+1} of $\mathcal{P}$, let h_i be the maximum distance from a point of $\mathcal{P}$ to line A_iA_{i+1}, $i = 1, \dots, n$. Denote by B_jB_{j+1} the side of $\mathcal{Q}$ such that $\overrightarrow{A_iA_{i+1}} = \overrightarrow{B_jB_{j+1}}$. Then the distance between B_jB_{j+1} and its opposite side in $\mathcal{Q}$ is equal to $2h_i$.

Proof. Choose a vertex A_k of $\mathcal{P}$ at distance h_i from line A_iA_{i+1}. Let $\mathbf{u}$ be the unit vector perpendicular to A_iA_{i+1} and pointing inside $\mathcal{P}$. Denoting by $\mathbf{x}\cdot\mathbf{y}$ the dot product of vectors $\mathbf{x}$ and $\mathbf{y}$, we have

$$h = \mathbf{u}\cdot\overrightarrow{A_iA_k} = \mathbf{u}\cdot(\overrightarrow{A_iA_{i+1}} + \dots + \overrightarrow{A_{k-1}A_k}) = \mathbf{u}\cdot(\overrightarrow{A_iA_{i-1}} + \dots + \overrightarrow{A_{k+1}A_k}).$$

In $\mathcal{Q}$, the distance H_i between the opposite sides B_jB_{j+1} and $B_{j+n}B_{j+n+1}$ is given by

$$H_i = \mathbf{u}\cdot(\overrightarrow{B_jB_{j+1}} + \dots + \overrightarrow{B_{j+n-1}B_{j+n}}) = \mathbf{u}\cdot(\overrightarrow{\mathbf{b}_j} + \overrightarrow{\mathbf{b}_{j+1}} + \dots + \overrightarrow{\mathbf{b}_{j+n-1}}).$$

The choice of vertex A_k implies that the n consecutive vectors $\overrightarrow{\mathbf{b}_j}, \overrightarrow{\mathbf{b}_{j+1}}, \dots, \overrightarrow{\mathbf{b}_{j+n-1}}$ are precisely $\overrightarrow{A_iA_{i+1}}, \dots, \overrightarrow{A_{k-1}A_k}$ and $\overrightarrow{A_iA_{i-1}}, \dots, \overrightarrow{A_{k+1}A_k}$, taken in some order. This implies $H_i = 2h_i$. □

For a proof of (1), apply the lemma to each side of $\mathcal{P}$. If O the centre of $\mathcal{Q}$ then, using the notation of the lemma,

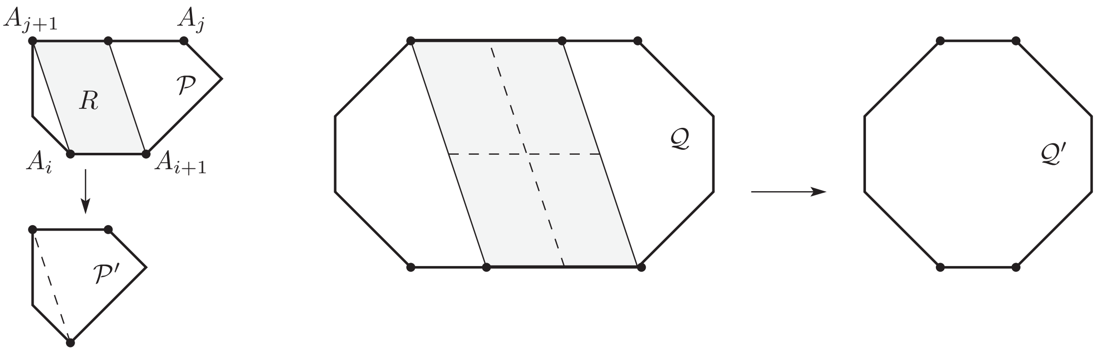
$$[B_jB_{j+1}O] = [B_{j+n}B_{j+n+1}O] = [A_iA_{i+1}A_k] = S_i\,.$$

Summation over all sides of $\mathcal{P}$ yields (1).

Set $d(\mathcal{P}) = [\mathcal{Q}] - 4[\mathcal{P}]$ for a convex polygon $\mathcal{P}$ with associate $\mathcal{Q}$. Inequality (2) means that $d(\mathcal{P}) \ge 0$ for each convex polygon $\mathcal{P}$. The last inequality will be proved by induction on the number ℓ of side directions of $\mathcal{P}$, i. e. the number of pairwise nonparallel lines each containing a side of $\mathcal{P}$.

We choose to start the induction with $\ell = 1$ as a base case, meaning that certain degenerate polygons are allowed. More exactly, we regard as *degenerate* convex polygons all closed polygonal lines of the form $X_1X_2 \dots X_kY_1Y_2 \dots Y_mX_1$, where $X_1, X_2, \dots, X_k$ are points in this order on a line segment X_1Y_1, and so are $Y_m, Y_{m-1}, \dots, Y_1$. The initial construction applies to degenerate polygons; their associates are also degenerate, and the value of d is zero. For the inductive step, consider a convex polygon $\mathcal{P}$ which determines ℓ side directions, assuming that $d(\mathcal{P}) \geq 0$ for polygons with smaller values of ℓ.

Suppose first that $\mathcal{P}$ *has a pair of parallel sides*, i. e. sides on distinct parallel lines. Let A_iA_{i+1} and A_jA_{j+1} be such a pair, and let $A_iA_{i+1} \leq A_jA_{j+1}$. Remove from $\mathcal{P}$ the parallelogram R determined by vectors $\overrightarrow{A_iA_{i+1}}$ and $\overrightarrow{A_iA_{j+1}}$. Two polygons are obtained in this way. Translating one of them by vector $\overrightarrow{A_iA_{i+1}}$ yields a new convex polygon $\mathcal{P}'$, of area $[\mathcal{P}] - [R]$ and with value of ℓ not exceeding the one of $\mathcal{P}$. The construction just described will be called operation $\mathbf{A}$.



The associate of $\mathcal{P}'$ is obtained from $\mathcal{Q}$ upon decreasing the lengths of two opposite sides by an amount of $2A_iA_{i+1}$. By the lemma, the distance between these opposite sides is twice the distance between A_iA_{i+1} and A_jA_{j+1}. Thus operation $\mathbf{A}$ decreases $[\mathcal{Q}]$ by the area of a parallelogram with base and respective altitude twice the ones of R, i. e. by $4[R]$. Hence $\mathbf{A}$ leaves the difference $d(\mathcal{P}) = [\mathcal{Q}] - 4[\mathcal{P}]$ unchanged.

Now, if $\mathcal{P}'$ also has a pair of parallel sides, apply operation $\mathbf{A}$ to it. Keep doing so with the subsequent polygons obtained for as long as possible. Now, $\mathbf{A}$ decreases the number p of pairs of parallel sides in $\mathcal{P}$. Hence its repeated applications gradually reduce p to 0, and further applications of $\mathbf{A}$ will be impossible after several steps. For clarity, let us denote by $\mathcal{P}$ again the polygon obtained at that stage.

The inductive step is complete if $\mathcal{P}$ is degenerate. Otherwise $\ell > 1$ and $p = 0$, i. e. there are no parallel sides in $\mathcal{P}$. Observe that then $\ell \geq 3$. Indeed, $\ell = 2$ means that the vertices of $\mathcal{P}$ all lie on the boundary of a parallelogram, implying $p > 0$.

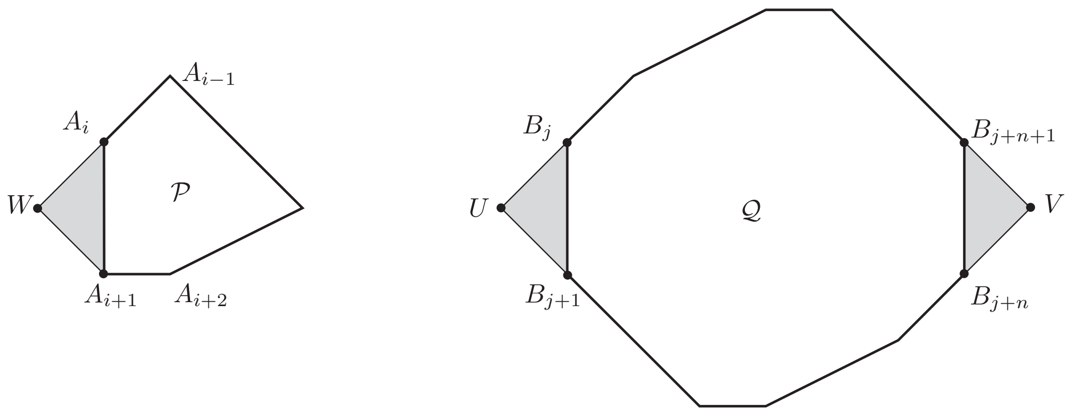
Furthermore, since $\mathcal{P}$ has no parallel sides, consecutive collinear vectors in the sequence $\left(\overrightarrow{\mathbf{b}_k}\right)$ (if any) correspond to consecutive $180°$-angles in $\mathcal{P}$. Removing the vertices of such angles, we obtain a convex polygon with the same value of $d(\mathcal{P})$.

In summary, if operation $\mathbf{A}$ is impossible for a nondegenerate polygon $\mathcal{P}$, then $\ell \geq 3$. In addition, one may assume that $\mathcal{P}$ has no angles of size $180°$.

The last two conditions then also hold for the associate $\mathcal{Q}$ of $\mathcal{P}$, and we perform the following construction. Since $\ell \geq 3$, there is a side B_jB_{j+1} of $\mathcal{Q}$ such that the sum of the angles at B_j and B_{j+1} is greater than $180°$. (Such a side exists in each convex k-gon for $k > 4$.) Naturally, $B_{j+n}B_{j+n+1}$ is a side with the same property. Extend the pairs of sides $B_{j-1}B_j, B_{j+1}B_{j+2}$

and $B_{j+n-1}B_{j+n}$, $B_{j+n+1}B_{j+n+2}$ to meet at U and V, respectively. Let $\mathcal{Q}'$ be the centrally symmetric convex $2(n+1)$-gon obtained from $\mathcal{Q}$ by inserting U and V into the sequence $B_1, \ldots, B_{2n}$ as new vertices between B_j, B_{j+1} and B_{j+n}, B_{j+n+1}, respectively. Informally, we adjoin to $\mathcal{Q}$ the congruent triangles $B_jB_{j+1}U$ and $B_{j+n}B_{j+n+1}V$. Note that B_j, B_{j+1}, B_{j+n} and B_{j+n+1} are kept as vertices of $\mathcal{Q}'$, although B_jB_{j+1} and $B_{j+n}B_{j+n+1}$ are no longer its sides.

Let A_iA_{i+1} be the side of $\mathcal{P}$ such that $\overrightarrow{A_iA_{i+1}} = \overrightarrow{B_jB_{j+1}} = \overrightarrow{\mathbf{b}_j}$. Consider the point W such that triangle $A_iA_{i+1}W$ is congruent to triangle $B_jB_{j+1}U$ and exterior to $\mathcal{P}$. Insert W into the sequence $A_1, A_2, \ldots, A_n$ as a new vertex between A_i and A_{i+1} to obtain an $(n+1)$-gon $\mathcal{P}'$. We claim that $\mathcal{P}'$ is convex and its associate is $\mathcal{Q}'$.



Vectors $\overrightarrow{A_iW}$ and $\overrightarrow{\mathbf{b}_{j-1}}$ are collinear and have the same direction, as well as vectors $\overrightarrow{WA_{i+1}}$ and $\overrightarrow{\mathbf{b}_{j+1}}$. Since $\overrightarrow{\mathbf{b}_{j-1}}, \overrightarrow{\mathbf{b}_j}, \overrightarrow{\mathbf{b}_{j+1}}$ are consecutive terms in the sequence $\left(\overrightarrow{\mathbf{b}_k}\right)$, the angle inequalities $\angle(\overrightarrow{\mathbf{b}_{j-1}}, \overrightarrow{\mathbf{b}_j}) \le \angle(\overrightarrow{A_{i-1}A_i}, \overrightarrow{\mathbf{b}_j})$ and $\angle(\overrightarrow{\mathbf{b}_j}, \overrightarrow{\mathbf{b}_{j+1}}) \le \angle(\overrightarrow{\mathbf{b}_j}, \overrightarrow{A_{i+1}A_{i+2}})$ hold true. They show that $\mathcal{P}'$ is a convex polygon. To construct its associate, vectors $\pm\overrightarrow{A_iA_{i+1}} = \pm\overrightarrow{\mathbf{b}_j}$ must be deleted from the defining sequence $\left(\overrightarrow{\mathbf{b}_k}\right)$ of $\mathcal{Q}$, and the vectors $\pm\overrightarrow{A_iW}$, $\pm\overrightarrow{WA_{i+1}}$ must be inserted appropriately into it. The latter can be done as follows:

$$\ldots,\ \overrightarrow{\mathbf{b}_{j-1}},\ \overrightarrow{A_iW},\ \overrightarrow{WA_{i+1}},\ \overrightarrow{\mathbf{b}_{j+1}},\ \ldots,\ -\overrightarrow{\mathbf{b}_{j-1}},\ -\overrightarrow{A_iW},\ -\overrightarrow{WA_{i+1}},\ -\overrightarrow{\mathbf{b}_{j+1}},\ \ldots .$$

This updated sequence produces $\mathcal{Q}'$ as the associate of $\mathcal{P}'$.

It follows from the construction that $[\mathcal{P}'] = [\mathcal{P}] + [A_iA_{i+1}W]$ and $[\mathcal{Q}'] = [\mathcal{Q}] + 2[A_iA_{i+1}W]$. Therefore $d(\mathcal{P}') = d(\mathcal{P}) - 2[A_iA_{i+1}W] < d(\mathcal{P})$.

To finish the induction, it remains to notice that the value of ℓ for $\mathcal{P}'$ is less than the one for $\mathcal{P}$. This is because side A_iA_{i+1} was removed. The newly added sides A_iW and WA_{i+1} do not introduce new side directions. Each one of them is either parallel to a side of $\mathcal{P}$ or lies on the line determined by such a side. The proof is complete.

Number Theory

N1. Determine all pairs (x, y) of integers satisfying the equation

$$1 + 2^x + 2^{2x+1} = y^2.$$

(USA)

Solution. If (x, y) is a solution then obviously $x \geq 0$ and $(x, -y)$ is a solution too. For $x = 0$ we get the two solutions $(0, 2)$ and $(0, -2)$.

Now let (x, y) be a solution with $x > 0$; without loss of generality confine attention to $y > 0$. The equation rewritten as

$$2^x(1 + 2^{x+1}) = (y - 1)(y + 1)$$

shows that the factors $y - 1$ and $y + 1$ are even, exactly one of them divisible by 4. Hence $x \geq 3$ and one of these factors is divisible by 2^{x-1} but not by 2^x. So

$$y = 2^{x-1}m + \epsilon, \qquad m \text{ odd}, \qquad \epsilon = \pm 1. \tag{1}$$

Plugging this into the original equation we obtain

$$2^x \left(1 + 2^{x+1}\right) = \left(2^{x-1}m + \epsilon\right)^2 - 1 = 2^{2x-2}m^2 + 2^x m\epsilon,$$

or, equivalently

$$1 + 2^{x+1} = 2^{x-2}m^2 + m\epsilon.$$

Therefore

$$1 - \epsilon m = 2^{x-2}(m^2 - 8). \tag{2}$$

For $\epsilon = 1$ this yields $m^2 - 8 \leq 0$, i.e., $m = 1$, which fails to satisfy (2).

For $\epsilon = -1$ equation (2) gives us

$$1 + m = 2^{x-2}(m^2 - 8) \geq 2(m^2 - 8),$$

implying $2m^2 - m - 17 \leq 0$. Hence $m \leq 3$; on the other hand m cannot be 1 by (2). Because m is odd, we obtain $m = 3$, leading to $x = 4$. From (1) we get $y = 23$. These values indeed satisfy the given equation. Recall that then $y = -23$ is also good. Thus we have the complete list of solutions (x, y): $(0, 2)$, $(0, -2)$, $(4, 23)$, $(4, -23)$.

N2. For $x \in (0,1)$ let $y \in (0,1)$ be the number whose nth digit after the decimal point is the (2^n)th digit after the decimal point of x. Show that if x is rational then so is y.

(Canada)

Solution. Since x is rational, its digits repeat periodically starting at some point. We wish to show that this is also true for the digits of y, implying that y is rational.

Let d be the length of the period of x and let $d = 2^u \cdot v$, where v is odd. There is a positive integer w such that

$$2^w \equiv 1 \pmod{v}.$$

(For instance, one can choose w to be $\varphi(v)$, the value of Euler's function at v.) Therefore

$$2^{n+w} = 2^n \cdot 2^w \equiv 2^n \pmod{v}$$

for each n. Also, for $n \geq u$ we have

$$2^{n+w} \equiv 2^n \equiv 0 \pmod{2^u}.$$

It follows that, for all $n \geq u$, the relation

$$2^{n+w} \equiv 2^n \pmod{d}$$

holds. Thus, for n sufficiently large, the 2^{n+w}th digit of x is in the same spot in the cycle of x as its 2^nth digit, and so these digits are equal. Hence the $(n+w)$th digit of y is equal to its nth digit. This means that the digits of y repeat periodically with period w from some point on, as required.

N3. The sequence $f(1), f(2), f(3), \ldots$ is defined by

$$f(n) = \frac{1}{n}\left(\left\lfloor\frac{n}{1}\right\rfloor + \left\lfloor\frac{n}{2}\right\rfloor + \cdots + \left\lfloor\frac{n}{n}\right\rfloor\right),$$

where $\lfloor x \rfloor$ denotes the integer part of x.

(a) Prove that $f(n+1) > f(n)$ infinitely often.

(b) Prove that $f(n+1) < f(n)$ infinitely often.

(South Africa)

Solution. Let $g(n) = nf(n)$ for $n \geq 1$ and $g(0) = 0$. We note that, for $k = 1, \ldots, n$,

$$\left\lfloor\frac{n}{k}\right\rfloor - \left\lfloor\frac{n-1}{k}\right\rfloor = 0$$

if k is not a divisor of n and

$$\left\lfloor\frac{n}{k}\right\rfloor - \left\lfloor\frac{n-1}{k}\right\rfloor = 1$$

if k divides n. It therefore follows that if $d(n)$ is the number of positive divisors of $n \geq 1$ then

$$\begin{aligned}
g(n) &= \left\lfloor\frac{n}{1}\right\rfloor + \left\lfloor\frac{n}{2}\right\rfloor + \cdots + \left\lfloor\frac{n}{n-1}\right\rfloor + \left\lfloor\frac{n}{n}\right\rfloor \\
&= \left\lfloor\frac{n-1}{1}\right\rfloor + \left\lfloor\frac{n-1}{2}\right\rfloor + \cdots + \left\lfloor\frac{n-1}{n-1}\right\rfloor + \left\lfloor\frac{n-1}{n}\right\rfloor + d(n) \\
&= g(n-1) + d(n).
\end{aligned}$$

Hence

$$g(n) = g(n-1) + d(n) = g(n-2) + d(n-1) + d(n) = \cdots = d(1) + d(2) + \cdots + d(n),$$

meaning that

$$f(n) = \frac{d(1) + d(2) + \cdots + d(n)}{n}.$$

In other words, $f(n)$ is equal to the arithmetic mean of $d(1), d(2), \ldots, d(n)$. In order to prove the claims, it is therefore sufficient to show that $d(n+1) > f(n)$ and $d(n+1) < f(n)$ both hold infinitely often.

We note that $d(1) = 1$. For $n > 1$, $d(n) \geq 2$ holds, with equality if and only if n is prime. Since $f(6) = 7/3 > 2$, it follows that $f(n) > 2$ holds for all $n \geq 6$.

Since there are infinitely many primes, $d(n+1) = 2$ holds for infinitely many values of n, and for each such $n \geq 6$ we have $d(n+1) = 2 < f(n)$. This proves claim (b).

To prove (a), notice that the sequence $d(1), d(2), d(3), \ldots$ is unbounded (e. g. $d(2^k) = k+1$ for all k). Hence $d(n+1) > \max\{d(1), d(2), \ldots, d(n)\}$ for infinitely many n. For all such n, we have $d(n+1) > f(n)$. This completes the solution.

N4. Let P be a polynomial of degree $n > 1$ with integer coefficients and let k be any positive integer. Consider the polynomial $Q(x) = P(P(\ldots P(P(x))\ldots))$, with k pairs of parentheses. Prove that Q has no more than n integer fixed points, i.e. integers satisfying the equation $Q(x) = x$.

(Romania)

Solution. The claim is obvious if every integer fixed point of Q is a fixed point of P itself. For the sequel assume that this is not the case. Take any integer x_0 such that $Q(x_0) = x_0$, $P(x_0) \neq x_0$ and define inductively $x_{i+1} = P(x_i)$ for $i = 0, 1, 2, \ldots$; then $x_k = x_0$.

It is evident that

$$P(u) - P(v) \text{ is divisible by } u - v \text{ for distinct integers } u, v. \tag{1}$$

(Indeed, if $P(x) = \sum a_i x^i$ then each $a_i(u^i - v^i)$ is divisible by $u - v$.) Therefore each term in the chain of (nonzero) differences

$$x_0 - x_1, \quad x_1 - x_2, \quad \ldots, \quad x_{k-1} - x_k, \quad x_k - x_{k+1} \tag{2}$$

is a divisor of the next one; and since $x_k - x_{k+1} = x_0 - x_1$, all these differences have equal absolute values. For $x_m = \min(x_1, \ldots, x_k)$ this means that $x_{m-1} - x_m = -(x_m - x_{m+1})$. Thus $x_{m-1} = x_{m+1} (\neq x_m)$. It follows that consecutive differences in the sequence (2) have opposite signs. Consequently, $x_0, x_1, x_2, \ldots$ is an alternating sequence of two distinct values. In other words, every integer fixed point of Q is a fixed point of the polynomial $P(P(x))$. Our task is to prove that there are at most n such points.

Let a be one of them so that $b = P(a) \neq a$ (we have assumed that such an a exists); then $a = P(b)$. Take any other integer fixed point α of $P(P(x))$ and let $P(\alpha) = \beta$, so that $P(\beta) = \alpha$; the numbers α and β need not be distinct (α can be a fixed point of P), but each of α, β is different from each of a, b. Applying property (1) to the four pairs of integers (α, a), (β, b), (α, b), (β, a) we get that the numbers $\alpha - a$ and $\beta - b$ divide each other, and also $\alpha - b$ and $\beta - a$ divide each other. Consequently

$$\alpha - b = \pm(\beta - a), \qquad \alpha - a = \pm(\beta - b). \tag{3}$$

Suppose we have a plus in both instances: $\alpha - b = \beta - a$ and $\alpha - a = \beta - b$. Subtraction yields $a - b = b - a$, a contradiction, as $a \neq b$. Therefore at least one equality in (3) holds with a minus sign. For each of them this means that $\alpha + \beta = a + b$; equivalently $a + b - \alpha - P(\alpha) = 0$.

Denote $a + b$ by C. We have shown that every integer fixed point of Q other that a and b is a root of the polynomial $F(x) = C - x - P(x)$. This is of course true for a and b as well. And since P has degree $n > 1$, the polynomial F has the same degree, so it cannot have more than n roots. Hence the result.

Comment. The first part of the solution, showing that integer fixed points of any iterate of P are in fact fixed points of the second iterate $P \circ P$ is standard; moreover, this fact has already appeared in contests. We however do not consider this as a major drawback to the problem because the only tricky moment comes up only at the next stage of the reasoning—to apply the divisibility property (1) to points from distinct 2-orbits of P. Yet maybe it would be more appropriate to state the problem in a version involving $k = 2$ only.

N5. Find all integer solutions of the equation

$$\frac{x^7-1}{x-1} = y^5 - 1.$$

(Russia)

Solution. The equation has no integer solutions. To show this, we first prove a lemma.

Lemma. If x is an integer and p is a prime divisor of $\dfrac{x^7-1}{x-1}$ then either $p \equiv 1 \pmod 7$ or $p = 7$.

Proof. Both x^7-1 and $x^{p-1}-1$ are divisible by p, by hypothesis and by Fermat's little theorem, respectively. Suppose that 7 does not divide $p-1$. Then $\gcd(p-1, 7) = 1$, so there exist integers k and m such that $7k + (p-1)m = 1$. We therefore have

$$x \equiv x^{7k+(p-1)m} \equiv (x^7)^k \cdot (x^{p-1})^m \equiv 1 \pmod p,$$

and so

$$\frac{x^7-1}{x-1} = 1 + x + \cdots + x^6 \equiv 7 \pmod p.$$

It follows that p divides 7, hence $p = 7$ must hold if $p \equiv 1 \pmod 7$ does not, as stated. $\square$

The lemma shows that each positive divisor d of $\dfrac{x^7-1}{x-1}$ satisfies either $d \equiv 0 \pmod 7$ or $d \equiv 1 \pmod 7$.

Now assume that (x, y) is an integer solution of the original equation. Notice that $y - 1 > 0$, because $\dfrac{x^7-1}{x-1} > 0$ for all $x \neq 1$. Since $y - 1$ divides $\dfrac{x^7-1}{x-1} = y^5 - 1$, we have $y \equiv 1 \pmod 7$ or $y \equiv 2 \pmod 7$ by the previous paragraph. In the first case, $1 + y + y^2 + y^3 + y^4 \equiv 5 \pmod 7$, and in the second $1 + y + y^2 + y^3 + y^4 \equiv 3 \pmod 7$. Both possibilities contradict the fact that the positive divisor $1 + y + y^2 + y^3 + y^4$ of $\dfrac{x^7-1}{x-1}$ is congruent to 0 or 1 modulo 7. So the given equation has no integer solutions.

N6. Let $a > b > 1$ be relatively prime positive integers. Define the *weight* of an integer c, denoted by $w(c)$, to be the minimal possible value of $|x| + |y|$ taken over all pairs of integers x and y such that

$$ax + by = c.$$

An integer c is called a *local champion* if $w(c) \geq w(c \pm a)$ and $w(c) \geq w(c \pm b)$.

Find all local champions and determine their number.

(USA)

Solution. Call the pair of integers (x, y) a *representation* of c if $ax + by = c$ and $|x| + |y|$ has the smallest possible value, i.e. $|x| + |y| = w(c)$.

We characterise the local champions by the following three observations.

Lemma 1. If (x, y) a representation of a local champion c then $xy < 0$.

Proof. Suppose indirectly that $x \geq 0$ and $y \geq 0$ and consider the values $w(c)$ and $w(c+a)$. All representations of the numbers c and $c + a$ in the form $au + bv$ can be written as

$$c = a(x - kb) + b(y + ka), \qquad c + a = a(x + 1 - kb) + b(y + ka)$$

where k is an arbitrary integer.

Since $|x| + |y|$ is minimal, we have

$$x + y = |x| + |y| \leq |x - kb| + |y + ka|$$

for all k. On the other hand, $w(c + a) \leq w(c)$, so there exists a k for which

$$|x + 1 - kb| + |y + ka| \leq |x| + |y| = x + y.$$

Then

$$(x + 1 - kb) + (y + ka) \leq |x + 1 - kb| + |y + ka| \leq x + y \leq |x - kb| + |y + ka|.$$

Comparing the first and the third expressions, we find $k(a - b) + 1 \leq 0$ implying $k < 0$. Comparing the second and fourth expressions, we get $|x + 1 - kb| \leq |x - kb|$, therefore $kb > x$; this is a contradiction.

If $x, y \leq 0$ then we can switch to $-c$, $-x$ and $-y$. $\square$

From this point, write $c = ax - by$ instead of $c = ax + by$ and consider only those cases where x and y are nonzero and have the same sign. By Lemma 1, there is no loss of generality in doing so.

Lemma 2. Let $c = ax - by$ where $|x| + |y|$ is minimal and x, y have the same sign. The number c is a local champion if and only if $|x| < b$ and $|x| + |y| = \left\lfloor \frac{a+b}{2} \right\rfloor$.

Proof. Without loss of generality we may assume $x, y > 0$.

The numbers $c - a$ and $c + b$ can be written as

$$c - a = a(x - 1) - by \quad \text{and} \quad c + b = ax - b(y - 1)$$

and trivially $w(c - a) \leq (x - 1) + y < w(c)$ and $w(c + b) \leq x + (y - 1) < w(c)$ in all cases.

Now assume that c is a local champion and consider $w(c + a)$. Since $w(c + a) \leq w(c)$, there exists an integer k such that

$$c + a = a(x + 1 - kb) - b(y - ka) \quad \text{and} \quad |x + 1 - kb| + |y - ka| \leq x + y.$$

This inequality cannot hold if $k \le 0$, therefore $k > 0$. We prove that we can choose $k = 1$.

Consider the function $f(t) = |x+1-bt| + |y-at| - (x+y)$. This is a convex function and we have $f(0) = 1$ and $f(k) \le 0$. By Jensen's inequality, $f(1) \le \left(1 - \frac{1}{k}\right)f(0) + \frac{1}{k}f(k) < 1$. But $f(1)$ is an integer. Therefore $f(1) \le 0$ and

$$|x+1-b| + |y-a| \le x+y.$$

Knowing $c = a(x-b) - b(y-a)$, we also have

$$x+y \le |x-b| + |y-a|.$$

Combining the two inequalities yields $|x+1-b| \le |x-b|$ which is equivalent to $x < b$.

Considering $w(c-b)$, we obtain similarly that $y < a$.

Now $|x-b| = b-x$, $|x+1-b| = b-x-1$ and $|y-a| = a-y$, therefore we have

$$(b-x-1) + (a-y) \le x+y \le (b-x) + (a-y),$$

$$\frac{a+b-1}{2} \le x+y \le \frac{a+b}{2}.$$

Hence $x+y = \left\lfloor \frac{a+b}{2} \right\rfloor$.

To prove the opposite direction, assume $0 < x < b$ and $x+y = \left\lfloor \frac{a+b}{2} \right\rfloor$. Since $a > b$, we also have $0 < y < a$. Then

$$w(c+a) \le |x+1-b| + |y-a| = a+b-1-(x+y) \le x+y = w(c)$$

and

$$w(c-b) \le |x-b| + |y+1-a| = a+b-1-(x+y) \le x+y = w(c)$$

therefore c is a local champion indeed. □

Lemma 3. Let $c = ax - by$ and assume that x and y have the same sign, $|x| < b$, $|y| < a$ and $|x| + |y| = \left\lfloor \frac{a+b}{2} \right\rfloor$. Then $w(c) = x+y$.

Proof. By definition $w(c) = \min\{|x-kb| + |y-ka| : \ k \in \mathbb{Z}\}$. If $k \le 0$ then obviously $|x-kb| + |y-ka| \ge x+y$. If $k \ge 1$ then

$$|x-kb| + |y-ka| = (kb-x) + (ka-y) = k(a+b) - (x+y) \ge (2k-1)(x+y) \ge x+y.$$

Therefore $w(c) = x+y$ indeed. □

Lemmas 1, 2 and 3 together yield that the set of local champions is

$$C = \left\{ \pm (ax-by) : \ 0 < x < b, \ x+y = \left\lfloor \frac{a+b}{2} \right\rfloor \right\}.$$

Denote by C^+ and C^- the two sets generated by the expressions $+(ax-by)$ and $-(ax-by)$, respectively. It is easy to see that both sets are arithmetic progressions of length $b-1$, with difference $a+b$.

If a and b are odd, then $C^+ = C^-$, because $a(-x) - b(-y) = a(b-x) - b(a-y)$ and $x+y = \frac{a+b}{2}$ is equivalent to $(b-x) + (a-y) = \frac{a+b}{2}$. In this case there exist $b-1$ local champions.

If a and b have opposite parities then the answer is different. For any $c_1 \in C^+$ and $c_2 \in C^-$,

$$2c_1 \equiv -2c_2 \equiv 2\left(a\frac{a+b-1}{2} - b \cdot 0\right) \equiv -a \pmod{a+b}$$

and

$$2c_1 - 2c_2 \equiv -2a \pmod{a+b}.$$

The number $a+b$ is odd and relatively prime to a, therefore the elements of C^+ and C^- belong to two different residue classes modulo $a+b$. Hence, the set C is the union of two disjoint arithmetic progressions and the number of all local champions is $2(b-1)$.

So the number of local champions is $b-1$ if both a and b are odd and $2(b-1)$ otherwise.

Comment. The original question, as stated by the proposer, was:

(a) Show that there exists only finitely many local champions;

(b) Show that there exists at least one local champion.

N7. Prove that, for every positive integer n, there exists an integer m such that $2^m + m$ is divisible by n.

(Estonia)

Solution. We will prove by induction on d that, for every positive integer N, there exist positive integers $b_0, b_1, \ldots, b_{d-1}$ such that, for each $i = 0, 1, 2, \ldots, d-1$, we have $b_i > N$ and

$$2^{b_i} + b_i \equiv i \pmod{d}.$$

This yields the claim for $m = b_0$.

The base case $d = 1$ is trivial. Take an $a > 1$ and assume that the statement holds for all $d < a$. Note that the remainders of 2^i modulo a repeat periodically starting with some exponent M. Let k be the length of the period; this means that $2^{M+k'} \equiv 2^M \pmod{a}$ holds only for those k' which are multiples of k. Note further that the period cannot contain all the a remainders, since 0 either is missing or is the only number in the period. Thus $k < a$.

Let $d = \gcd(a, k)$ and let $a' = a/d$, $k' = k/d$. Since $0 < k < a$, we also have $0 < d < a$. By the induction hypothesis, there exist positive integers $b_0, b_1, \ldots, b_{d-1}$ such that $b_i > \max(2^M, N)$ and

$$2^{b_i} + b_i \equiv i \pmod{d} \quad \text{for} \quad i = 0, 1, 2, \ldots, d-1. \tag{1}$$

For each $i = 0, 1, \ldots, d-1$ consider the sequence

$$2^{b_i} + b_i, \;\; 2^{b_i+k} + (b_i + k), \;\; \ldots, \;\; 2^{b_i+(a'-1)k} + (b_i + (a'-1)k). \tag{2}$$

Modulo a, these numbers are congruent to

$$2^{b_i} + b_i, \;\; 2^{b_i} + (b_i + k), \;\; \ldots, \;\; 2^{b_i} + (b_i + (a'-1)k),$$

respectively. The d sequences contain $a'd = a$ numbers altogether. We shall now prove that no two of these numbers are congruent modulo a.

Suppose that

$$2^{b_i} + (b_i + mk) \equiv 2^{b_j} + (b_j + nk) \pmod{a} \tag{3}$$

for some values of $i, j \in \{0, 1, \ldots, d-1\}$ and $m, n \in \{0, 1, \ldots, a'-1\}$. Since d is a divisor of a, we also have

$$2^{b_i} + (b_i + mk) \equiv 2^{b_j} + (b_j + nk) \pmod{d}.$$

Because d is a divisor of k and in view of (1), we obtain $i \equiv j \pmod{d}$. As $i, j \in \{0, 1, \ldots, d-1\}$, this just means that $i = j$. Substituting this into (3) yields $mk \equiv nk \pmod{a}$. Therefore $mk' \equiv nk' \pmod{a'}$; and since a' and k' are coprime, we get $m \equiv n \pmod{a'}$. Hence also $m = n$.

It follows that the a numbers that make up the d sequences (2) satisfy all the requirements; they are certainly all greater than N because we chose each $b_i > \max(2^M, N)$. So the statement holds for a, completing the induction.

48th International Mathematical Olympiad

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Shortlisted Problems

with Solutions

July 19-31, 2007

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Vietnam 2007

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Contributing Countries

Austria, Australia, Belgium, Bulgaria, Canada, Croatia, Czech Republic, Estonia, Finland, Greece, India, Indonesia, Iran, Japan, Korea (North), Korea (South), Lithuania, Luxembourg, Mexico, Moldova, Netherlands, New Zealand, Poland, Romania, Russia, Serbia, South Africa, Sweden, Thailand, Taiwan, Turkey, Ukraine, United Kingdom, United States of America

Problem Selection Committee

Ha Huy Khoai
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Algebra

A1. Given a sequence $a_1, a_2, \ldots, a_n$ of real numbers. For each i $(1 \leq i \leq n)$ define

$$d_i = \max\{a_j : 1 \leq j \leq i\} - \min\{a_j : i \leq j \leq n\}$$

and let

$$d = \max\{d_i : 1 \leq i \leq n\}.$$

(a) Prove that for arbitrary real numbers $x_1 \leq x_2 \leq \ldots \leq x_n$,

$$\max\bigl\{|x_i - a_i| : 1 \leq i \leq n\bigr\} \geq \frac{d}{2}. \tag{1}$$

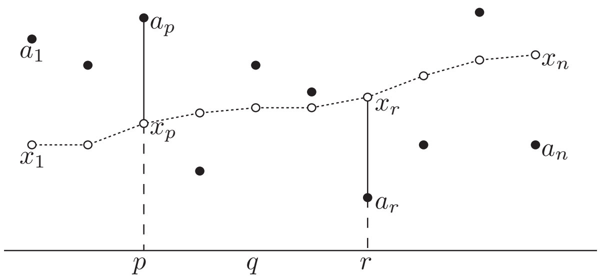
(b) Show that there exists a sequence $x_1 \leq x_2 \leq \ldots \leq x_n$ of real numbers such that we have equality in (1).

(New Zealand)

Solution 1. (a) Let $1 \leq p \leq q \leq r \leq n$ be indices for which

$$d = d_q, \qquad a_p = \max\{a_j : 1 \leq j \leq q\}, \quad a_r = \min\{a_j : q \leq j \leq n\}$$

and thus $d = a_p - a_r$. (These indices are not necessarily unique.)



For arbitrary real numbers $x_1 \leq x_2 \leq \ldots \leq x_n$, consider just the two quantities $|x_p - a_p|$ and $|x_r - a_r|$. Since

$$(a_p - x_p) + (x_r - a_r) = (a_p - a_r) + (x_r - x_p) \geq a_p - a_r = d,$$

we have either $a_p - x_p \geq \dfrac{d}{2}$ or $x_r - a_r \geq \dfrac{d}{2}$. Hence,

$$\max\{|x_i - a_i| : 1 \leq i \leq n\} \geq \max\bigl\{|x_p - a_p|, |x_r - a_r|\bigr\} \geq \max\{a_p - x_p, x_r - a_r\} \geq \frac{d}{2}.$$

(b) Define the sequence (x_k) as

$$x_1 = a_1 - \frac{d}{2}, \qquad x_k = \max\left\{x_{k-1},\ a_k - \frac{d}{2}\right\} \quad \text{for } 2 \le k \le n.$$

We show that we have equality in (1) for this sequence.

By the definition, sequence (x_k) is non-decreasing and $x_k - a_k \ge -\frac{d}{2}$ for all $1 \le k \le n$. Next we prove that

$$x_k - a_k \le \frac{d}{2} \quad \text{for all } 1 \le k \le n. \tag{2}$$

Consider an arbitrary index $1 \le k \le n$. Let $\ell \le k$ be the smallest index such that $x_k = x_\ell$. We have either $\ell = 1$, or $\ell \ge 2$ and $x_\ell > x_{\ell-1}$. In both cases,

$$x_k = x_\ell = a_\ell - \frac{d}{2}. \tag{3}$$

Since

$$a_\ell - a_k \le \max\{a_j : 1 \le j \le k\} - \min\{a_j :\ k \le j \le n\} = d_k \le d,$$

equality (3) implies

$$x_k - a_k = a_\ell - a_k - \frac{d}{2} \le d - \frac{d}{2} = \frac{d}{2}.$$

We obtained that $-\frac{d}{2} \le x_k - a_k \le \frac{d}{2}$ for all $1 \le k \le n$, so

$$\max\left\{|x_i - a_i| :\ 1 \le i \le n\right\} \le \frac{d}{2}.$$

We have equality because $|x_1 - a_1| = \frac{d}{2}$.

Solution 2. We present another construction of a sequence (x_i) for part (b).

For each $1 \le i \le n$, let

$$M_i = \max\{a_j : 1 \le j \le i\} \quad \text{and} \quad m_i = \min\{a_j :\ i \le j \le n\}.$$

For all $1 \le i < n$, we have

$$M_i = \max\{a_1, \ldots, a_i\} \le \max\{a_1, \ldots, a_i, a_{i+1}\} = M_{i+1}$$

and

$$m_i = \min\{a_i, a_{i+1}, \ldots, a_n\} \le \min\{a_{i+1}, \ldots, a_n\} = m_{i+1}.$$

Therefore sequences (M_i) and (m_i) are non-decreasing. Moreover, since a_i is listed in both definitions,

$$m_i \le a_i \le M_i.$$

To achieve equality in (1), set

$$x_i = \frac{M_i + m_i}{2}.$$

Since sequences (M_i) and (m_i) are non-decreasing, this sequence is non-decreasing as well.

From $d_i = M_i - m_i$ we obtain that

$$-\frac{d_i}{2} = \frac{m_i - M_i}{2} = x_i - M_i \le x_i - a_i \le x_i - m_i = \frac{M_i - m_i}{2} = \frac{d_i}{2}.$$

Therefore

$$\max\big\{|x_i - a_i| : 1 \le i \le n\big\} \le \max\left\{\frac{d_i}{2} : 1 \le i \le n\right\} = \frac{d}{2}.$$

Since the opposite inequality has been proved in part (a), we must have equality.

A2. Consider those functions $f\colon \mathbb{N} \to \mathbb{N}$ which satisfy the condition

$$f(m+n) \geq f(m) + f\big(f(n)\big) - 1 \tag{1}$$

for all $m, n \in \mathbb{N}$. Find all possible values of $f(2007)$.

($\mathbb{N}$ denotes the set of all positive integers.)

(Bulgaria)

Answer. 1, 2, ..., 2008.

Solution. Suppose that a function $f\colon \mathbb{N} \to \mathbb{N}$ satisfies (1). For arbitrary positive integers $m > n$, by (1) we have

$$f(m) = f\big(n + (m-n)\big) \geq f(n) + f\big(f(m-n)\big) - 1 \geq f(n),$$

so f is nondecreasing.

Function $f \equiv 1$ is an obvious solution. To find other solutions, assume that $f \not\equiv 1$ and take the smallest $a \in \mathbb{N}$ such that $f(a) > 1$. Then $f(b) \geq f(a) > 1$ for all integer $b \geq a$.

Suppose that $f(n) > n$ for some $n \in \mathbb{N}$. Then we have

$$f\big(f(n)\big) = f\Big(\big(f(n)-n\big)+n\Big) \geq f\big(f(n)-n\big) + f\big(f(n)\big) - 1,$$

so $f\big(f(n)-n\big) \leq 1$ and hence $f(n)-n < a$. Then there exists a maximal value of the expression $f(n)-n$; denote this value by c, and let $f(k)-k = c \geq 1$. Applying the monotonicity together with (1), we get

$$\begin{aligned}2k + c \geq f(2k) = f(k+k) &\geq f(k) + f\big(f(k)\big) - 1 \\ &\geq f(k) + f(k) - 1 = 2(k+c) - 1 = 2k + (2c-1),\end{aligned}$$

hence $c \leq 1$ and $f(n) \leq n+1$ for all $n \in \mathbb{N}$. In particular, $f(2007) \leq 2008$.

Now we present a family of examples showing that all values from 1 to 2008 can be realized. Let

$$f_j(n) = \max\{1, n+j-2007\} \quad \text{for } j = 1, 2, \ldots, 2007; \qquad f_{2008}(n) = \begin{cases} n, & 2007 \nmid n, \\ n+1, & 2007 \mid n. \end{cases}$$

We show that these functions satisfy the condition (1) and clearly $f_j(2007) = j$.

To check the condition (1) for the function f_j ($j \leq 2007$), note first that f_j is nondecreasing and $f_j(n) \leq n$, hence $f_j\big(f_j(n)\big) \leq f_j(n) \leq n$ for all $n \in \mathbb{N}$. Now, if $f_j(m) = 1$, then the inequality (1) is clear since $f_j(m+n) \geq f_j(n) \geq f_j\big(f_j(n)\big) = f_j(m) + f_j\big(f_j(n)\big) - 1$. Otherwise,

$$f_j(m) + f_j\big(f_j(n)\big) - 1 \leq (m+j-2007) + n = (m+n) + j - 2007 = f_j(m+n).$$

In the case $j = 2008$, clearly $n+1 \geq f_{2008}(n) \geq n$ for all $n \in \mathbb{N}$; moreover, $n+1 \geq f_{2008}\big(f_{2008}(n)\big)$ as well. Actually, the latter is trivial if $f_{2008}(n) = n$; otherwise, $f_{2008}(n) = n+1$, which implies $2007 \nmid n+1$ and hence $n+1 = f_{2008}(n+1) = f_{2008}\big(f_{2008}(n)\big)$.

So, if $2007 \mid m+n$, then

$$f_{2008}(m+n) = m+n+1 = (m+1) + (n+1) - 1 \geq f_{2008}(m) + f_{2008}\big(f_{2008}(n)\big) - 1.$$

Otherwise, $2007 \nmid m+n$, hence $2007 \nmid m$ or $2007 \nmid n$. In the former case we have $f_{2008}(m) = m$, while in the latter one $f_{2008}\big(f_{2008}(n)\big) = f_{2008}(n) = n$, providing

$$f_{2008}(m) + f_{2008}\big(f_{2008}(n)\big) - 1 \leq (m+n+1) - 1 = f_{2008}(m+n).$$

Comment. The examples above are not unique. The values $1, 2, \ldots, 2008$ can be realized in several ways. Here we present other two constructions for $j \le 2007$, without proof:

$$g_j(n) = \begin{cases} 1, & n < 2007, \\ j, & n = 2007, \\ n, & n > 2007; \end{cases} \qquad h_j(n) = \max\left\{1, \left\lfloor \frac{jn}{2007} \right\rfloor \right\}.$$

Also the example for $j = 2008$ can be generalized. In particular, choosing a divisor $d > 1$ of 2007, one can set

$$f_{2008,d}(n) = \begin{cases} n, & d \nmid n, \\ n+1, & d \mid n. \end{cases}$$

A3. Let n be a positive integer, and let x and y be positive real numbers such that $x^n+y^n=1$. Prove that

$$\left(\sum_{k=1}^{n}\frac{1+x^{2k}}{1+x^{4k}}\right)\left(\sum_{k=1}^{n}\frac{1+y^{2k}}{1+y^{4k}}\right)<\frac{1}{(1-x)(1-y)}.$$

(Estonia)

Solution 1. For each real $t\in(0,1)$,

$$\frac{1+t^2}{1+t^4}=\frac{1}{t}-\frac{(1-t)(1-t^3)}{t(1+t^4)}<\frac{1}{t}.$$

Substituting $t=x^k$ and $t=y^k$,

$$0<\sum_{k=1}^{n}\frac{1+x^{2k}}{1+x^{4k}}<\sum_{k=1}^{n}\frac{1}{x^k}=\frac{1-x^n}{x^n(1-x)}\quad\text{and}\quad 0<\sum_{k=1}^{n}\frac{1+y^{2k}}{1+y^{4k}}<\sum_{k=1}^{n}\frac{1}{y^k}=\frac{1-y^n}{y^n(1-y)}.$$

Since $1-y^n=x^n$ and $1-x^n=y^n$,

$$\frac{1-x^n}{x^n(1-x)}=\frac{y^n}{x^n(1-x)},\qquad \frac{1-y^n}{y^n(1-y)}=\frac{x^n}{y^n(1-y)}$$

and therefore

$$\left(\sum_{k=1}^{n}\frac{1+x^{2k}}{1+x^{4k}}\right)\left(\sum_{k=1}^{n}\frac{1+y^{2k}}{1+y^{4k}}\right)<\frac{y^n}{x^n(1-x)}\cdot\frac{x^n}{y^n(1-y)}=\frac{1}{(1-x)(1-y)}.$$

Solution 2. We prove

$$\left(\sum_{k=1}^{n}\frac{1+x^{2k}}{1+x^{4k}}\right)\left(\sum_{k=1}^{n}\frac{1+y^{2k}}{1+y^{4k}}\right)<\frac{\left(\frac{1+\sqrt{2}}{2}\ln 2\right)^2}{(1-x)(1-y)}<\frac{0.7001}{(1-x)(1-y)}. \tag{1}$$

The idea is to estimate each term on the left-hand side with the same constant. To find the upper bound for the expression $\frac{1+x^{2k}}{1+x^{4k}}$, consider the function $f(t)=\frac{1+t}{1+t^2}$ in interval $(0,1)$. Since

$$f'(t)=\frac{1-2t-t^2}{(1+t^2)^2}=\frac{(\sqrt{2}+1+t)(\sqrt{2}-1-t)}{(1+t^2)^2},$$

the function increases in interval $(0,\sqrt{2}-1]$ and decreases in $[\sqrt{2}-1,1)$. Therefore the maximum is at point $t_0=\sqrt{2}-1$ and

$$f(t)=\frac{1+t}{1+t^2}\le f(t_0)=\frac{1+\sqrt{2}}{2}=\alpha.$$

Applying this to each term on the left-hand side of (1), we obtain

$$\left(\sum_{k=1}^{n}\frac{1+x^{2k}}{1+x^{4k}}\right)\left(\sum_{k=1}^{n}\frac{1+y^{2k}}{1+y^{4k}}\right)\le n\alpha\cdot n\alpha=(n\alpha)^2. \tag{2}$$

To estimate $(1-x)(1-y)$ on the right-hand side, consider the function

$$g(t)=\ln(1-t^{1/n})+\ln\bigl(1-(1-t)^{1/n}\bigr).$$

Substituting s for $1-t$, we have

$$-ng'(t) = \frac{t^{1/n-1}}{1-t^{1/n}} - \frac{s^{1/n-1}}{1-s^{1/n}} = \frac{1}{st}\left(\frac{(1-t)t^{1/n}}{1-t^{1/n}} - \frac{(1-s)s^{1/n}}{1-s^{1/n}}\right) = \frac{h(t)-h(s)}{st}.$$

The function

$$h(t) = t^{1/n}\frac{1-t}{1-t^{1/n}} = \sum_{i=1}^{n} t^{i/n}$$

is obviously increasing for $t \in (0,1)$, hence for these values of t we have

$$g'(t) > 0 \iff h(t) < h(s) \iff t < s = 1-t \iff t < \frac{1}{2}.$$

Then, the maximum of $g(t)$ in $(0,1)$ is attained at point $t_1 = 1/2$ and therefore

$$g(t) \le g\left(\frac{1}{2}\right) = 2\ln(1-2^{-1/n}), \qquad t \in (0,1).$$

Substituting $t = x^n$, we have $1-t = y^n$, $(1-x)(1-y) = \exp g(t)$ and hence

$$(1-x)(1-y) = \exp g(t) \le (1-2^{-1/n})^2. \tag{3}$$

Combining (2) and (3), we get

$$\left(\sum_{k=1}^{n}\frac{1+x^{2k}}{1+x^{4k}}\right)\left(\sum_{k=1}^{n}\frac{1+y^{2k}}{1+y^{4k}}\right) \le (\alpha n)^2 \cdot 1 \le (\alpha n)^2\frac{(1-2^{-1/n})^2}{(1-x)(1-y)} = \frac{\left(\alpha n(1-2^{-1/n})\right)^2}{(1-x)(1-y)}.$$

Applying the inequality $1-\exp(-t) < t$ for $t = \dfrac{\ln 2}{n}$, we obtain

$$\alpha n(1-2^{-1/n}) = \alpha n\left(1-\exp\left(-\frac{\ln 2}{n}\right)\right) < \alpha n \cdot \frac{\ln 2}{n} = \alpha\ln 2 = \frac{1+\sqrt{2}}{2}\ln 2.$$

Hence,

$$\left(\sum_{k=1}^{n}\frac{1+x^{2k}}{1+x^{4k}}\right)\left(\sum_{k=1}^{n}\frac{1+y^{2k}}{1+y^{4k}}\right) < \frac{\left(\frac{1+\sqrt{2}}{2}\ln 2\right)^2}{(1-x)(1-y)}.$$

Comment. It is a natural idea to compare the sum $S_n(x) = \displaystyle\sum_{k=1}^{n}\frac{1+x^{2k}}{1+x^{4k}}$ with the integral $I_n(x) = \displaystyle\int_0^n \frac{1+x^{2t}}{1+x^{4t}}\mathrm{d}t$. Though computing the integral is quite standard, many difficulties arise. First, the integrand $\dfrac{1+x^{2k}}{1+x^{4k}}$ has an increasing segment and, depending on x, it can have a decreasing segment as well. So comparing $S_n(x)$ and $I_n(x)$ is not completely obvious. We can add a term to fix the estimate, e.g. $S_n \le I_n + (\alpha-1)$, but then the final result will be weak for the small values of n. Second, we have to minimize $(1-x)(1-y)I_n(x)I_n(y)$ which leads to very unpleasant computations.

However, by computer search we found that the maximum of $I_n(x)I_n(y)$ is at $x = y = 2^{-1/n}$, as well as the maximum of $S_n(x)S_n(y)$, and the latter is less. Hence, one can conjecture that the exact constant which can be put into the numerator on the right-hand side of (1) is

$$\left(\ln 2 \cdot \int_0^1 \frac{1+4^{-t}}{1+16^{-t}}\mathrm{d}t\right)^2 = \frac{1}{4}\left(\frac{1}{2}\ln\frac{17}{2} + \arctan 4 - \frac{\pi}{4}\right)^2 \approx 0.6484.$$

A4. Find all functions $f\colon \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$f\bigl(x + f(y)\bigr) = f(x + y) + f(y) \tag{1}$$

for all $x, y \in \mathbb{R}^+$. (Symbol $\mathbb{R}^+$ denotes the set of all positive real numbers.)

(Thaliand)

Answer. $f(x) = 2x$.

Solution 1. First we show that $f(y) > y$ for all $y \in \mathbb{R}^+$. Functional equation (1) yields $f\bigl(x + f(y)\bigr) > f(x + y)$ and hence $f(y) \neq y$ immediately. If $f(y) < y$ for some y, then setting $x = y - f(y)$ we get

$$f(y) = f\Bigl(\bigl(y - f(y)\bigr) + f(y)\Bigr) = f\Bigl(\bigl(y - f(y)\bigr) + y\Bigr) + f(y) > f(y),$$

contradiction. Therefore $f(y) > y$ for all $y \in \mathbb{R}^+$.

For $x \in \mathbb{R}^+$ define $g(x) = f(x) - x$; then $f(x) = g(x) + x$ and, as we have seen, $g(x) > 0$. Transforming (1) for function $g(x)$ and setting $t = x + y$,

$$f\bigl(t + g(y)\bigr) = f(t) + f(y),$$

$$g\bigl(t + g(y)\bigr) + t + g(y) = \bigl(g(t) + t\bigr) + \bigl(g(y) + y\bigr)$$

and therefore

$$g\bigl(t + g(y)\bigr) = g(t) + y \qquad \text{for all } t > y > 0. \tag{2}$$

Next we prove that function $g(x)$ is injective. Suppose that $g(y_1) = g(y_2)$ for some numbers $y_1, y_2 \in \mathbb{R}^+$. Then by (2),

$$g(t) + y_1 = g\bigl(t + g(y_1)\bigr) = g\bigl(t + g(y_2)\bigr) = g(t) + y_2$$

for all $t > \max\{y_1, y_2\}$. Hence, $g(y_1) = g(y_2)$ is possible only if $y_1 = y_2$.

Now let u, v be arbitrary positive numbers and $t > u + v$. Applying (2) three times,

$$g\bigl(t + g(u) + g(v)\bigr) = g\bigl(t + g(u)\bigr) + v = g(t) + u + v = g\bigl(t + g(u + v)\bigr).$$

By the injective property we conclude that $t + g(u) + g(v) = t + g(u + v)$, hence

$$g(u) + g(v) = g(u + v). \tag{3}$$

Since function $g(v)$ is positive, equation (3) also shows that g is an increasing function.

Finally we prove that $g(x) = x$. Combining (2) and (3), we obtain

$$g(t) + y = g\bigl(t + g(y)\bigr) = g(t) + g\bigl(g(y)\bigr)$$

and hence

$$g\bigl(g(y)\bigr) = y.$$

Suppose that there exists an $x \in \mathbb{R}^+$ such that $g(x) \neq x$. By the monotonicity of g, if $x > g(x)$ then $g(x) > g\bigl(g(x)\bigr) = x$. Similarly, if $x < g(x)$ then $g(x) < g\bigl(g(x)\bigr) = x$. Both cases lead to contradiction, so there exists no such x.

We have proved that $g(x) = x$ and therefore $f(x) = g(x) + x = 2x$ for all $x \in \mathbb{R}^+$. This function indeed satisfies the functional equation (1).

Comment. It is well-known that the additive property (3) together with $g(x) \geq 0$ (for $x > 0$) imply $g(x) = cx$. So, after proving (3), it is sufficient to test functions $f(x) = (c+1)x$.

Solution 2. We prove that $f(y) > y$ and introduce function $g(x) = f(x) - x > 0$ in the same way as in Solution 1.

For arbitrary $t > y > 0$, substitute $x = t - y$ into (1) to obtain

$$f\big(t + g(y)\big) = f(t) + f(y)$$

which, by induction, implies

$$f\big(t + ng(y)\big) = f(t) + nf(y) \qquad \text{for all } t > y > 0,\ n \in \mathbb{N}. \tag{4}$$

Take two arbitrary positive reals y and z and a third fixed number $t > \max\{y, z\}$. For each positive integer k, let $\ell_k = \left\lfloor k\dfrac{g(y)}{g(z)} \right\rfloor$. Then $t + kg(y) - \ell_k g(z) \geq t > z$ and, applying (4) twice,

$$f\big(t + kg(y) - \ell_k g(z)\big) + \ell_k f(z) = f\big(t + kg(y)\big) = f(t) + kf(y),$$
$$0 < \frac{1}{k} f\big(t + kg(y) - \ell_k g(z)\big) = \frac{f(t)}{k} + f(y) - \frac{\ell_k}{k} f(z).$$

As $k \to \infty$ we get

$$0 \leq \lim_{k\to\infty} \left(\frac{f(t)}{k} + f(y) - \frac{\ell_k}{k} f(z) \right) = f(y) - \frac{g(y)}{g(z)} f(z) = f(y) - \frac{f(y) - y}{f(z) - z} f(z)$$

and therefore

$$\frac{f(y)}{y} \leq \frac{f(z)}{z}.$$

Exchanging variables y and z, we obtain the reverse inequality. Hence, $\dfrac{f(y)}{y} = \dfrac{f(z)}{z}$ for arbitrary y and z; so function $\dfrac{f(x)}{x}$ is constant, $f(x) = cx$.

Substituting back into (1), we find that $f(x) = cx$ is a solution if and only if $c = 2$. So the only solution for the problem is $f(x) = 2x$.

A5. Let $c > 2$, and let $a(1), a(2), \ldots$ be a sequence of nonnegative real numbers such that

$$a(m+n) \leq 2a(m) + 2a(n) \quad \text{for all } m, n \geq 1, \tag{1}$$

and

$$a(2^k) \leq \frac{1}{(k+1)^c} \quad \text{for all } k \geq 0. \tag{2}$$

Prove that the sequence $a(n)$ is bounded.

(Croatia)

Solution 1. For convenience, define $a(0) = 0$; then condition (1) persists for all pairs of nonnegative indices.

Lemma 1. For arbitrary nonnegative indices $n_1, \ldots, n_k$, we have

$$a\left(\sum_{i=1}^{k} n_i\right) \leq \sum_{i=1}^{k} 2^i a(n_i) \tag{3}$$

and

$$a\left(\sum_{i=1}^{k} n_i\right) \leq 2k \sum_{i=1}^{k} a(n_i). \tag{4}$$

Proof. Inequality (3) is proved by induction on k. The base case $k = 1$ is trivial, while the induction step is provided by

$$a\left(\sum_{i=1}^{k+1} n_i\right) = a\left(n_1 + \sum_{i=2}^{k+1} n_i\right) \leq 2a(n_1) + 2a\left(\sum_{i=1}^{k} n_{i+1}\right) \leq 2a(n_1) + 2\sum_{i=1}^{k} 2^i a(n_{i+1}) = \sum_{i=1}^{k+1} 2^i a(n_i).$$

To establish (4), first the inequality

$$a\left(\sum_{i=1}^{2^d} n_i\right) \leq 2^d \sum_{i=1}^{2^d} a(n_i)$$

can be proved by an obvious induction on d. Then, turning to (4), we find an integer d such that $2^{d-1} < k \leq 2^d$ to obtain

$$a\left(\sum_{i=1}^{k} n_i\right) = a\left(\sum_{i=1}^{k} n_i + \sum_{i=k+1}^{2^d} 0\right) \leq 2^d\left(\sum_{i=1}^{k} a(n_i) + \sum_{i=k+1}^{2^d} a(0)\right) = 2^d \sum_{i=1}^{k} a(n_i) \leq 2k \sum_{i=1}^{k} a(n_i).$$

$\square$

Fix an increasing unbounded sequence $0 = M_0 < M_1 < M_2 < \ldots$ of real numbers; the exact values will be defined later. Let n be an arbitrary positive integer and write

$$n = \sum_{i=0}^{d} \varepsilon_i \cdot 2^i, \qquad \text{where } \varepsilon_i \in \{0, 1\}.$$

Set $\varepsilon_i = 0$ for $i > d$, and take some positive integer f such that $M_f > d$. Applying (3), we get

$$a(n) = a\left(\sum_{k=1}^{f} \sum_{M_{k-1} \leq i < M_k} \varepsilon_i \cdot 2^i\right) \leq \sum_{k=1}^{f} 2^k a\left(\sum_{M_{k-1} \leq i < M_k} \varepsilon_i \cdot 2^i\right).$$

Note that there are less than $M_k - M_{k-1} + 1$ integers in interval $[M_{k-1}, M_k)$; hence, using (4) we have

$$\begin{aligned}
a(n) &\le \sum_{k=1}^{f} 2^k \cdot 2(M_k - M_{k-1} + 1) \sum_{M_{k-1} \le i < M_k} \varepsilon_i \cdot a(2^i) \\
&\le \sum_{k=1}^{f} 2^k \cdot 2(M_k - M_{k-1} + 1)^2 \max_{M_{k-1} \le i < M_k} a(2^i) \\
&\le \sum_{k=1}^{f} 2^{k+1}(M_k + 1)^2 \cdot \frac{1}{(M_{k-1}+1)^c} = \sum_{k=1}^{f} \left(\frac{M_k + 1}{M_{k-1}+1}\right)^2 \frac{2^{k+1}}{(M_{k-1}+1)^{c-2}}.
\end{aligned}$$

Setting $M_k = 4^{k/(c-2)} - 1$, we obtain

$$a(n) \le \sum_{k=1}^{f} 4^{2/(c-2)} \frac{2^{k+1}}{\left(4^{(k-1)/(c-2)}\right)^{c-2}} = 8 \cdot 4^{2/(c-2)} \sum_{k=1}^{f} \left(\frac{1}{2}\right)^k < 8 \cdot 4^{2/(c-2)},$$

and the sequence $a(n)$ is bounded.

Solution 2.

Lemma 2. Suppose that $s_1, \ldots, s_k$ are positive integers such that

$$\sum_{i=1}^{k} 2^{-s_i} \le 1.$$

Then for arbitrary positive integers $n_1, \ldots, n_k$ we have

$$a\left(\sum_{i=1}^{k} n_i\right) \le \sum_{i=1}^{k} 2^{s_i} a(n_i).$$

Proof. Apply an induction on k. The base cases are $k = 1$ (trivial) and $k = 2$ (follows from the condition (1)). Suppose that $k > 2$. We can assume that $s_1 \le s_2 \le \cdots \le s_k$. Note that

$$\sum_{i=1}^{k-1} 2^{-s_i} \le 1 - 2^{-s_{k-1}},$$

since the left-hand side is a fraction with the denominator $2^{s_{k-1}}$, and this fraction is less than 1. Define $s'_{k-1} = s_{k-1} - 1$ and $n'_{k-1} = n_{k-1} + n_k$; then we have

$$\sum_{i=1}^{k-2} 2^{-s_i} + 2^{-s'_{k-1}} \le (1 - 2 \cdot 2^{-s_{k-1}}) + 2^{1-s_{k-1}} = 1.$$

Now, the induction hypothesis can be applied to achieve

$$\begin{aligned}
a\left(\sum_{i=1}^{k} n_i\right) = a\left(\sum_{i=1}^{k-2} n_i + n'_{k-1}\right) &\le \sum_{i=1}^{k-2} 2^{s_i} a(n_i) + 2^{s'_{k-1}} a(n'_{k-1}) \\
&\le \sum_{i=1}^{k-2} 2^{s_i} a(n_i) + 2^{s_{k-1}-1} \cdot 2\big(a(n_{k-1}) + a(n_k)\big) \\
&\le \sum_{i=1}^{k-2} 2^{s_i} a(n_i) + 2^{s_{k-1}} a(n_{k-1}) + 2^{s_k} a(n_k).
\end{aligned}$$

$\square$

Let $q = c/2 > 1$. Take an arbitrary positive integer n and write

$$n = \sum_{i=1}^{k} 2^{u_i}, \qquad 0 \le u_1 < u_2 < \cdots < u_k.$$

Choose $s_i = \lfloor \log_2 (u_i+1)^q \rfloor + d$ $(i = 1, \ldots, k)$ for some integer d. We have

$$\sum_{i=1}^{k} 2^{-s_i} = 2^{-d} \sum_{i=1}^{k} 2^{-\lfloor \log_2 (u_i+1)^q \rfloor},$$

and we choose d in such a way that

$$\frac{1}{2} < \sum_{i=1}^{k} 2^{-s_i} \le 1.$$

In particular, this implies

$$2^d < 2 \sum_{i=1}^{k} 2^{-\lfloor \log_2 (u_i+1)^q \rfloor} < 4 \sum_{i=1}^{k} \frac{1}{(u_i+1)^q}.$$

Now, by Lemma 2 we obtain

$$\begin{aligned} a(n) = a\left(\sum_{i=1}^{k} 2^{u_i}\right) &\le \sum_{i=1}^{k} 2^{s_i} a(2^{u_i}) \le \sum_{i=1}^{k} 2^d (u_i+1)^q \cdot \frac{1}{(u_i+1)^{2q}} \\ &= 2^d \sum_{i=1}^{k} \frac{1}{(u_i+1)^q} < 4 \left(\sum_{i=1}^{k} \frac{1}{(u_i+1)^q}\right)^2, \end{aligned}$$

which is bounded since $q > 1$.

Comment 1. In fact, Lemma 2 (applied to the case $n_i = 2^{u_i}$ only) provides a sharp bound for any $a(n)$. Actually, let $b(k) = \dfrac{1}{(k+1)^c}$ and consider the sequence

$$a(n) = \min\left\{ \sum_{i=1}^{k} 2^{s_i} b(u_i) \;\middle|\; k \in \mathbb{N}, \quad \sum_{i=1}^{k} 2^{-s_i} \le 1, \quad \sum_{i=1}^{k} 2^{u_i} = n \right\}. \tag{5}$$

We show that this sequence satisfies the conditions of the problem. Take two arbitrary indices m and n. Let

$$\begin{aligned} a(m) &= \sum_{i=1}^{k} 2^{s_i} b(u_i), & \sum_{i=1}^{k} 2^{-s_i} &\le 1, & \sum_{i=1}^{k} 2^{u_i} &= m; \\ a(n) &= \sum_{i=1}^{l} 2^{r_i} b(w_i), & \sum_{i=1}^{l} 2^{-r_i} &\le 1, & \sum_{i=1}^{l} 2^{w_i} &= n. \end{aligned}$$

Then we have

$$\sum_{i=1}^{k} 2^{-1-s_i} + \sum_{i=1}^{l} 2^{-1-r_i} \le \frac{1}{2} + \frac{1}{2} = 1, \qquad \sum_{i=1}^{k} 2^{u_i} + \sum_{i=1}^{l} 2^{w_i} = m + n,$$

so by (5) we obtain

$$a(n+m) \le \sum_{i=1}^{k} 2^{1+s_i} b(u_i) + \sum_{i=1}^{l} 2^{1+r_i} b(w_i) = 2a(m) + 2a(n).$$

Comment 2. The condition $c > 2$ is sharp; we show that the sequence (5) is not bounded if $c \leq 2$.

First, we prove that for an arbitrary n the minimum in (5) is attained with a sequence (u_i) consisting of distinct numbers. To the contrary, assume that $u_{k-1} = u_k$. Replace u_{k-1} and u_k by a single number $u'_{k-1} = u_k + 1$, and s_{k-1} and s_k by $s'_{k-1} = \min\{s_{k-1}, s_k\}$. The modified sequences provide a better bound since

$$2^{s'_{k-1}} b(u'_{k-1}) = 2^{s'_{k-1}} b(u_k + 1) < 2^{s_{k-1}} b(u_{k-1}) + 2^{s_k} b(u_k)$$

(we used the fact that $b(k)$ is decreasing). This is impossible.

Hence, the claim is proved, and we can assume that the minimum is attained with $u_1 < \cdots < u_k$; then

$$n = \sum_{i=1}^{k} 2^{u_i}$$

is simply the binary representation of n. (In particular, it follows that $a(2^n) = b(n)$ for each n.)

Now we show that the sequence $\bigl(a(2^k - 1)\bigr)$ is not bounded. For some $s_1, \dots, s_k$ we have

$$a(2^k - 1) = a\left(\sum_{i=1}^{k} 2^{i-1}\right) = \sum_{i=1}^{k} 2^{s_i} b(i-1) = \sum_{i=1}^{k} \frac{2^{s_i}}{i^c}.$$

By the Cauchy–Schwarz inequality we get

$$a(2^k - 1) = a(2^k - 1) \cdot 1 \geq \left(\sum_{i=1}^{k} \frac{2^{s_i}}{i^c}\right)\left(\sum_{i=1}^{k} \frac{1}{2^{s_i}}\right) \geq \left(\sum_{i=1}^{k} \frac{1}{i^{c/2}}\right)^2,$$

which is unbounded.

For $c \leq 2$, it is also possible to show a concrete counterexample. Actually, one can prove that the sequence

$$a\left(\sum_{i=1}^{k} 2^{u_i}\right) = \sum_{i=1}^{k} \frac{i}{(u_i + 1)^2} \qquad (0 \leq u_1 < \ldots < u_k)$$

satisfies (1) and (2) but is not bounded.

A6. Let $a_1, a_2, \ldots, a_{100}$ be nonnegative real numbers such that $a_1^2 + a_2^2 + \ldots + a_{100}^2 = 1$. Prove that

$$a_1^2 a_2 + a_2^2 a_3 + \ldots + a_{100}^2 a_1 < \frac{12}{25}.$$

(Poland)

Solution. Let $S = \sum_{k=1}^{100} a_k^2 a_{k+1}$. (As usual, we consider the indices modulo 100, e.g. we set $a_{101} = a_1$ and $a_{102} = a_2$.)

Applying the Cauchy-Schwarz inequality to sequences (a_{k+1}) and $(a_k^2 + 2a_{k+1}a_{k+2})$, and then the AM-GM inequality to numbers a_{k+1}^2 and a_{k+2}^2,

$$\begin{aligned}
(3S)^2 &= \left(\sum_{k=1}^{100} a_{k+1}(a_k^2 + 2a_{k+1}a_{k+2})\right)^2 \le \left(\sum_{k=1}^{100} a_{k+1}^2\right)\left(\sum_{k=1}^{100}(a_k^2 + 2a_{k+1}a_{k+2})^2\right) \qquad (1)\\
&= 1 \cdot \sum_{k=1}^{100}(a_k^2 + 2a_{k+1}a_{k+2})^2 = \sum_{k=1}^{100}(a_k^4 + 4a_k^2 a_{k+1}a_{k+2} + 4a_{k+1}^2 a_{k+2}^2)\\
&\le \sum_{k=1}^{100}\bigl(a_k^4 + 2a_k^2(a_{k+1}^2 + a_{k+2}^2) + 4a_{k+1}^2 a_{k+2}^2\bigr) = \sum_{k=1}^{100}\bigl(a_k^4 + 6a_k^2 a_{k+1}^2 + 2a_k^2 a_{k+2}^2\bigr).
\end{aligned}$$

Applying the trivial estimates

$$\sum_{k=1}^{100}(a_k^4 + 2a_k^2 a_{k+1}^2 + 2a_k^2 a_{k+2}^2) \le \left(\sum_{k=1}^{100} a_k^2\right)^2 \quad \text{and} \quad \sum_{k=1}^{100} a_k^2 a_{k+1}^2 \le \left(\sum_{i=1}^{50} a_{2i-1}^2\right)\left(\sum_{j=1}^{50} a_{2j}^2\right),$$

we obtain that

$$(3S)^2 \le \left(\sum_{k=1}^{100} a_k^2\right)^2 + 4\left(\sum_{i=1}^{50} a_{2i-1}^2\right)\left(\sum_{j=1}^{50} a_{2j}^2\right) \le 1 + \left(\sum_{i=1}^{50} a_{2i-1}^2 + \sum_{j=1}^{50} a_{2j}^2\right)^2 = 2,$$

hence

$$S \le \frac{\sqrt{2}}{3} \approx 0.4714 < \frac{12}{25} = 0.48.$$

Comment 1. By applying the Lagrange multiplier method, one can see that the maximum is attained at values of a_i satisfying

$$a_{k-1}^2 + 2a_k a_{k+1} = 2\lambda a_k \qquad (2)$$

for all $k = 1, 2, \ldots, 100$. Though this system of equations seems hard to solve, it can help to find the estimate above; it may suggest to have a closer look at the expression $a_{k-1}^2 a_k + 2a_k^2 a_{k+1}$.

Moreover, if the numbers $a_1, \ldots, a_{100}$ satisfy (2), we have equality in (1). (See also Comment 3.)

Comment 2. It is natural to ask what is the best constant c_n in the inequality

$$a_1^2 a_2 + a_2^2 a_3 + \ldots + a_n^2 a_1 \le c_n\bigl(a_1^2 + a_2^2 + \ldots + a_n^2\bigr)^{3/2}. \qquad (3)$$

For $1 \le n \le 4$ one may prove $c_n = 1/\sqrt{n}$ which is achieved when $a_1 = a_2 = \ldots = a_n$. However, the situation changes completely if $n \ge 5$. In this case we do not know the exact value of c_n. By computer search it can be found that $c_n \approx 0.4514$ and it is realized for example if

$$a_1 \approx 0.5873, \quad a_2 \approx 0.6771, \quad a_3 \approx 0.4224, \quad a_4 \approx 0.1344, \quad a_5 \approx 0.0133$$

and $a_k \approx 0$ for $k \ge 6$. This example also proves that $c_n > 0.4513$.

Comment 3. The solution can be improved in several ways to give somewhat better bounds for c_n. Here we show a variant which proves $c_n < 0.4589$ for $n \geq 5$.

The value of c_n does not change if negative values are also allowed in (3). So the problem is equivalent to maximizing

$$f(a_1, a_2, \ldots, a_n) = a_1^2 a_2 + a_2^2 a_3 + \ldots + a_n^2 a_1$$

on the unit sphere $a_1^2 + a_2^2 + \ldots + a_n^2 = 1$ in $\mathbb{R}^n$. Since the unit sphere is compact, the function has a maximum and we can apply the Lagrange multiplier method; for each maximum point there exists a real number λ such that

$$a_{k-1}^2 + 2a_k a_{k+1} = \lambda \cdot 2a_k \qquad \text{for all } k = 1, 2, \ldots, n.$$

Then

$$3S = \sum_{k=1}^{n} \left(a_{k-1}^2 a_k + 2a_k^2 a_{k+1}\right) = \sum_{k=1}^{n} 2\lambda a_k^2 = 2\lambda$$

and therefore

$$a_{k-1}^2 + 2a_k a_{k+1} = 3Sa_k \qquad \text{for all } k = 1, 2, \ldots, n. \tag{4}$$

From (4) we can derive

$$9S^2 = \sum_{k=1}^{n} (3Sa_k)^2 = \sum_{k=1}^{n} \left(a_{k-1}^2 + 2a_k a_{k+1}\right)^2 = \sum_{k=1}^{n} a_k^4 + 4\sum_{k=1}^{n} a_k^2 a_{k+1}^2 + 4\sum_{k=1}^{n} a_k^2 a_{k+1} a_{k+2} \tag{5}$$

and

$$3S^2 = \sum_{k=1}^{n} 3Sa_{k-1}^2 a_k = \sum_{k=1}^{n} a_{k-1}^2 \left(a_{k-1}^2 + 2a_k a_{k+1}\right) = \sum_{k=1}^{n} a_k^4 + 2\sum_{k=1}^{n} a_k^2 a_{k+1} a_{k+2}. \tag{6}$$

Let p be a positive number. Combining (5) and (6) and applying the AM-GM inequality,

$$\begin{aligned}
(9+3p)S^2 &= (1+p)\sum_{k=1}^{n} a_k^4 + 4\sum_{k=1}^{n} a_k^2 a_{k+1}^2 + (4+2p)\sum_{k=1}^{n} a_k^2 a_{k+1} a_{k+2} \\
&\leq (1+p)\sum_{k=1}^{n} a_k^4 + 4\sum_{k=1}^{n} a_k^2 a_{k+1}^2 + \sum_{k=1}^{n}\left(2(1+p)a_k^2 a_{k+2}^2 + \frac{(2+p)^2}{2(1+p)} a_k^2 a_{k+1}^2\right) \\
&= (1+p)\sum_{k=1}^{n}\left(a_k^4 + 2a_k^2 a_{k+1}^2 + 2a_k^2 a_{k+2}^2\right) + \left(4 + \frac{(2+p)^2}{2(1+p)} - 2(1+p)\right)\sum_{k=1}^{n} a_k^2 a_{k+1}^2 \\
&\leq (1+p)\left(\sum_{k=1}^{n} a_k^2\right)^2 + \frac{8+4p-3p^2}{2(1+p)}\sum_{k=1}^{n} a_k^2 a_{k+1}^2 \\
&= (1+p) + \frac{8+4p-3p^2}{2(1+p)}\sum_{k=1}^{n} a_k^2 a_{k+1}^2.
\end{aligned}$$

Setting $p = \dfrac{2+2\sqrt{7}}{3}$ which is the positive root of $8 + 4p - 3p^2 = 0$, we obtain

$$S \leq \sqrt{\frac{1+p}{9+3p}} = \sqrt{\frac{5+2\sqrt{7}}{33+6\sqrt{7}}} \approx 0.458879.$$

A7. Let $n > 1$ be an integer. In the space, consider the set

$$S = \{(x, y, z) \mid x, y, z \in \{0, 1, \dots, n\},\ x + y + z > 0\}.$$

Find the smallest number of planes that jointly contain all $(n+1)^3 - 1$ points of S but none of them passes through the origin.

(Netherlands)

Answer. $3n$ planes.

Solution. It is easy to find $3n$ such planes. For example, planes $x = i$, $y = i$ or $z = i$ $(i = 1, 2, \dots, n)$ cover the set S but none of them contains the origin. Another such collection consists of all planes $x + y + z = k$ for $k = 1, 2, \dots, 3n$.

We show that $3n$ is the smallest possible number.

Lemma 1. Consider a nonzero polynomial $P(x_1, \dots, x_k)$ in k variables. Suppose that P vanishes at all points $(x_1, \dots, x_k)$ such that $x_1, \dots, x_k \in \{0, 1, \dots, n\}$ and $x_1 + \cdots + x_k > 0$, while $P(0, 0, \dots, 0) \neq 0$. Then $\deg P \geq kn$.

Proof. We use induction on k. The base case $k = 0$ is clear since $P \neq 0$. Denote for clarity $y = x_k$.

Let $R(x_1, \dots, x_{k-1}, y)$ be the residue of P modulo $Q(y) = y(y-1)\dots(y-n)$. Polynomial $Q(y)$ vanishes at each $y = 0, 1, \dots, n$, hence $P(x_1, \dots, x_{k-1}, y) = R(x_1, \dots, x_{k-1}, y)$ for all $x_1, \dots, x_{k-1}, y \in \{0, 1, \dots, n\}$. Therefore, R also satisfies the condition of the Lemma; moreover, $\deg_y R \leq n$. Clearly, $\deg R \leq \deg P$, so it suffices to prove that $\deg R \geq nk$.

Now, expand polynomial R in the powers of y:

$$R(x_1, \dots, x_{k-1}, y) = R_n(x_1, \dots, x_{k-1})y^n + R_{n-1}(x_1, \dots, x_{k-1})y^{n-1} + \cdots + R_0(x_1, \dots, x_{k-1}).$$

We show that polynomial $R_n(x_1, \dots, x_{k-1})$ satisfies the condition of the induction hypothesis.

Consider the polynomial $T(y) = R(0, \dots, 0, y)$ of degree $\leq n$. This polynomial has n roots $y = 1, \dots, n$; on the other hand, $T(y) \not\equiv 0$ since $T(0) \neq 0$. Hence $\deg T = n$, and its leading coefficient is $R_n(0, 0, \dots, 0) \neq 0$. In particular, in the case $k = 1$ we obtain that coefficient R_n is nonzero.

Similarly, take any numbers $a_1, \dots, a_{k-1} \in \{0, 1, \dots, n\}$ with $a_1 + \cdots + a_{k-1} > 0$. Substituting $x_i = a_i$ into $R(x_1, \dots, x_{k-1}, y)$, we get a polynomial in y which vanishes at all points $y = 0, \dots, n$ and has degree $\leq n$. Therefore, this polynomial is null, hence $R_i(a_1, \dots, a_{k-1}) = 0$ for all $i = 0, 1, \dots, n$. In particular, $R_n(a_1, \dots, a_{k-1}) = 0$.

Thus, the polynomial $R_n(x_1, \dots, x_{k-1})$ satisfies the condition of the induction hypothesis. So, we have $\deg R_n \geq (k-1)n$ and $\deg P \geq \deg R \geq \deg R_n + n \geq kn$. □

Now we can finish the solution. Suppose that there are N planes covering all the points of S but not containing the origin. Let their equations be $a_i x + b_i y + c_i z + d_i = 0$. Consider the polynomial

$$P(x, y, z) = \prod_{i=1}^{N} (a_i x + b_i y + c_i z + d_i).$$

It has total degree N. This polynomial has the property that $P(x_0, y_0, z_0) = 0$ for any $(x_0, y_0, z_0) \in S$, while $P(0, 0, 0) \neq 0$. Hence by Lemma 1 we get $N = \deg P \geq 3n$, as desired.

Comment 1. There are many other collections of $3n$ planes covering the set S but not covering the origin.

Solution 2. We present a different proof of the main Lemma 1. Here we confine ourselves to the case $k = 3$, which is applied in the solution, and denote the variables by x, y and z. (The same proof works for the general statement as well.)

The following fact is known with various proofs; we provide one possible proof for the completeness.

Lemma 2. For arbitrary integers $0 \le m < n$ and for an arbitrary polynomial $P(x)$ of degree m,

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} P(k) = 0. \tag{1}$$

Proof. We use an induction on n. If $n = 1$, then $P(x)$ is a constant polynomial, hence $P(1) - P(0) = 0$, and the base is proved.

For the induction step, define $P_1(x) = P(x+1) - P(x)$. Then clearly $\deg P_1 = \deg P - 1 = m - 1 < n - 1$, hence by the induction hypothesis we get

$$\begin{aligned}
0 &= -\sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} P_1(k) = \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} \big(P(k) - P(k+1)\big) \\
&= \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} P(k) - \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} P(k+1) \\
&= \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} P(k) + \sum_{k=1}^{n} (-1)^k \binom{n-1}{k-1} P(k) \\
&= P(0) + \sum_{k=1}^{n-1} (-1)^k \left(\binom{n-1}{k-1} + \binom{n-1}{k} \right) P(k) + (-1)^n P(n) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} P(k). \quad \square
\end{aligned}$$

Now return to the proof of Lemma 1. Suppose, to the contrary, that $\deg P = N < 3n$. Consider the sum

$$\Sigma = \sum_{i=0}^{n} \sum_{j=0}^{n} \sum_{k=0}^{n} (-1)^{i+j+k} \binom{n}{i} \binom{n}{j} \binom{n}{k} P(i,j,k).$$

The only nonzero term in this sum is $P(0,0,0)$ and its coefficient is $\binom{n}{0}^3 = 1$; therefore $\Sigma = P(0,0,0) \neq 0$.

On the other hand, if $P(x,y,z) = \sum\limits_{\alpha+\beta+\gamma \le N} p_{\alpha,\beta,\gamma} x^\alpha y^\beta z^\gamma$, then

$$\begin{aligned}
\Sigma &= \sum_{i=0}^{n} \sum_{j=0}^{n} \sum_{k=0}^{n} (-1)^{i+j+k} \binom{n}{i} \binom{n}{j} \binom{n}{k} \sum_{\alpha+\beta+\gamma \le N} p_{\alpha,\beta,\gamma} i^\alpha j^\beta k^\gamma \\
&= \sum_{\alpha+\beta+\gamma \le N} p_{\alpha,\beta,\gamma} \left(\sum_{i=0}^{n} (-1)^i \binom{n}{i} i^\alpha \right) \left(\sum_{j=0}^{n} (-1)^j \binom{n}{j} j^\beta \right) \left(\sum_{k=0}^{n} (-1)^k \binom{n}{k} k^\gamma \right).
\end{aligned}$$

Consider an arbitrary term in this sum. We claim that it is zero. Since $N < 3n$, one of three inequalities $\alpha < n$, $\beta < n$ or $\gamma < n$ is valid. For the convenience, suppose that $\alpha < n$. Applying Lemma 2 to polynomial x^α, we get $\sum\limits_{i=0}^{n} (-1)^i \binom{n}{i} i^\alpha = 0$, hence the term is zero as required.

This yields $\Sigma = 0$ which is a contradiction. Therefore, $\deg P \ge 3n$.

Comment 2. The proof does not depend on the concrete coefficients in Lemma 2. Instead of this Lemma, one can simply use the fact that there exist numbers $\alpha_0, \alpha_1, \ldots, \alpha_n$ $(\alpha_0 \neq 0)$ such that

$$\sum_{k=0}^{n} \alpha_k k^m = 0 \qquad \text{for every } 0 \leq m < n.$$

This is a system of homogeneous linear equations in variables α_i. Since the number of equations is less than the number of variables, the only nontrivial thing is that there exists a solution with $\alpha_0 \neq 0$. It can be shown in various ways.

Combinatorics

C1. Let $n>1$ be an integer. Find all sequences $a_1, a_2, \ldots, a_{n^2+n}$ satisfying the following conditions:

(a) $a_i \in \{0,1\}$ for all $1 \le i \le n^2+n$;

(b) $a_{i+1}+a_{i+2}+\ldots+a_{i+n} < a_{i+n+1}+a_{i+n+2}+\ldots+a_{i+2n}$ for all $0 \le i \le n^2-n$.

(Serbia)

Answer. Such a sequence is unique. It can be defined as follows:

$$a_{u+vn} = \begin{cases} 0, & u+v \le n, \\ 1, & u+v \ge n+1 \end{cases} \qquad \text{for all } 1 \le u \le n \text{ and } 0 \le v \le n. \tag{1}$$

The terms can be arranged into blocks of length n as

$$(\underbrace{0\ \ldots\ 0}_{n})\,(\underbrace{0\ \ldots\ 0}_{n-1}\,1)\,(\underbrace{0\ \ldots\ 0}_{n-2}\,1\,1)\ldots(\underbrace{0\ \ldots\ 0}_{n-v}\,\underbrace{1\ \ldots\ 1}_{v})\ldots(0\,\underbrace{1\ \ldots\ 1}_{n-1})\,(\underbrace{1\ \ldots\ 1}_{n}).$$

Solution 1. Consider a sequence (a_i) satisfying the conditions. For arbitrary integers $0 \le k \le l \le n^2+n$ denote $S(k,l] = a_{k+1}+\cdots+a_l$. (If $k=l$ then $S(k,l]=0$.) Then condition (b) can be rewritten as $S(i,i+n] < S(i+n,i+2n]$ for all $0 \le i \le n^2-n$. Notice that for $0 \le k \le l \le m \le n^2+n$ we have $S(k,m] = S(k,l]+S(l,m]$.

By condition (b),

$$0 \le S(0,n] < S(n,2n] < \cdots < S(n^2,n^2+n] \le n.$$

We have only $n+1$ distinct integers in the interval $[0,n]$; hence,

$$S\big(vn,(v+1)n\big] = v \qquad \text{for all } 0 \le v \le n. \tag{2}$$

In particular, $S(0,n]=0$ and $S(n^2,n^2+n]=n$, therefore

$$a_1 = a_2 = \ldots = a_n = 0, \tag{3}$$
$$a_{n^2+1} = a_{n^2+2} = \ldots = a_{n^2+n} = 1. \tag{4}$$

Subdivide sequence (a_i) into $n+1$ blocks, each consisting of n consecutive terms, and number them from 0 to n. We show by induction on v that the vth blocks has the form

$$(\underbrace{0\ \ldots\ 0}_{n-v}\,\underbrace{1\ \ldots\ 1}_{v}).$$

The base case $v=0$ is provided by (3).

Consider the vth block for $v > 0$. By (2), it contains some "ones". Let the first "one" in this block be at the uth position (that is, $a_{u+vn} = 1$). By the induction hypothesis, the $(v-1)$th and vth blocks of (a_i) have the form

$$(\underbrace{0\,\dots\,0\,\dots\,0}_{n-v+1}\,\underbrace{1\,\dots\,1}_{v-1})\,(\underbrace{0\,\dots\,0}_{u-1}\,1\,*\,\dots\,*),$$

where each star can appear to be any binary digit. Observe that $u \le n - v + 1$, since the sum in this block is v. Then, the fragment of length n bracketed above has exactly $(v-1)+1$ ones, i. e. $S\big(u + (v-1)n, u + vn\big] = v$. Hence,

$$v = S\big(u + (v-1)n, u + vn\big] < S\big(u + vn, u + (v+1)n\big] < \cdots < S\big(u + (n-1)n, u + n^2\big] \le n;$$

we have $n - v + 1$ distinct integers in the interval $[v, n]$, therefore $S(u + (t-1)n, u + tn] = t$ for each $t = v, \dots, n$.

Thus, the end of sequence (a_i) looks as following:

$$\underbrace{(\overbrace{0\,\dots\,0}^{u\text{ zeroes}}\,0\,\dots\,0\,1\,\dots\,1)}_{\sum = v-1}\,\underbrace{(0\,\dots\,0\,1\,*\,\dots\,*)}_{\sum = v}\,\underbrace{(*\,\dots\,*\,*\,\dots\,*)}_{\sum = v+1}\,\dots\,\underbrace{(1\,\dots\,1\,\overbrace{1\,\dots\,1}^{n-u\text{ ones}})}_{\sum = n}$$

(each bracketed fragment contains n terms). Computing in two ways the sum of all digits above, we obtain $n - u = v - 1$ and $u = n - v + 1$. Then, the first $n - v$ terms in the vth block are zeroes, and the next v terms are ones, due to the sum of all terms in this block. The statement is proved.

We are left to check that the sequence obtained satisfies the condition. Notice that $a_i \le a_{i+n}$ for all $1 \le i \le n^2$. Moreover, if $1 \le u \le n$ and $0 \le v \le n-1$, then $a_{u+vn} < a_{u+vn+n}$ exactly when $u + v = n$. In this case we have $u + vn = n + v(n-1)$.

Consider now an arbitrary index $0 \le i \le n^2 - n$. Clearly, there exists an integer v such that $n + v(n-1) \in [i+1, i+n]$. Then, applying the above inequalities we obtain that condition (b) is valid.

Solution 2. Similarly to Solution 1, we introduce the notation $S(k, l]$ and obtain (2), (3), and (4) in the same way. The sum of all elements of the sequence can be computed as

$$S(0, n^2 + n] = S(0, n] + S(n, 2n] + \ldots + S(n^2, n^2 + n] = 0 + 1 + \ldots + n.$$

For an arbitrary integer $0 \le u \le n$, consider the numbers

$$S(u, u+n] < S(u+n, u+2n] < \ldots < S\big(u + (n-1)n, u + n^2\big]. \tag{5}$$

They are n distinct integers from the $n+1$ possible values $0, 1, 2, \dots, n$. Denote by m the "missing" value which is not listed. We determine m from $S(0, n^2 + n]$. Write this sum as

$$S(0, n^2+n] = S(0, u] + S(u, u+n] + S(u+n, u+2n] + \ldots + S(u+(n-1)n, u+n^2] + S(u+n^2, n^2+n].$$

Since $a_1 = a_2 = \ldots = a_u = 0$ and $a_{u+n^2+1} = \ldots = a_{n^2+n} = 1$, we have $S(0, u] = 0$ and $S(u + n^2, n + n^2] = n - u$. Then

$$0 + 1 + \ldots + n = S(0, n^2 + n] = 0 + \big((0 + 1 + \ldots + n) - m\big) + (n - u),$$

so $m = n - u$.

Hence, the numbers listed in (5) are $0, 1, \ldots, n-u-1$ and $n-u+1, \ldots, n$, respectively, therefore

$$S\big(u+vn, u+(v+1)n\big] = \begin{cases} v, & v \le n-u-1, \\ v+1, & v \ge n-u \end{cases} \qquad \text{for all } 0 \le u \le n,\ 0 \le v \le n-1. \tag{6}$$

Conditions (6), together with (3), provide a system of linear equations in variables a_i. Now we solve this system and show that the solution is unique and satisfies conditions (a) and (b).

First, observe that any solution of the system (3), (6) satisfies the condition (b). By the construction, equations (6) immediately imply (5). On the other hand, all inequalities mentioned in condition (b) are included into the chain (5) for some value of u.

Next, note that the system (3), (6) is redundant. The numbers $S\big(kn, (k+1)n\big]$, where $1 \le k \le n-1$, appear twice in (6). For $u = 0$ and $v = k$ we have $v \le n-u-1$, and (6) gives $S\big(kn, (k+1)n\big] = v = k$. For $u = n$ and $v = k-1$ we have $v \ge n-u$ and we obtain the same value, $S\big(kn, (k+1)n\big] = v+1 = k$. Therefore, deleting one equation from each redundant pair, we can make every sum $S(k, k+n]$ appear exactly once on the left-hand side in (6).

Now, from (3), (6), the sequence (a_i) can be reconstructed inductively by

$$a_1 = a_2 = \ldots = a_{n-1} = 0, \qquad a_{k+n} = S(k, k+n] - (a_{k+1} + a_{k+2} + \ldots + a_{k+n-1}) \quad (0 \le k \le n^2),$$

taking the values of $S(k, k+n]$ from (6). This means first that there exists at most one solution of our system. Conversely, the constructed sequence obviously satisfies all equations (3), (6) (the only missing equation is $a_n = 0$, which follows from $S(0, n] = 0$). Hence it satisfies condition (b), and we are left to check condition (a) only.

For arbitrary integers $1 \le u, t \le n$ we get

$$\begin{aligned} a_{u+tn} - a_{u+(t-1)n} &= S\big(u+(t-1)n, u+tn\big] - S\big((u-1)+(t-1)n, (u-1)+tn\big] \\ &= \begin{cases} (t-1)-(t-1) = 0, & t \le n-u, \\ t-(t-1) = 1, & t = n-u+1, \\ t-t = 0, & t \ge n-u+2. \end{cases} \end{aligned}$$

Since $a_u = 0$, we have

$$a_{u+vn} = a_{u+vn} - a_u = \sum_{t=1}^{v} (a_{u+tn} - a_{u+(t-1)n})$$

for all $1 \le u, v \le n$. If $v < n-u+1$ then all terms are 0 on the right-hand side. If $v \ge n-u+1$, then variable t attains the value $n-u+1$ once. Hence,

$$a_{u+vn} = \begin{cases} 0, & u+v \le n, \\ 1, & u+v \ge n+1, \end{cases}$$

according with (1). Note that the formula is valid for $v = 0$ as well.

Finally, we presented the direct formula for (a_i), and we have proved that it satisfies condition (a). So, the solution is complete.

C2. A unit square is dissected into $n > 1$ rectangles such that their sides are parallel to the sides of the square. Any line, parallel to a side of the square and intersecting its interior, also intersects the interior of some rectangle. Prove that in this dissection, there exists a rectangle having no point on the boundary of the square.

(Japan)

Solution 1. Call the directions of the sides of the square horizontal and vertical. A horizontal or vertical line, which intersects the interior of the square but does not intersect the interior of any rectangle, will be called a *splitting line*. A rectangle having no point on the boundary of the square will be called an *interior rectangle*.

Suppose, to the contrary, that there exists a dissection of the square into more than one rectangle, such that no interior rectangle and no splitting line appear. Consider such a dissection with the least possible number of rectangles. Notice that this number of rectangles is greater than 2, otherwise their common side provides a splitting line.

If there exist two rectangles having a common side, then we can replace them by their union (see Figure 1). The number of rectangles was greater than 2, so in a new dissection it is greater than 1. Clearly, in the new dissection, there is also no splitting line as well as no interior rectangle. This contradicts the choice of the original dissection.

Denote the initial square by $ABCD$, with A and B being respectively the lower left and lower right vertices. Consider those two rectangles a and b containing vertices A and B, respectively. (Note that $a \neq b$, otherwise its top side provides a splitting line.) We can assume that the height of a is not greater than that of b. Then consider the rectangle c neighboring to the lower right corner of a (it may happen that $c = b$). By aforementioned, the heights of a and c are distinct. Then two cases are possible.

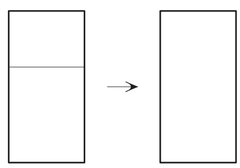


Figure 1

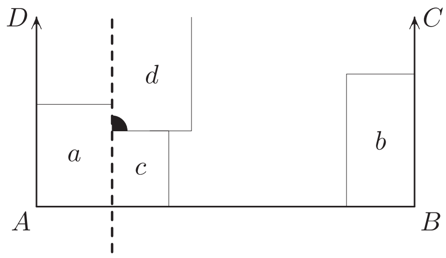


Figure 2

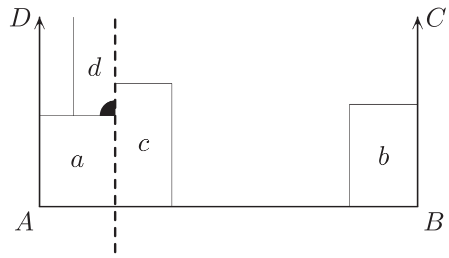


Figure 3

Case 1. The height of c is less than that of a. Consider the rectangle d which is adjacent to both a and c, i.e. the one containing the angle marked in Figure 2. This rectangle has no common point with BC (since a is not higher than b), as well as no common point with AB or with AD (obviously). Then d has a common point with CD, and its left side provides a splitting line. Contradiction.

Case 2. The height of c is greater than that of a. Analogously, consider the rectangle d containing the angle marked on Figure 3. It has no common point with AD (otherwise it has a common side with a), as well as no common point with AB or with BC (obviously). Then d has a common point with CD. Hence its right side provides a splitting line, and we get the contradiction again.

Solution 2. Again, we suppose the contrary. Consider an arbitrary counterexample. Then we know that each rectangle is attached to at least one side of the square. Observe that a rectangle cannot be attached to two opposite sides, otherwise one of its sides lies on a splitting line.

We say that two rectangles are *opposite* if they are attached to opposite sides of $ABCD$. We claim that there exist two opposite rectangles having a common point.

Consider the union L of all rectangles attached to the left. Assume, to the contrary, that L has no common point with the rectangles attached to the right. Take a polygonal line p connecting the top and the bottom sides of the square and passing close from the right to the boundary of L (see Figure 4). Then all its points belong to the rectangles attached either to the top or to the bottom. Moreover, the upper end-point of p belongs to a rectangle attached to the top, and the lower one belongs to an other rectangle attached to the bottom. Hence, there is a point on p where some rectangles attached to the top and to the bottom meet each other. So, there always exists a pair of neighboring opposite rectangles.

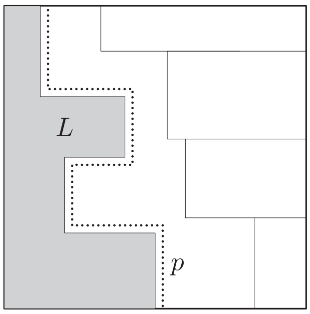


Figure 4

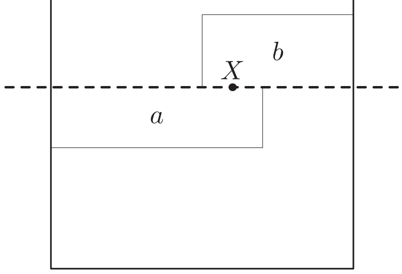


Figure 5

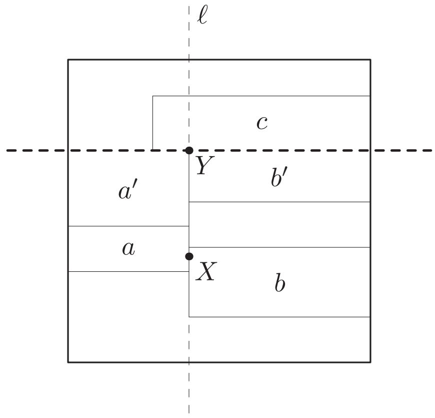


Figure 6

Now, take two opposite neighboring rectangles a and b. We can assume that a is attached to the left and b is attached to the right. Let X be their common point. If X belongs to their horizontal sides (in particular, X may appear to be a common vertex of a and b), then these sides provide a splitting line (see Figure 5). Otherwise, X lies on the vertical sides. Let ℓ be the line containing these sides.

Since ℓ is not a splitting line, it intersects the interior of some rectangle. Let c be such a rectangle, closest to X; we can assume that c lies above X. Let Y be the common point of ℓ and the bottom side of c (see Figure 6). Then Y is also a vertex of two rectangles lying below c.

So, let Y be the upper-right and upper-left corners of the rectangles a' and b', respectively. Then a' and b' are situated not lower than a and b, respectively (it may happen that $a = a'$ or $b = b'$). We claim that a' is attached to the left. If $a = a'$ then of course it is. If $a \neq a'$ then a' is above a, below c and to the left from b'. Hence, it can be attached to the left only.

Analogously, b' is attached to the right. Now, the top sides of these two rectangles pass through Y, hence they provide a splitting line again. This last contradiction completes the proof.

C3. Find all positive integers n, for which the numbers in the set $S=\{1,2,\ldots,n\}$ can be colored red and blue, with the following condition being satisfied: the set $S\times S\times S$ contains exactly 2007 ordered triples (x,y,z) such that (i) x,y,z are of the same color and (ii) $x+y+z$ is divisible by n.

(Netherlands)

Answer. $n=69$ and $n=84$.

Solution. Suppose that the numbers $1,2,\ldots,n$ are colored red and blue. Denote by R and B the sets of red and blue numbers, respectively; let $|R|=r$ and $|B|=b=n-r$. Call a triple $(x,y,z)\in S\times S\times S$ *monochromatic* if x,y,z have the same color, and *bichromatic* otherwise. Call a triple (x,y,z) *divisible* if $x+y+z$ is divisible by n. We claim that there are exactly r^2-rb+b^2 divisible monochromatic triples.

For any pair $(x,y)\in S\times S$ there exists a unique $z_{x,y}\in S$ such that the triple $(x,y,z_{x,y})$ is divisible; so there are exactly n^2 divisible triples. Furthermore, if a divisible triple (x,y,z) is bichromatic, then among x, y, z there are either one blue and two red numbers, or vice versa. In both cases, exactly one of the pairs (x,y), (y,z) and (z,x) belongs to the set $R\times B$. Assign such pair to the triple (x,y,z).

Conversely, consider any pair $(x,y)\in R\times B$, and denote $z=z_{x,y}$. Since $x\neq y$, the triples (x,y,z), (y,z,x) and (z,x,y) are distinct, and (x,y) is assigned to each of them. On the other hand, if (x,y) is assigned to some triple, then this triple is clearly one of those mentioned above. So each pair in $R\times B$ is assigned exactly three times.

Thus, the number of bichromatic divisible triples is three times the number of elements in $R\times B$, and the number of monochromatic ones is $n^2-3rb=(r+b)^2-3rb=r^2-rb+b^2$, as claimed.

So, to find all values of n for which the desired coloring is possible, we have to find all n, for which there exists a decomposition $n=r+b$ with $r^2-rb+b^2=2007$. Therefore, $9\mid r^2-rb+b^2=(r+b)^2-3rb$. From this it consequently follows that $3\mid r+b$, $3\mid rb$, and then $3\mid r$, $3\mid b$. Set $r=3s$, $b=3c$. We can assume that $s\geq c$. We have $s^2-sc+c^2=223$.

Furthermore,

$$892=4(s^2-sc+c^2)=(2c-s)^2+3s^2\geq 3s^2\geq 3s^2-3c(s-c)=3(s^2-sc+c^2)=669,$$

so $297\geq s^2\geq 223$ and $17\geq s\geq 15$. If $s=15$ then

$$c(15-c)=c(s-c)=s^2-(s^2-sc+c^2)=15^2-223=2$$

which is impossible for an integer c. In a similar way, if $s=16$ then $c(16-c)=33$, which is also impossible. Finally, if $s=17$ then $c(17-c)=66$, and the solutions are $c=6$ and $c=11$. Hence, $(r,b)=(51,18)$ or $(r,b)=(51,33)$, and the possible values of n are $n=51+18=69$ and $n=51+33=84$.

Comment. After the formula for the number of monochromatic divisible triples is found, the solution can be finished in various ways. The one presented is aimed to decrease the number of considered cases.

C4. Let $A_0 = (a_1, \dots, a_n)$ be a finite sequence of real numbers. For each $k \geq 0$, from the sequence $A_k = (x_1, \dots, x_n)$ we construct a new sequence A_{k+1} in the following way.

1. We choose a partition $\{1, \dots, n\} = I \cup J$, where I and J are two disjoint sets, such that the expression

$$\left|\sum_{i\in I} x_i - \sum_{j\in J} x_j\right|$$

attains the smallest possible value. (We allow the sets I or J to be empty; in this case the corresponding sum is 0.) If there are several such partitions, one is chosen arbitrarily.

2. We set $A_{k+1} = (y_1, \dots, y_n)$, where $y_i = x_i + 1$ if $i \in I$, and $y_i = x_i - 1$ if $i \in J$.

Prove that for some k, the sequence A_k contains an element x such that $|x| \geq n/2$.

(Iran)

Solution.

Lemma. Suppose that all terms of the sequence $(x_1, \dots, x_n)$ satisfy the inequality $|x_i| < a$. Then there exists a partition $\{1, 2, \dots, n\} = I \cup J$ into two disjoint sets such that

$$\left|\sum_{i\in I} x_i - \sum_{j\in J} x_j\right| < a. \tag{1}$$

Proof. Apply an induction on n. The base case $n = 1$ is trivial. For the induction step, consider a sequence $(x_1, \dots, x_n)$ $(n > 1)$. By the induction hypothesis there exists a splitting $\{1, \dots, n-1\} = I' \cup J'$ such that

$$\left|\sum_{i\in I'} x_i - \sum_{j\in J'} x_j\right| < a.$$

For convenience, suppose that $\sum\limits_{i\in I'} x_i \geq \sum\limits_{j\in J'} x_j$. If $x_n \geq 0$ then choose $I = I'$, $J = J \cup \{n\}$; otherwise choose $I = I' \cup \{n\}$, $J = J'$. In both cases, we have $\sum\limits_{i\in I'} x_i - \sum\limits_{j\in J'} x_j \in [0, a)$ and $|x_n| \in [0, a)$; hence

$$\sum_{i\in I} x_i - \sum_{j\in J} x_j = \sum_{i\in I'} x_i - \sum_{j\in J'} x_j - |x_n| \in (-a, a),$$

as desired. □

Let us turn now to the problem. To the contrary, assume that for all k, all the numbers in A_k lie in interval $(-n/2, n/2)$. Consider an arbitrary sequence $A_k = (b_1, \dots, b_n)$. To obtain the term b_i, we increased and decreased number a_i by one several times. Therefore $b_i - a_i$ is always an integer, and there are not more than n possible values for b_i. So, there are not more than n^n distinct possible sequences A_k, and hence two of the sequences $A_1, A_2, \dots, A_{n^n+1}$ should be identical, say $A_p = A_q$ for some $p < q$.

For any positive integer k, let S_k be the sum of squares of elements in A_k. Consider two consecutive sequences $A_k = (x_1, \dots, x_n)$ and $A_{k+1} = (y_1, \dots, y_n)$. Let $\{1, 2, \dots, n\} = I \cup J$ be the partition used in this step — that is, $y_i = x_i + 1$ for all $i \in I$ and $y_j = x_j - 1$ for all $j \in J$. Since the value of $\left|\sum\limits_{i\in I} x_i - \sum\limits_{j\in J} x_j\right|$ is the smallest possible, the Lemma implies that it is less than $n/2$. Then we have

$$S_{k+1} - S_k = \sum_{i\in I}\left((x_i+1)^2 - x_i^2\right) + \sum_{j\in J}\left((x_j-1)^2 - x_j^2\right) = n + 2\left(\sum_{i\in I} x_i - \sum_{j\in J} x_j\right) > n - 2\cdot\frac{n}{2} = 0.$$

Thus we obtain $S_q > S_{q-1} > \dots > S_p$. This is impossible since $A_p = A_q$ and hence $S_p = S_q$.

C5. In the Cartesian coordinate plane define the strip $S_n = \{(x, y) \mid n \le x < n+1\}$ for every integer n. Assume that each strip S_n is colored either red or blue, and let a and b be two distinct positive integers. Prove that there exists a rectangle with side lengths a and b such that its vertices have the same color.

(Romania)

Solution. If S_n and S_{n+a} have the same color for some integer n, then we can choose the rectangle with vertices $(n, 0) \in S_n$, $(n, b) \in S_n$, $(n+a, 0) \in S_{n+a}$, and $(n+a, b) \in S_{n+a}$, and we are done. So it can be assumed that S_n and S_{n+a} have opposite colors for each n.

Similarly, it also can be assumed that S_n and S_{n+b} have opposite colors. Then, by induction on $|p| + |q|$, we obtain that for arbitrary integers p and q, strips S_n and $S_{n+pa+qb}$ have the same color if $p+q$ is even, and these two strips have opposite colors if $p+q$ is odd.

Let $d = \gcd(a, b)$, $a_1 = a/d$ and $b_1 = b/d$. Apply the result above for $p = b_1$ and $q = -a_1$. The strips S_0 and $S_{0+b_1 a - a_1 b}$ are identical and therefore they have the same color. Hence, $a_1 + b_1$ is even. By the construction, a_1 and b_1 are coprime, so this is possible only if both are odd.

Without loss of generality, we can assume $a > b$. Then $a_1 > b_1 \ge 1$, so $a_1 \ge 3$.

Choose integers k and ℓ such that $ka_1 - \ell b_1 = 1$ and therefore $ka - \ell b = d$. Since a_1 and b_1 are odd, $k + \ell$ is odd as well. Hence, for every integer n, strips S_n and $S_{n+ka-\ell b} = S_{n+d}$ have opposite colors. This also implies that the coloring is periodic with period $2d$, i.e. strips S_n and S_{n+2d} have the same color for every n.

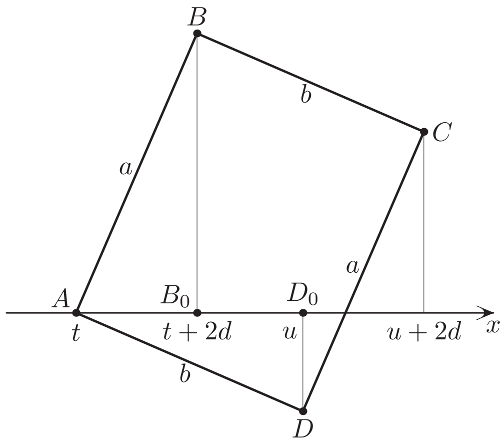


Figure 1

We will construct the desired rectangle $ABCD$ with $AB = CD = a$ and $BC = AD = b$ in a position such that vertex A lies on the x-axis, and the projection of side AB onto the x-axis is of length $2d$ (see Figure 1). This is possible since $a = a_1 d > 2d$. The coordinates of the vertices will have the forms

$$A = (t, 0), \quad B = (t + 2d, y_1), \quad C = (u + 2d, y_2), \quad D = (u, y_3).$$

Let $\varphi = \sqrt{a_1^2 - 4}$. By Pythagoras' theorem,

$$y_1 = BB_0 = \sqrt{a^2 - 4d^2} = d\sqrt{a_1^2 - 4} = d\varphi.$$

So, by the similar triangles ADD_0 and BAB_0, we have the constraint

$$u - t = AD_0 = \frac{AD}{AB} \cdot BB_0 = \frac{bd}{a}\varphi \tag{1}$$

for numbers t and u. Computing the numbers y_2 and y_3 is not required since they have no effect to the colors.

Observe that the number φ is irrational, because φ^2 is an integer, but φ is not: $a_1 > \varphi \geq \sqrt{a_1^2 - 2a_1 + 2} > a_1 - 1$.

By the periodicity, points A and B have the same color; similarly, points C and D have the same color. Furthermore, these colors depend only on the values of t and u. So it is sufficient to choose numbers t and u such that vertices A and D have the same color.

Let w be the largest positive integer such that there exist w consecutive strips $S_{n_0}, S_{n_0+1}, \ldots, S_{n_0+w-1}$ with the same color, say red. (Since S_{n_0+d} must be blue, we have $w \leq d$.) We will choose t from the interval $(n_0, n_0 + w)$.

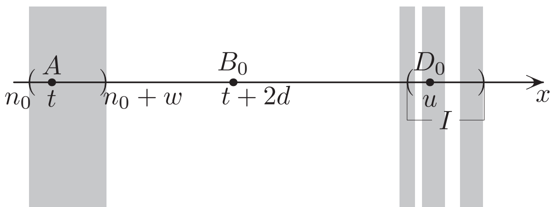


Figure 2

Consider the interval $I = \left(n_0 + \frac{bd}{a}\varphi, n_0 + \frac{bd}{a}\varphi + w\right)$ on the x-axis (see Figure 2). Its length is w, and the end-points are irrational. Therefore, this interval intersects $w + 1$ consecutive strips. Since at most w consecutive strips may have the same color, interval I must contain both red and blue points. Choose $u \in I$ such that the line $x = u$ is red and set $t = u - \frac{bd}{a}\varphi$, according to the constraint (1). Then $t \in (n_0, n_0 + w)$ and $A = (t, 0)$ is red as well as $D = (u, y_3)$.

Hence, variables u and t can be set such that they provide a rectangle with four red vertices.

Comment. The statement is false for squares, i.e. in the case $a = b$. If strips S_{2ka}, S_{2ka+1}, ..., $S_{(2k+1)a-1}$ are red, and strips $S_{(2k+1)a}$, $S_{(2k+1)a+1}$, ..., $S_{(2k+2)a-1}$ are blue for every integer k, then each square of size $a \times a$ has at least one red and at least one blue vertex as well.

C6. In a mathematical competition some competitors are friends; friendship is always mutual. Call a group of competitors a *clique* if each two of them are friends. The number of members in a clique is called its *size*.

It is known that the largest size of cliques is even. Prove that the competitors can be arranged in two rooms such that the largest size of cliques in one room is the same as the largest size of cliques in the other room.

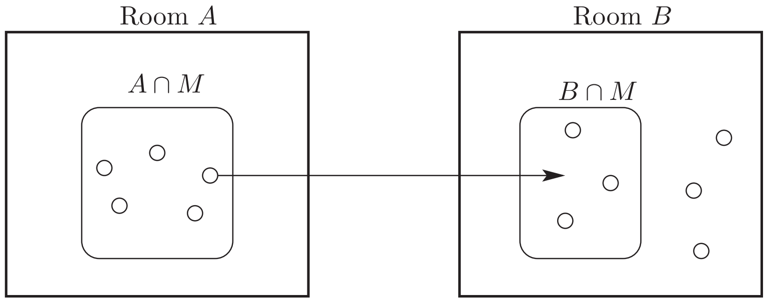
(Russia)

Solution. We present an algorithm to arrange the competitors. Let the two rooms be *Room A* and *Room B*. We start with an initial arrangement, and then we modify it several times by sending one person to the other room. At any state of the algorithm, A and B denote the sets of the competitors in the rooms, and $c(A)$ and $c(B)$ denote the largest sizes of cliques in the rooms, respectively.

Step 1. *Let M be one of the cliques of largest size, $|M| = 2m$. Send all members of M to Room A and all other competitors to Room B.*

Since M is a clique of the largest size, we have $c(A) = |M| \geq c(B)$.

Step 2. *While $c(A) > c(B)$, send one person from Room A to Room B.*



Note that $c(A) > c(B)$ implies that Room A is not empty.

In each step, $c(A)$ decreases by one and $c(B)$ increases by at most one. So at the end we have $c(A) \leq c(B) \leq c(A) + 1$.

We also have $c(A) = |A| \geq m$ at the end. Otherwise we would have at least $m+1$ members of M in Room B and at most $m-1$ in Room A, implying $c(B)-c(A) \geq (m+1)-(m-1) = 2$.

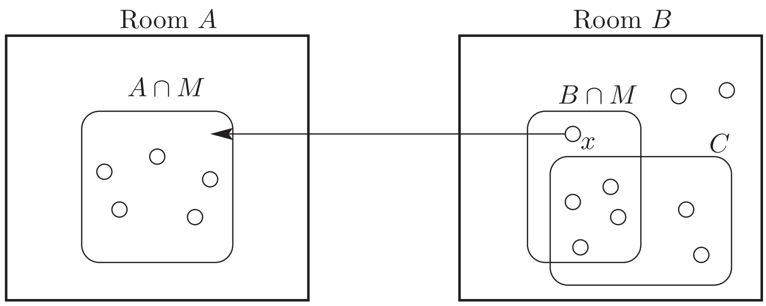
Step 3. *Let $k = c(A)$. If $c(B) = k$ then STOP.*

If we reached $c(A) = c(B) = k$ then we have found the desired arrangement.

In all other cases we have $c(B) = k + 1$.

From the estimate above we also know that $k = |A| = |A \cap M| \geq m$ and $|B \cap M| \leq m$.

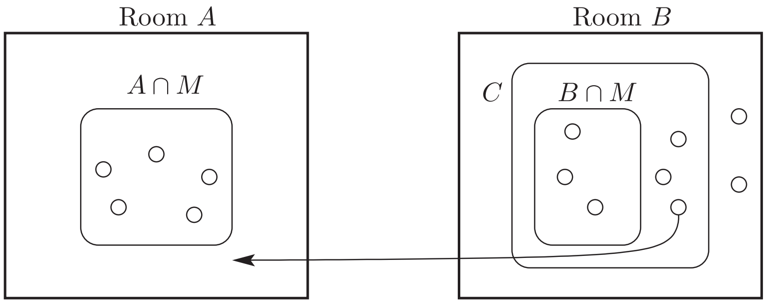
Step 4. *If there exists a competitor $x \in B \cap M$ and a clique $C \subset B$ such that $|C| = k + 1$ and $x \notin C$, then move x to Room A and STOP.*



After moving x back to Room A, we will have $k + 1$ members of M in Room A, thus $c(A) = k + 1$. Due to $x \notin C$, $c(B) = |C|$ is not decreased, and after this step we have $c(A) = c(B) = k + 1$.

If there is no such competitor x, then in Room B, all cliques of size $k+1$ contain $B \cap M$ as a subset.

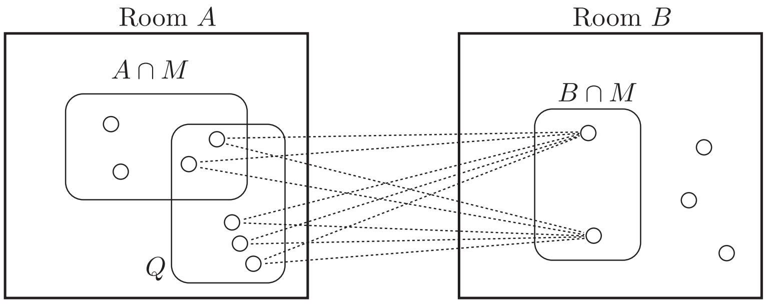
Step 5. *While $c(B) = k+1$, choose a clique $C \subset B$ such that $|C| = k+1$ and move one member of $C \setminus M$ to Room A.*



Note that $|C| = k+1 > m \geq |B \cap M|$, so $C \setminus M$ cannot be empty.

Every time we move a single person from Room B to Room A, so $c(B)$ decreases by at most 1. Hence, at the end of this loop we have $c(B) = k$.

In Room A we have the clique $A \cap M$ with size $|A \cap M| = k$ thus $c(A) \geq k$. We prove that there is no clique of larger size there. Let $Q \subset A$ be an arbitrary clique. We show that $|Q| \leq k$.



In Room A, and specially in set Q, there can be two types of competitors:

– Some members of M. Since M is a clique, they are friends with all members of $B \cap M$.

– Competitors which were moved to Room A in Step 5. Each of them has been in a clique with $B \cap M$ so they are also friends with all members of $B \cap M$.

Hence, all members of Q are friends with all members of $B \cap M$. Sets Q and $B \cap M$ are cliques themselves, so $Q \cup (B \cap M)$ is also a clique. Since M is a clique of the largest size,

$$|M| \geq |Q \cup (B \cap M)| = |Q| + |B \cap M| = |Q| + |M| - |A \cap M|,$$

therefore

$$|Q| \leq |A \cap M| = k.$$

Finally, after Step 5 we have $c(A) = c(B) = k$.

Comment. Obviously, the statement is false without the assumption that the largest clique size is even.

C7. Let $\alpha < \dfrac{3-\sqrt{5}}{2}$ be a positive real number. Prove that there exist positive integers n and $p > \alpha \cdot 2^n$ for which one can select $2p$ pairwise distinct subsets $S_1, \ldots, S_p, T_1, \ldots, T_p$ of the set $\{1, 2, \ldots, n\}$ such that $S_i \cap T_j \neq \varnothing$ for all $1 \leq i, j \leq p$.

(Austria)

Solution. Let k and m be positive integers (to be determined later) and set $n = km$. Decompose the set $\{1, 2, \ldots, n\}$ into k disjoint subsets, each of size m; denote these subsets by $A_1, \ldots, A_k$. Define the following families of sets:

$$\mathcal{S} = \big\{S \subset \{1, 2, \ldots, n\} : \ \forall i \ S \cap A_i \neq \varnothing\big\},$$
$$\mathcal{T}_1 = \big\{T \subset \{1, 2, \ldots, n\} : \ \exists i \ A_i \subset T\big\}, \qquad \mathcal{T} = \mathcal{T}_1 \setminus \mathcal{S}.$$

For each set $T \in \mathcal{T} \subset \mathcal{T}_1$, there exists an index $1 \leq i \leq k$ such that $A_i \subset T$. Then for all $S \in \mathcal{S}$, $S \cap T \supset S \cap A_i \neq \varnothing$. Hence, each $S \in \mathcal{S}$ and each $T \in \mathcal{T}$ have at least one common element.

Below we show that the numbers m and k can be chosen such that $|\mathcal{S}|, |\mathcal{T}| > \alpha \cdot 2^n$. Then, choosing $p = \min\big\{|\mathcal{S}|, |\mathcal{T}|\big\}$, one can select the desired $2p$ sets $S_1, \ldots, S_p$ and $T_1, \ldots, T_p$ from families $\mathcal{S}$ and $\mathcal{T}$, respectively. Since families $\mathcal{S}$ and $\mathcal{T}$ are disjoint, sets S_i and T_j will be pairwise distinct.

To count the sets $S \in \mathcal{S}$, observe that each A_i has $2^m - 1$ nonempty subsets so we have $2^m - 1$ choices for $S \cap A_i$. These intersections uniquely determine set S, so

$$|\mathcal{S}| = (2^m - 1)^k. \tag{1}$$

Similarly, if a set $H \subset \{1, 2, \ldots, n\}$ does not contain a certain set A_i then we have $2^m - 1$ choices for $H \cap A_i$: all subsets of A_i, except A_i itself. Therefore, the complement of $\mathcal{T}_1$ contains $(2^m - 1)^k$ sets and

$$|\mathcal{T}_1| = 2^{km} - (2^m - 1)^k. \tag{2}$$

Next consider the family $\mathcal{S} \setminus \mathcal{T}_1$. If a set S intersects all A_i but does not contain any of them, then there exists $2^m - 2$ possible values for each $S \cap A_i$: all subsets of A_i except $\varnothing$ and A_i. Therefore the number of such sets S is $(2^m - 2)^k$, so

$$|\mathcal{S} \setminus \mathcal{T}_1| = (2^m - 2)^k. \tag{3}$$

From (1), (2), and (3) we obtain

$$|\mathcal{T}| = |\mathcal{T}_1| - |\mathcal{S} \cap \mathcal{T}_1| = |\mathcal{T}_1| - \big(|\mathcal{S}| - |\mathcal{S} \setminus \mathcal{T}_1|\big) = 2^{km} - 2(2^m - 1)^k + (2^m - 2)^k.$$

Let $\delta = \dfrac{3-\sqrt{5}}{2}$ and $k = k(m) = \big[2^m \log \frac{1}{\delta}\big]$. Then

$$\lim_{m \to \infty} \frac{|\mathcal{S}|}{2^{km}} = \lim_{m \to \infty} \left(1 - \frac{1}{2^m}\right)^k = \exp\left(-\lim_{m \to \infty} \frac{k}{2^m}\right) = \delta$$

and similarly

$$\lim_{m \to \infty} \frac{|\mathcal{T}|}{2^{km}} = 1 - 2 \lim_{m \to \infty} \left(1 - \frac{1}{2^m}\right)^k + \lim_{m \to \infty} \left(1 - \frac{2}{2^m}\right)^k = 1 - 2\delta + \delta^2 = \delta.$$

Hence, if m is sufficiently large then $\dfrac{|\mathcal{S}|}{2^{mk}}$ and $\dfrac{|\mathcal{T}|}{2^{mk}}$ are greater than α (since $\alpha < \delta$). So $|\mathcal{S}|, |\mathcal{T}| > \alpha \cdot 2^{mk} = \alpha \cdot 2^n$.

Comment. It can be proved that the constant $\dfrac{3-\sqrt{5}}{2}$ is sharp. Actually, if $S_1, \ldots, S_p, T_1, \ldots, T_p$ are distinct subsets of $\{1, 2, \ldots, n\}$ such that each S_i intersects each T_j, then $p < \dfrac{3-\sqrt{5}}{2} \cdot 2^n$.

C8. Given a convex n-gon P in the plane. For every three vertices of P, consider the triangle determined by them. Call such a triangle *good* if all its sides are of unit length.

Prove that there are not more than $\frac{2}{3}n$ good triangles.

(Ukraine)

Solution. Consider all good triangles containing a certain vertex A. The other two vertices of any such triangle lie on the circle ω_A with unit radius and center A. Since P is convex, all these vertices lie on an arc of angle less than $180°$. Let $L_A R_A$ be the shortest such arc, oriented clockwise (see Figure 1). Each of segments AL_A and AR_A belongs to a unique good triangle. We say that the good triangle with side AL_A is *assigned counterclockwise* to A, and the second one, with side AR_A, is *assigned clockwise* to A. In those cases when there is a single good triangle containing vertex A, this triangle is assigned to A twice.

There are at most two assignments to each vertex of the polygon. (Vertices which do not belong to any good triangle have no assignment.) So the number of assignments is at most $2n$.

Consider an arbitrary good triangle ABC, with vertices arranged clockwise. We prove that ABC is assigned to its vertices at least three times. Then, denoting the number of good triangles by t, we obtain that the number K of all assignments is at most $2n$, while it is not less than $3t$. Then $3t \leq K \leq 2n$, as required.

Actually, we prove that triangle ABC is assigned either counterclockwise to C or clockwise to B. Then, by the cyclic symmetry of the vertices, we obtain that triangle ABC is assigned either counterclockwise to A or clockwise to C, and either counterclockwise to B or clockwise to A, providing the claim.

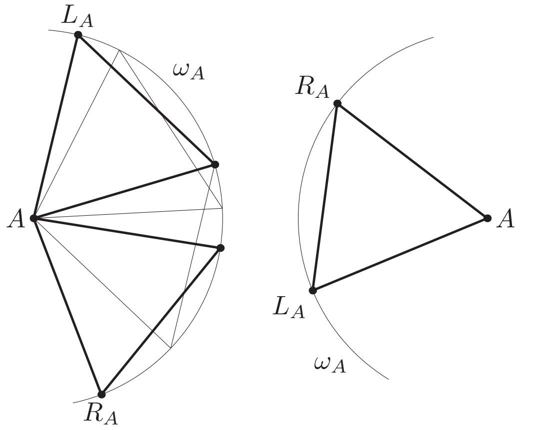


Figure 1

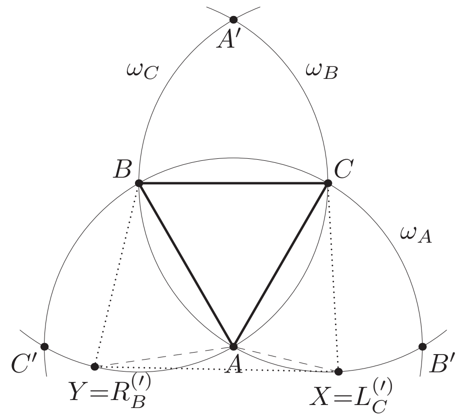


Figure 2

Assume, to the contrary, that $L_C \neq A$ and $R_B \neq A$. Denote by A', B', C' the intersection points of circles ω_A, ω_B and ω_C, distinct from A, B, C (see Figure 2). Let $CL_C L'_C$ be the good triangle containing CL_C. Observe that the angle of arc $L_C A$ is less than $120°$. Then one of the points L_C and L'_C belongs to arc $B'A$ of ω_C; let this point be X. In the case when $L_C = B'$ and $L'_C = A$, choose $X = B'$.

Analogously, considering the good triangle $BR'_B R_B$ which contains BR_B as an edge, we see that one of the points R_B and R'_B lies on arc AC' of ω_B. Denote this point by Y, $Y \neq A$. Then angles XAY, YAB, BAC and CAX (oriented clockwise) are not greater than $180°$. Hence, point A lies in quadrilateral $XYBC$ (either in its interior or on segment XY). This is impossible, since all these five points are vertices of P.

Hence, each good triangle has at least three assignments, and the statement is proved.

Comment 1. Considering a diameter AB of the polygon, one can prove that every good triangle containing either A or B has at least four assignments. This observation leads to $t \leq \left\lfloor \frac{2}{3}(n-1) \right\rfloor$.

Comment 2. The result $t \leq \left\lfloor \frac{2}{3}(n-1) \right\rfloor$ is sharp. To construct a polygon with $n = 3k+1$ vertices and $t = 2k$ triangles, take a rhombus $AB_1C_1D_1$ with unit side length and $\angle B_1 = 60^\circ$. Then rotate it around A by small angles obtaining rhombi $AB_2C_2D_2, \ldots, AB_kC_kD_k$ (see Figure 3). The polygon $AB_1 \ldots B_kC_1 \ldots C_kD_1 \ldots D_k$ has $3k+1$ vertices and contains $2k$ good triangles.

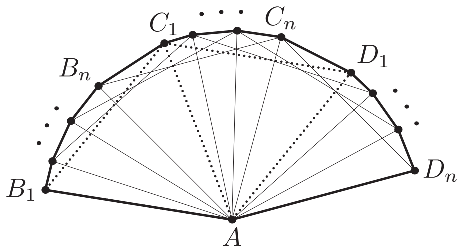


Figure 3

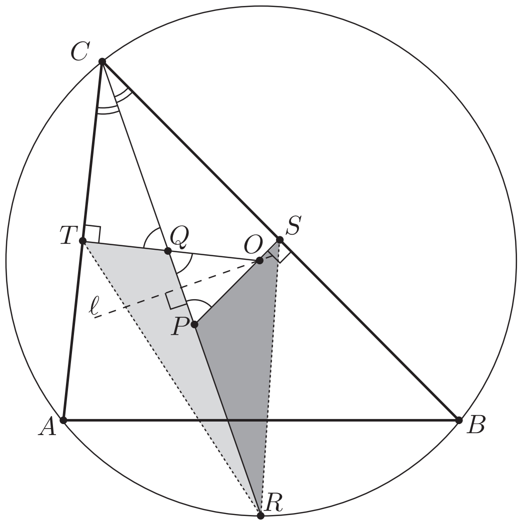
The construction for $n = 3k$ and $n = 3k-1$ can be obtained by deleting vertices D_n and D_{n-1}.

Geometry

G1. In triangle ABC, the angle bisector at vertex C intersects the circumcircle and the perpendicular bisectors of sides BC and CA at points R, P, and Q, respectively. The midpoints of BC and CA are S and T, respectively. Prove that triangles RQT and RPS have the same area.

(Czech Republic)

Solution 1. If $AC = BC$ then triangle ABC is isosceles, triangles RQT and RPS are symmetric about the bisector CR and the statement is trivial. If $AC \neq BC$ then it can be assumed without loss of generality that $AC < BC$.



Denote the circumcenter by O. The right triangles CTQ and CSP have equal angles at vertex C, so they are similar, $\angle CPS = \angle CQT = \angle OQP$ and

$$\frac{QT}{PS} = \frac{CQ}{CP}. \tag{1}$$

Let ℓ be the perpendicular bisector of chord CR; of course, ℓ passes through the circumcenter O. Due to the equal angles at P and Q, triangle OPQ is isosceles with $OP = OQ$. Then line ℓ is the axis of symmetry in this triangle as well. Therefore, points P and Q lie symmetrically on line segment CR,

$$RP = CQ \quad \text{and} \quad RQ = CP. \tag{2}$$

Triangles RQT and RPS have equal angles at vertices Q and P, respectively. Then

$$\frac{\operatorname{area}(RQT)}{\operatorname{area}(RPS)} = \frac{\frac{1}{2} \cdot RQ \cdot QT \cdot \sin \angle RQT}{\frac{1}{2} \cdot RP \cdot PS \cdot \sin \angle RPS} = \frac{RQ}{RP} \cdot \frac{QT}{PS}.$$

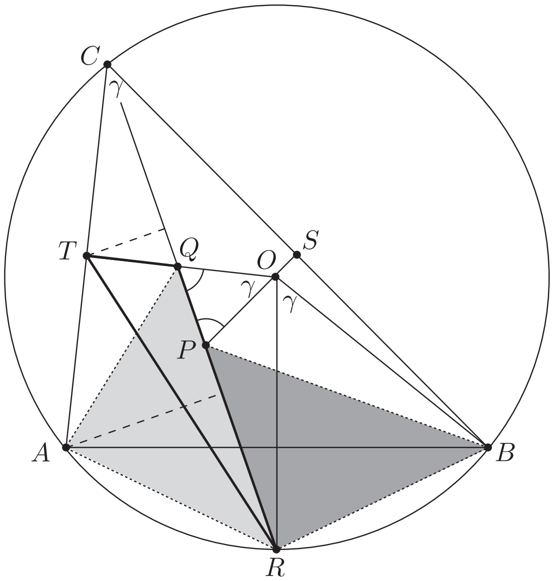
Substituting (1) and (2),

$$\frac{\operatorname{area}(RQT)}{\operatorname{area}(RPS)} = \frac{RQ}{RP} \cdot \frac{QT}{PS} = \frac{CP}{CQ} \cdot \frac{CQ}{CP} = 1.$$

Hence, $\operatorname{area}(RQT) = \operatorname{area}(RSP)$.

Solution 2. Assume again $AC < BC$. Denote the circumcenter by O, and let γ be the angle at C. Similarly to the first solution, from right triangles CTQ and CSP we obtain that $\angle OPQ = \angle OQP = 90^\circ - \frac{\gamma}{2}$. Then triangle OPQ is isosceles, $OP = OQ$ and moreover $\angle POQ = \gamma$.

As is well-known, point R is the midpoint of arc AB and $\angle ROA = \angle BOR = \gamma$.



Consider the rotation around point O by angle γ. This transform moves A to R, R to B and Q to P; hence triangles RQA and BPR are congruent and they have the same area.

Triangles RQT and RQA have RQ as a common side, so the ratio between their areas is

$$\frac{\text{area}(RQT)}{\text{area}(RQA)} = \frac{d(T, CR)}{d(A, CR)} = \frac{CT}{CA} = \frac{1}{2}.$$

($d(X, YZ)$ denotes the distance between point X and line YZ).

It can be obtained similarly that

$$\frac{\text{area}(RPS)}{\text{area}(BPR)} = \frac{CS}{CB} = \frac{1}{2}.$$

Now the proof can be completed as

$$\text{area}(RQT) = \frac{1}{2}\,\text{area}(RQA) = \frac{1}{2}\,\text{area}(BPR) = \text{area}(RPS).$$

G2. Given an isosceles triangle ABC with $AB = AC$. The midpoint of side BC is denoted by M. Let X be a variable point on the shorter arc MA of the circumcircle of triangle ABM. Let T be the point in the angle domain BMA, for which $\angle TMX = 90°$ and $TX = BX$. Prove that $\angle MTB - \angle CTM$ does not depend on X.

(Canada)

Solution 1. Let N be the midpoint of segment BT (see Figure 1). Line XN is the axis of symmetry in the isosceles triangle BXT, thus $\angle TNX = 90°$ and $\angle BXN = \angle NXT$. Moreover, in triangle BCT, line MN is the midline parallel to CT; hence $\angle CTM = \angle NMT$.

Due to the right angles at points M and N, these points lie on the circle with diameter XT. Therefore,

$$\angle MTB = \angle MTN = \angle MXN \qquad \text{and} \qquad \angle CTM = \angle NMT = \angle NXT = \angle BXN.$$

Hence

$$\angle MTB - \angle CTM = \angle MXN - \angle BXN = \angle MXB = \angle MAB$$

which does not depend on X.

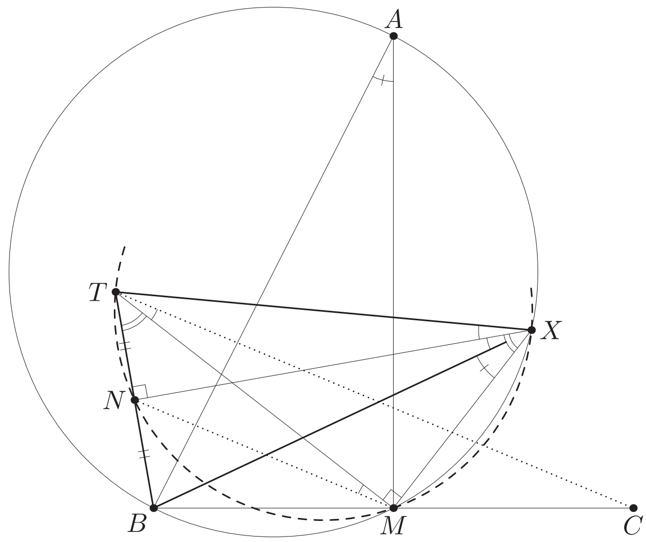


Figure 1

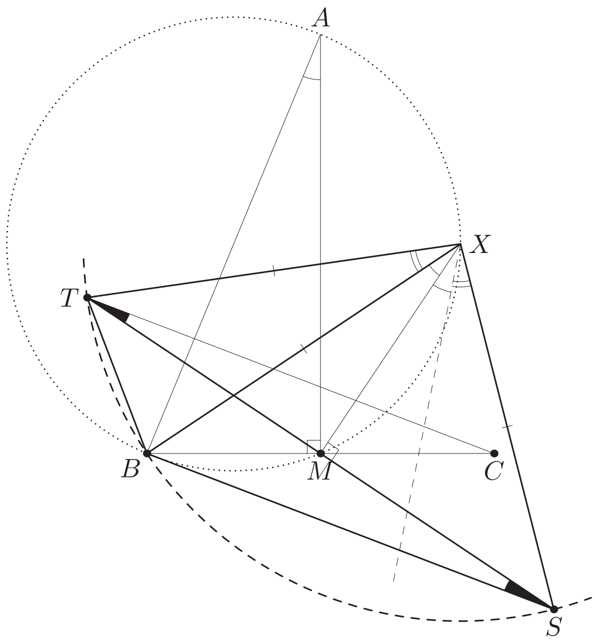


Figure 2

Solution 2. Let S be the reflection of point T over M (see Figure 2). Then XM is the perpendicular bisector of TS, hence $XB = XT = XS$, and X is the circumcenter of triangle BST. Moreover, $\angle BSM = \angle CTM$ since they are symmetrical about M. Then

$$\angle MTB - \angle CTM = \angle STB - \angle BST = \frac{\angle SXB - \angle BXT}{2}.$$

Observe that $\angle SXB = \angle SXT - \angle BXT = 2\angle MXT - \angle BXT$, so

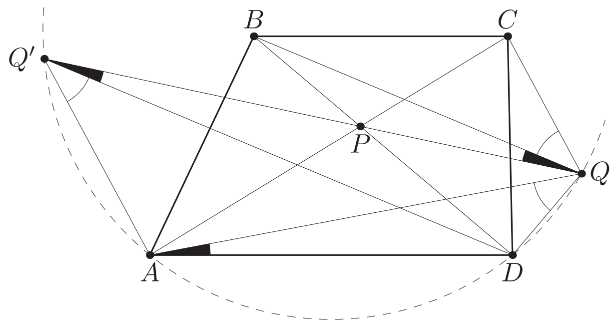
$$\angle MTB - \angle CTM = \frac{2\angle MXT - 2\angle BXT}{2} = \angle MXB = \angle MAB,$$

which is constant.

G3. The diagonals of a trapezoid $ABCD$ intersect at point P. Point Q lies between the parallel lines BC and AD such that $\angle AQD = \angle CQB$, and line CD separates points P and Q. Prove that $\angle BQP = \angle DAQ$.

(Ukraine)

Solution. Let $t = \dfrac{AD}{BC}$. Consider the homothety h with center P and scale $-t$. Triangles PDA and PBC are similar with ratio t, hence $h(B) = D$ and $h(C) = A$.



Let $Q' = h(Q)$ (see Figure 1). Then points Q, P and Q' are obviously collinear. Points Q and P lie on the same side of AD, as well as on the same side of BC; hence Q' and P are also on the same side of $h(BC) = AD$, and therefore Q and Q' are on the same side of AD. Moreover, points Q and C are on the same side of BD, while Q' and A are on the opposite side (see Figure above).

By the homothety, $\angle AQ'D = \angle CQB = \angle AQD$, hence quadrilateral $AQ'QD$ is cyclic. Then

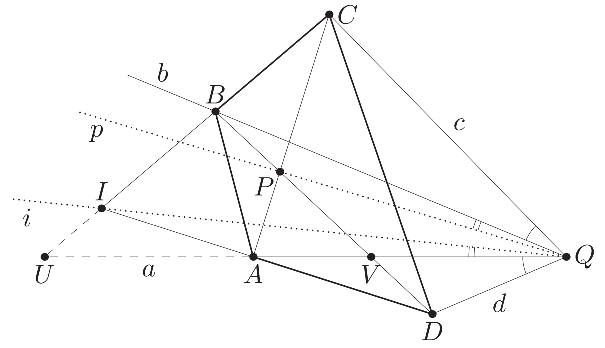
$$\angle DAQ = \angle DQ'Q = \angle DQ'P = \angle BQP$$

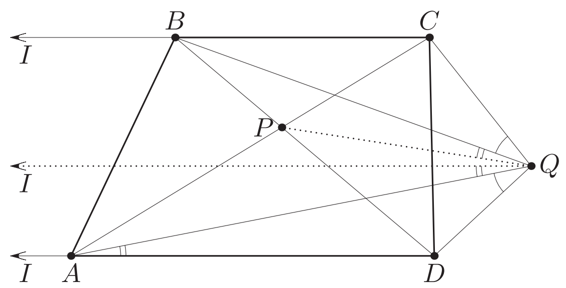
(the latter equality is valid by the homothety again).

Comment. The statement of the problem is a limit case of the following result.

In an arbitrary quadrilateral $ABCD$, let $P = AC \cap BD$, $I = AD \cap BC$, and let Q be an arbitrary point which is not collinear with any two of points A, B, C, D. Then $\angle AQD = \angle CQB$ if and only if $\angle BQP = \angle IQA$ (angles are oriented; see Figure below to the left).

In the special case of the trapezoid, I is an ideal point and $\angle DAQ = \angle IQA = \angle BQP$.





Let $a = QA$, $b = QB$, $c = QC$, $d = QD$, $i = QI$ and $p = QP$. Let line QA intersect lines BC and BD at points U and V, respectively. On lines BC and BD we have

$$(abci) = (UBCI) \qquad \text{and} \qquad (badp) = (abpd) = (VBPD).$$

Projecting from A, we get

$$(abci) = (UBCI) = (VBPD) = (badp).$$

Suppose that $\angle AQD = \angle CQB$. Let line p' be the reflection of line i about the bisector of angle AQB. Then by symmetry we have $(badp') = (abci) = (badp)$. Hence $p = p'$, as desired.

The converse statement can be proved analogously.

G4. Consider five points A, B, C, D, E such that $ABCD$ is a parallelogram and $BCED$ is a cyclic quadrilateral. Let ℓ be a line passing through A, and let ℓ intersect segment DC and line BC at points F and G, respectively. Suppose that $EF = EG = EC$. Prove that ℓ is the bisector of angle DAB.

(Luxembourg)

Solution. If $CF = CG$, then $\angle FGC = \angle GFC$, hence $\angle GAB = \angle GFC = \angle FGC = \angle FAD$, and ℓ is a bisector.

Assume that $CF < GC$. Let EK and EL be the altitudes in the isosceles triangles ECF and EGC, respectively. Then in the right triangles EKF and ELC we have $EF = EC$ and

$$KF = \frac{CF}{2} < \frac{GC}{2} = LC,$$

so

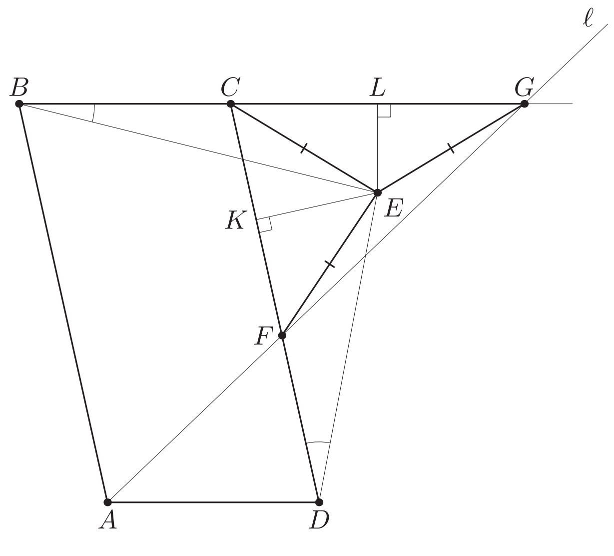
$$KE = \sqrt{EF^2 - KF^2} > \sqrt{EC^2 - LC^2} = LE.$$

Since quadrilateral $BCED$ is cyclic, we have $\angle EDC = \angle EBC$, so the right triangles BEL and DEK are similar. Then $KE > LE$ implies $DK > BL$, and hence

$$DF = DK - KF > BL - LC = BC = AD.$$

But triangles ADF and GCF are similar, so we have $1 > \dfrac{AD}{DF} = \dfrac{GC}{CF}$; this contradicts our assumption.

The case $CF > GC$ is completely similar. We consequently obtain the converse inequalities $KF > LC$, $KE < LE$, $DK < BL$, $DF < AD$, hence $1 < \dfrac{AD}{DF} = \dfrac{GC}{CF}$; a contradiction.



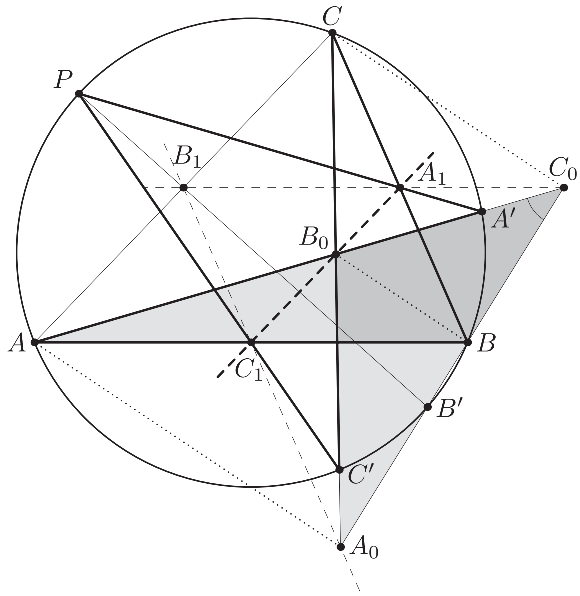
G5. Let ABC be a fixed triangle, and let A_1, B_1, C_1 be the midpoints of sides BC, CA, AB, respectively. Let P be a variable point on the circumcircle. Let lines PA_1, PB_1, PC_1 meet the circumcircle again at A', B', C' respectively. Assume that the points A, B, C, A', B', C' are distinct, and lines AA', BB', CC' form a triangle. Prove that the area of this triangle does not depend on P.

(United Kingdom)

Solution 1. Let A_0, B_0, C_0 be the points of intersection of the lines AA', BB' and CC' (see Figure). We claim that $\operatorname{area}(A_0B_0C_0) = \frac{1}{2}\operatorname{area}(ABC)$, hence it is constant.

Consider the inscribed hexagon $ABCC'PA'$. By Pascal's theorem, the points of intersection of its opposite sides (or of their extensions) are collinear. These points are $AB \cap C'P = C_1$, $BC \cap PA' = A_1$, $CC' \cap A'A = B_0$. So point B_0 lies on the midline A_1C_1 of triangle ABC. Analogously, points A_0 and C_0 lie on lines B_1C_1 and A_1B_1, respectively.

Lines AC and A_1C_1 are parallel, so triangles $B_0C_0A_1$ and AC_0B_1 are similar; hence we have



$$\frac{B_0C_0}{AC_0} = \frac{A_1C_0}{B_1C_0}.$$

Analogously, from $BC \parallel B_1C_1$ we obtain

$$\frac{A_1C_0}{B_1C_0} = \frac{BC_0}{A_0C_0}.$$

Combining these equalities, we get

$$\frac{B_0C_0}{AC_0} = \frac{BC_0}{A_0C_0},$$

or

$$A_0C_0 \cdot B_0C_0 = AC_0 \cdot BC_0.$$

Hence we have

$$\operatorname{area}(A_0B_0C_0) = \frac{1}{2}A_0C_0 \cdot B_0C_0 \sin\angle A_0C_0B_0 = \frac{1}{2}AC_0 \cdot BC_0 \sin\angle AC_0B = \operatorname{area}(ABC_0).$$

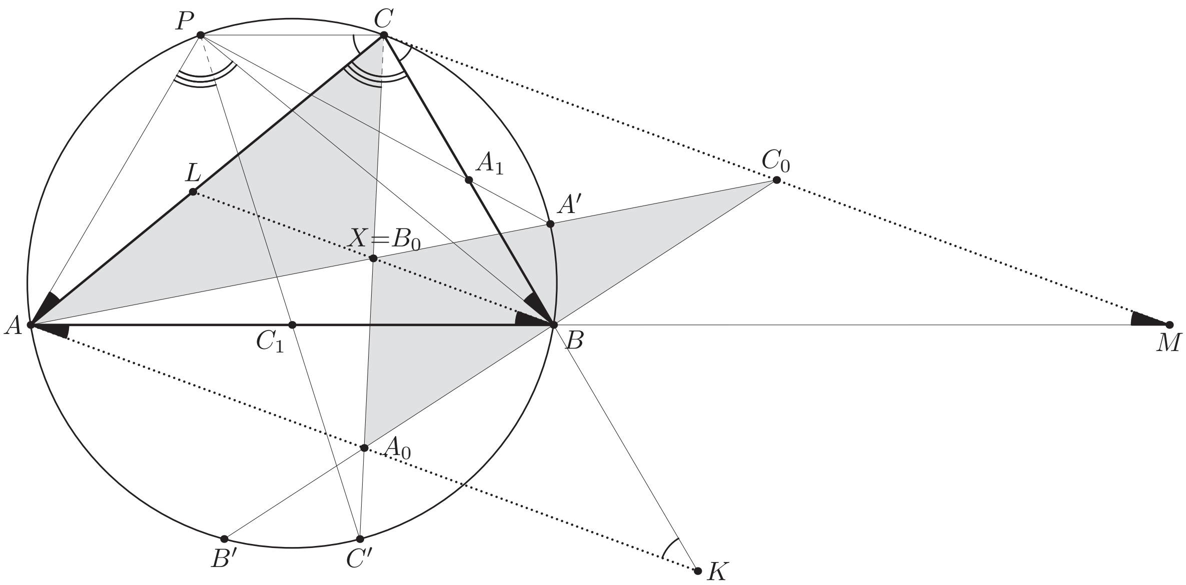
Since C_0 lies on the midline, we have $d(C_0, AB) = \frac{1}{2}d(C, AB)$ (we denote by $d(X, YZ)$ the distance between point X and line YZ). Then we obtain

$$\operatorname{area}(A_0B_0C_0) = \operatorname{area}(ABC_0) = \frac{1}{2}AB \cdot d(C_0, AB) = \frac{1}{4}AB \cdot d(C, AB) = \frac{1}{2}\operatorname{area}(ABC).$$

Solution 2. Again, we prove that $\operatorname{area}(A_0B_0C_0) = \frac{1}{2}\operatorname{area}(ABC)$.

We can assume that P lies on arc AC. Mark a point L on side AC such that $\angle CBL = \angle PBA$; then $\angle LBA = \angle CBA - \angle CBL = \angle CBA - \angle PBA = \angle CBP$. Note also that $\angle BAL = \angle BAC = \angle BPC$ and $\angle LCB = \angle APB$. Hence, triangles BAL and BPC are similar, and so are triangles LCB and APB.

Analogously, mark points K and M respectively on the extensions of sides CB and AB beyond point B, such that $\angle KAB = \angle CAP$ and $\angle BCM = \angle PCA$. For analogous reasons, $\angle KAC = \angle BAP$ and $\angle ACM = \angle PCB$. Hence $\triangle ABK \sim \triangle APC \sim \triangle MBC$, $\triangle ACK \sim \triangle APB$, and $\triangle MAC \sim \triangle BPC$. From these similarities, we have $\angle CMB = \angle KAB = \angle CAP$, while we have seen that $\angle CAP = \angle CBP = \angle LBA$. Hence, $AK \parallel BL \parallel CM$.



Let line CC' intersect BL at point X. Note that $\angle LCX = \angle ACC' = \angle APC' = \angle APC_1$, and PC_1 is a median in triangle APB. Since triangles APB and LCB are similar, CX is a median in triangle LCB, and X is a midpoint of BL. For the same reason, AA' passes through this midpoint, so $X = B_0$. Analogously, A_0 and C_0 are the midpoints of AK and CM.

Now, from $AA_0 \parallel CC_0$, we have

$$\text{area}(A_0B_0C_0) = \text{area}(AC_0A_0) - \text{area}(AB_0A_0) = \text{area}(ACA_0) - \text{area}(AB_0A_0) = \text{area}(ACB_0).$$

Finally,

$$\text{area}(A_0B_0C_0) = \text{area}(ACB_0) = \frac{1}{2}B_0L \cdot AC \sin ALB_0 = \frac{1}{4}BL \cdot AC \sin ALB = \frac{1}{2}\text{area}(ABC).$$

Comment 1. The equality $\text{area}(A_0B_0C_0) = \text{area}(ACB_0)$ in Solution 2 does not need to be proved since the following fact is frequently known.

Suppose that the lines KL and MN are parallel, while the lines KM and LN intersect in a point E. Then $\text{area}(KEN) = \text{area}(MEL)$.

Comment 2. It follows immediately from both solutions that $AA_0 \parallel BB_0 \parallel CC_0$. These lines pass through an ideal point which is isogonally conjugate to P. It is known that they are parallel to the Simson line of point Q which is opposite to P on the circumcircle.

Comment 3. If $A = A'$, then one can define the line AA' to be the tangent to the circumcircle at point A. Then the statement of the problem is also valid in this case.

G6. Determine the smallest positive real number k with the following property.

Let $ABCD$ be a convex quadrilateral, and let points A_1, B_1, C_1 and D_1 lie on sides AB, BC, CD and DA, respectively. Consider the areas of triangles AA_1D_1, BB_1A_1, CC_1B_1, and DD_1C_1; let S be the sum of the two smallest ones, and let S_1 be the area of quadrilateral $A_1B_1C_1D_1$. Then we always have $kS_1 \geq S$.

(U.S.A.)

Answer. $k = 1$.

Solution. Throughout the solution, triangles AA_1D_1, BB_1A_1, CC_1B_1, and DD_1C_1 will be referred to as *border triangles*. We will denote by $[\mathcal{R}]$ the area of a region $\mathcal{R}$.

First, we show that $k \geq 1$. Consider a triangle ABC with unit area; let A_1, B_1, K be the midpoints of its sides AB, BC, AC, respectively. Choose a point D on the extension of BK, close to K. Take points C_1 and D_1 on sides CD and DA close to D (see Figure 1). We have $[BB_1A_1] = \frac{1}{4}$. Moreover, as $C_1, D_1, D \to K$, we get $[A_1B_1C_1D_1] \to [A_1B_1K] = \frac{1}{4}$, $[AA_1D_1] \to [AA_1K] = \frac{1}{4}$, $[CC_1B_1] \to [CKB_1] = \frac{1}{4}$ and $[DD_1C_1] \to 0$. Hence, the sum of the two smallest areas of border triangles tends to $\frac{1}{4}$, as well as $[A_1B_1C_1D_1]$; therefore, their ratio tends to 1, and $k \geq 1$.

We are left to prove that $k = 1$ satisfies the desired property.

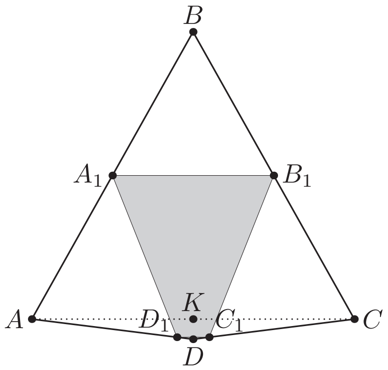


Figure 1

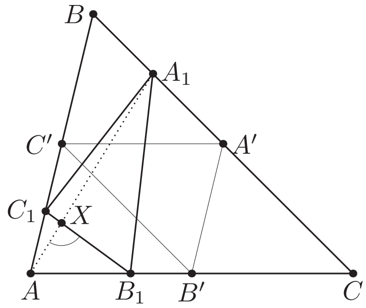


Figure 2

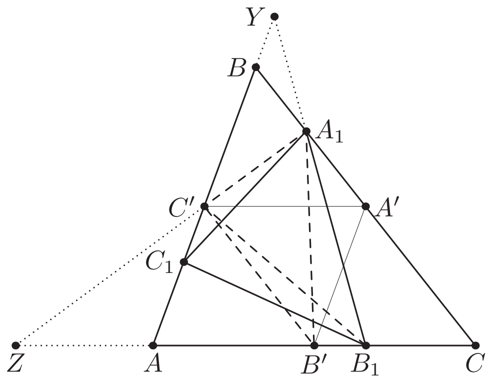


Figure 3

Lemma. Let points A_1, B_1, C_1 lie respectively on sides BC, CA, AB of a triangle ABC. Then $[A_1B_1C_1] \geq \min\{[AC_1B_1], [BA_1C_1], [CB_1A_1]\}$.

Proof. Let A', B', C' be the midpoints of sides BC, CA and AB, respectively.

Suppose that two of points A_1, B_1, C_1 lie in one of triangles $AC'B'$, $BA'C'$ and $CB'A'$ (for convenience, let points B_1 and C_1 lie in triangle $AC'B'$; see Figure 2). Let segments B_1C_1 and AA_1 intersect at point X. Then X also lies in triangle $AC'B'$. Hence $A_1X \geq AX$, and we have

$$\frac{[A_1B_1C_1]}{[AC_1B_1]} = \frac{\frac{1}{2}A_1X \cdot B_1C_1 \cdot \sin\angle A_1XC_1}{\frac{1}{2}AX \cdot B_1C_1 \cdot \sin\angle AXB_1} = \frac{A_1X}{AX} \geq 1,$$

as required.

Otherwise, each one of triangles $AC'B'$, $BA'C'$, $CB'A'$ contains exactly one of points A_1, B_1, C_1, and we can assume that $BA_1 < BA'$, $CB_1 < CB'$, $AC_1 < AC'$ (see Figure 3). Then lines B_1A_1 and AB intersect at a point Y on the extension of AB beyond point B, hence $\dfrac{[A_1B_1C_1]}{[A_1B_1C']} = \dfrac{C_1Y}{C'Y} > 1$; also, lines A_1C' and CA intersect at a point Z on the extension of CA beyond point A, hence $\dfrac{[A_1B_1C']}{[A_1B'C']} = \dfrac{B_1Z}{B'Z} > 1$. Finally, since $A_1A' \parallel B'C'$, we have $[A_1B_1C_1] > [A_1B_1C'] > [A_1B'C'] = [A'B'C'] = \frac{1}{4}[ABC]$.

Now, from $[A_1B_1C_1] + [AC_1B_1] + [BA_1C_1] + [CB_1A_1] = [ABC]$ we obtain that one of the remaining triangles AC_1B_1, BA_1C_1, CB_1A_1 has an area less than $\frac{1}{4}[ABC]$, so it is less than $[A_1B_1C_1]$. $\square$

Now we return to the problem. We say that triangle $A_1B_1C_1$ is *small* if $[A_1B_1C_1]$ is less than each of $[BB_1A_1]$ and $[CC_1B_1]$; otherwise this triangle is *big* (the similar notion is introduced for triangles $B_1C_1D_1$, $C_1D_1A_1$, $D_1A_1B_1$). If both triangles $A_1B_1C_1$ and $C_1D_1A_1$ are big, then $[A_1B_1C_1]$ is not less than the area of some border triangle, and $[C_1D_1A_1]$ is not less than the area of another one; hence, $S_1 = [A_1B_1C_1] + [C_1D_1A_1] \geq S$. The same is valid for the pair of $B_1C_1D_1$ and $D_1A_1B_1$. So it is sufficient to prove that in one of these pairs both triangles are big.

Suppose the contrary. Then there is a small triangle in each pair. Without loss of generality, assume that triangles $A_1B_1C_1$ and $D_1A_1B_1$ are small. We can assume also that $[A_1B_1C_1] \leq [D_1A_1B_1]$. Note that in this case ray D_1C_1 intersects line BC.

Consider two cases.

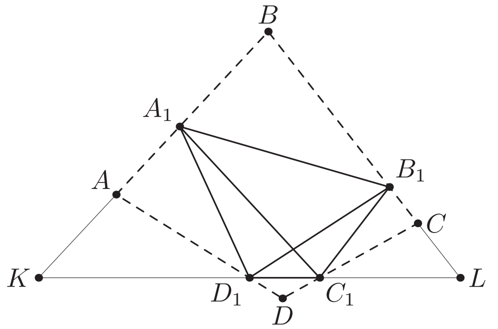


Figure 4

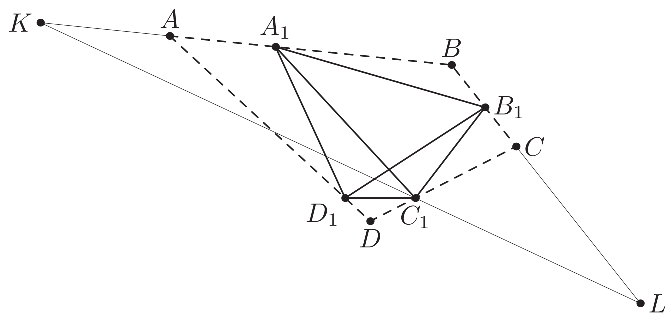


Figure 5

Case 1. Ray C_1D_1 intersects line AB at some point K. Let ray D_1C_1 intersect line BC at point L (see Figure 4). Then we have $[A_1B_1C_1] < [CC_1B_1] < [LC_1B_1]$, $[A_1B_1C_1] < [BB_1A_1]$ (both — since $[A_1B_1C_1]$ is small), and $[A_1B_1C_1] \leq [D_1A_1B_1] < [AA_1D_1] < [KA_1D_1] < [KA_1C_1]$ (since triangle $D_1A_1B_1$ is small). This contradicts the Lemma, applied for triangle $A_1B_1C_1$ inside LKB.

Case 2. Ray C_1D_1 does not intersect AB. Then choose a "sufficiently far" point K on ray BA such that $[KA_1C_1] > [A_1B_1C_1]$, and that ray KC_1 intersects line BC at some point L (see Figure 5). Since ray C_1D_1 does not intersect line AB, the points A and D_1 are on different sides of KL; then A and D are also on different sides, and C is on the same side as A and B. Then analogously we have $[A_1B_1C_1] < [CC_1B_1] < [LC_1B_1]$ and $[A_1B_1C_1] < [BB_1A_1]$ since triangle $A_1B_1C_1$ is small. This (together with $[A_1B_1C_1] < [KA_1C_1]$) contradicts the Lemma again.

G7. Given an acute triangle ABC with angles α, β and γ at vertices A, B and C, respectively, such that $\beta > \gamma$. Point I is the incenter, and R is the circumradius. Point D is the foot of the altitude from vertex A. Point K lies on line AD such that $AK = 2R$, and D separates A and K. Finally, lines DI and KI meet sides AC and BC at E and F, respectively.

Prove that if $IE = IF$ then $\beta \leq 3\gamma$.

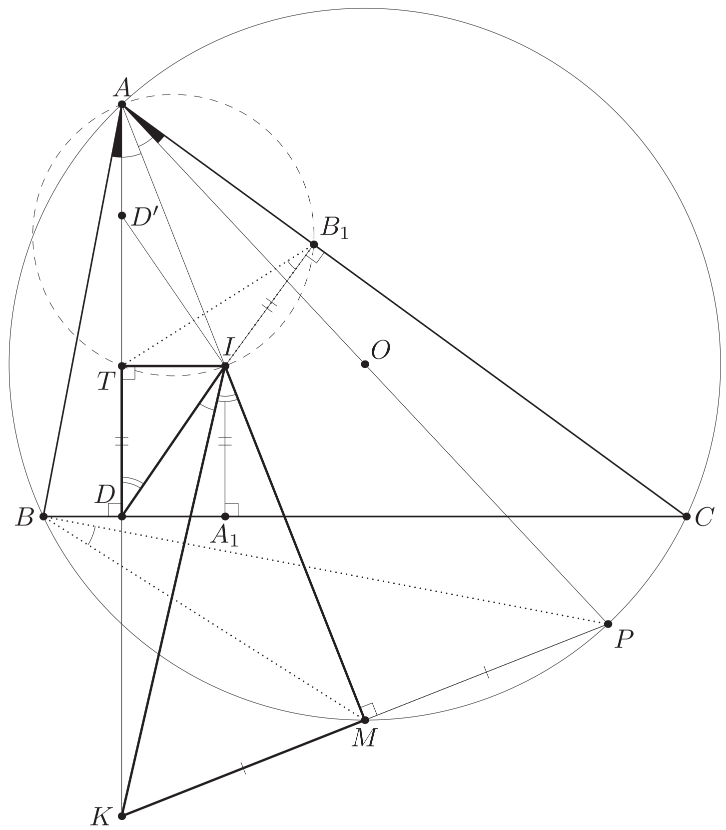
(Iran)

Solution 1. We first prove that

$$\angle KID = \frac{\beta - \gamma}{2} \tag{1}$$

even without the assumption that $IE = IF$. Then we will show that the statement of the problem is a consequence of this fact.

Denote the circumcenter by O. On the circumcircle, let P be the point opposite to A, and let the angle bisector AI intersect the circle again at M. Since $AK = AP = 2R$, triangle AKP is isosceles. It is known that $\angle BAD = \angle CAO$, hence $\angle DAI = \angle BAI - \angle BAD = \angle CAI - \angle CAO = \angle OAI$, and AM is the bisector line in triangle AKP. Therefore, points K and P are symmetrical about AM, and $\angle AMK = \angle AMP = 90°$. Thus, M is the midpoint of KP, and AM is the perpendicular bisector of KP.



Denote the perpendicular feet of incenter I on lines BC, AC, and AD by A_1, B_1, and T, respectively. Quadrilateral DA_1IT is a rectangle, hence $TD = IA_1 = IB_1$.

Due to the right angles at T and B_1, quadrilateral AB_1IT is cyclic. Hence $\angle B_1TI = \angle B_1AI = \angle CAM = \angle BAM = \angle BPM$ and $\angle IB_1T = \angle IAT = \angle MAK = \angle MAP = \angle MBP$. Therefore, triangles B_1TI and BPM are similar and $\dfrac{IT}{IB_1} = \dfrac{MP}{MB}$.

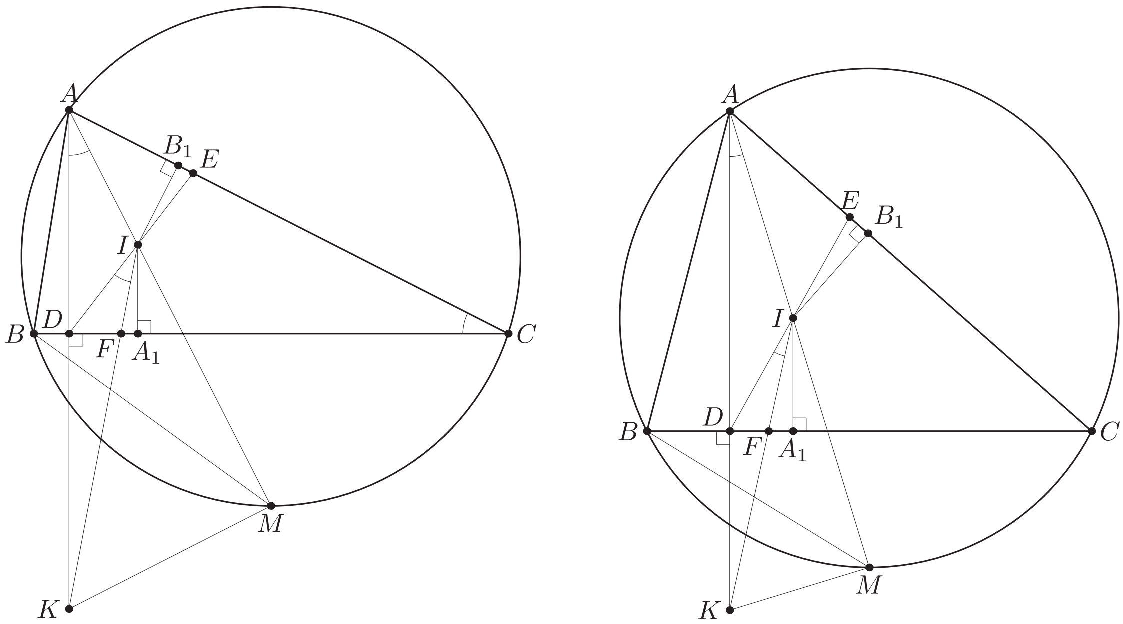
It is well-known that $MB = MC = MI$. Then right triangles ITD and KMI are also similar, because $\dfrac{IT}{TD} = \dfrac{IT}{IB_1} = \dfrac{MP}{MB} = \dfrac{KM}{MI}$. Hence, $\angle KIM = \angle IDT = \angle IDA$, and

$$\angle KID = \angle MID - \angle KIM = (\angle IAD + \angle IDA) - \angle IDA = \angle IAD.$$

Finally, from the right triangle ADB we can compute

$$\angle KID = \angle IAD = \angle IAB - \angle DAB = \frac{\alpha}{2} - (90^\circ - \beta) = \frac{\alpha}{2} - \frac{\alpha+\beta+\gamma}{2} + \beta = \frac{\beta-\gamma}{2}.$$

Now let us turn to the statement and suppose that $IE = IF$. Since $IA_1 = IB_1$, the right triangles IEB_1 and IFA_1 are congruent and $\angle IEB_1 = \angle IFA_1$. Since $\beta > \gamma$, A_1 lies in the interior of segment CD and F lies in the interior of A_1D. Hence, $\angle IFC$ is acute. Then two cases are possible depending on the order of points A, C, B_1 and E.



If point E lies between C and B_1 then $\angle IFC = \angle IEA$, hence quadrilateral $CEIF$ is cyclic and $\angle FCE = 180^\circ - \angle EIF = \angle KID$. By (1), in this case we obtain $\angle FCE = \gamma = \angle KID = \dfrac{\beta-\gamma}{2}$ and $\beta = 3\gamma$.

Otherwise, if point E lies between A and B_1, quadrilateral $CEIF$ is a deltoid such that $\angle IEC = \angle IFC < 90^\circ$. Then we have $\angle FCE > 180^\circ - \angle EIF = \angle KID$. Therefore, $\angle FCE = \gamma > \angle KID = \dfrac{\beta-\gamma}{2}$ and $\beta < 3\gamma$.

Comment 1. In the case when quadrilateral $CEIF$ is a deltoid, one can prove the desired inequality without using (1). Actually, from $\angle IEC = \angle IFC < 90^\circ$ it follows that $\angle ADI = 90^\circ - \angle EDC < \angle AED - \angle EDC = \gamma$. Since the incircle lies inside triangle ABC, we have $AD > 2r$ (here r is the inradius), which implies $DT < TA$ and $DI < AI$; hence $\dfrac{\beta-\gamma}{2} = \angle IAD < \angle ADI < \gamma$.

Solution 2. We give a different proof for (1). Then the solution can be finished in the same way as above.

Define points M and P again; it can be proved in the same way that AM is the perpendicular bisector of KP. Let J be the center of the excircle touching side BC. It is well-known that points B, C, I, J lie on a circle with center M; denote this circle by ω_1.

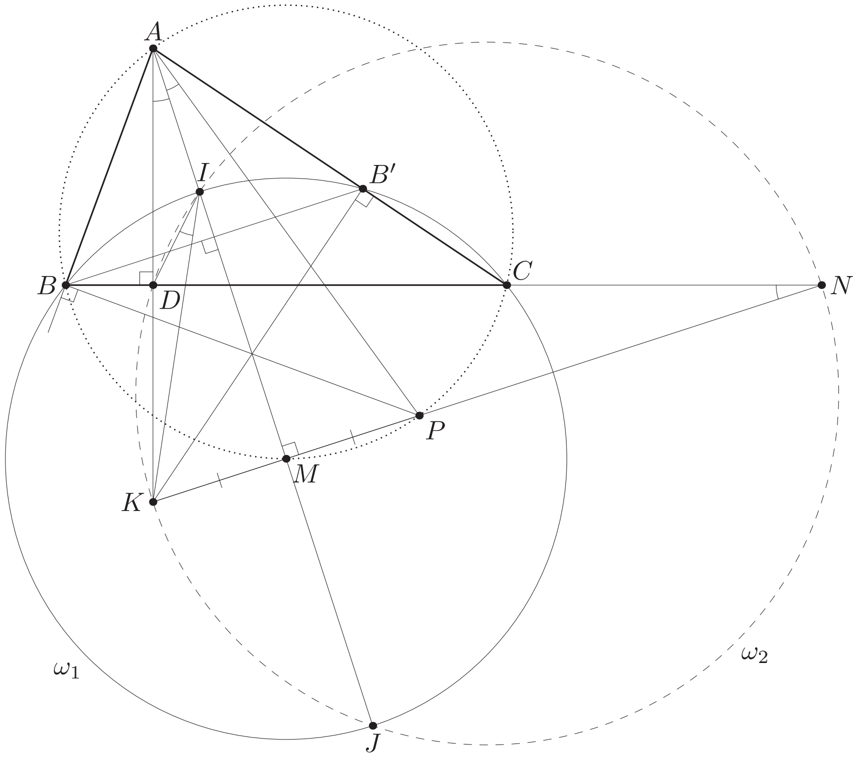
Let B' be the reflection of point B about the angle bisector AM. By the symmetry, B' is the second intersection point of circle ω_1 and line AC. Triangles PBA and $KB'A$ are symmetrical with respect to line AM, therefore $\angle KB'A = \angle PBA = 90^\circ$. By the right angles at D and B', points K, D, B', C are concyclic and

$$AD \cdot AK = AB' \cdot AC.$$

From the cyclic quadrilateral $IJCB'$ we obtain $AB' \cdot AC = AI \cdot AJ$ as well, therefore

$$AD \cdot AK = AB' \cdot AC = AI \cdot AJ$$

and points I, J, K, D are also concyclic. Denote circle $IDKJ$ by ω_2.



Let N be the point on circle ω_2 which is opposite to K. Since $\angle NDK = 90^\circ = \angle CDK$, point N lies on line BC. Point M, being the center of circle ω_1, is the midpoint of segment IJ, and KM is perpendicular to IJ. Therefore, line KM is the perpendicular bisector of IJ and hence it passes through N.

From the cyclic quadrilateral $IDKN$ we obtain

$$\angle KID = \angle KND = 90^\circ - \angle DKN = 90^\circ - \angle AKM = \angle MAK = \frac{\beta - \gamma}{2}.$$

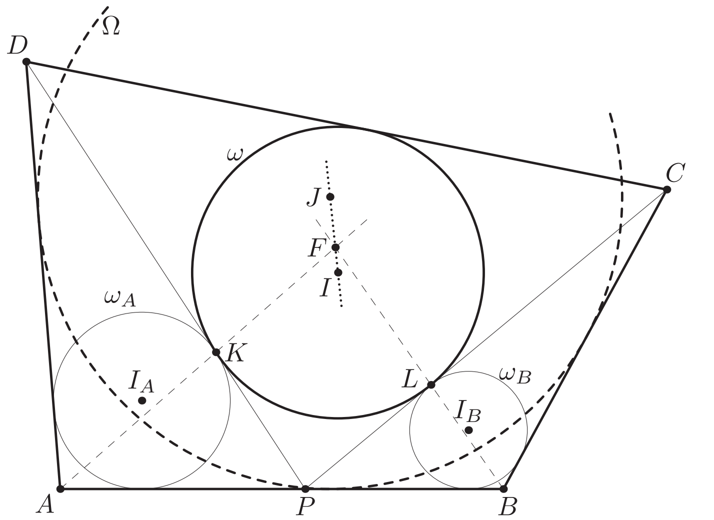
Comment 2. The main difficulty in the solution is finding (1). If someone can guess this fact, he or she can compute it in a relatively short way.

One possible way is finding and applying the relation $AI^2 = 2R(h_a - 2r)$, where $h_a = AD$ is the length of the altitude. Using this fact, one can see that triangles AKI and AID' are similar (here D' is the point symmetrical to D about T). Hence, $\angle MIK = \angle DD'I = \angle IDD'$. The proof can be finished as in Solution 1.

G8. Point P lies on side AB of a convex quadrilateral $ABCD$. Let ω be the incircle of triangle CPD, and let I be its incenter. Suppose that ω is tangent to the incircles of triangles APD and BPC at points K and L, respectively. Let lines AC and BD meet at E, and let lines AK and BL meet at F. Prove that points E, I, and F are collinear.

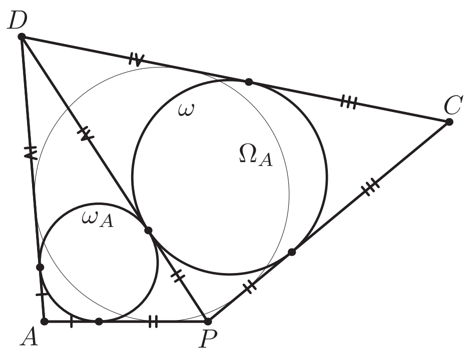
(Poland)

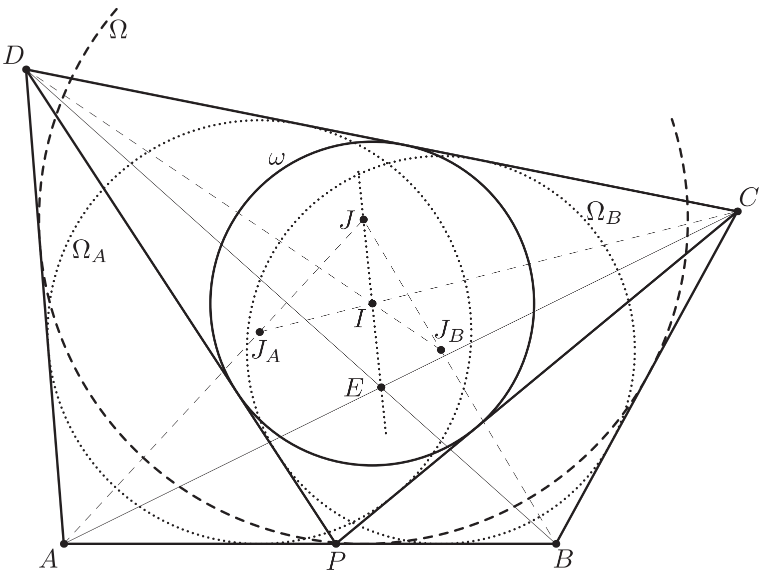
Solution. Let Ω be the circle tangent to segment AB and to rays AD and BC; let J be its center. We prove that points E and F lie on line IJ.



Denote the incircles of triangles ADP and BCP by ω_A and ω_B. Let h_1 be the homothety with a negative scale taking ω to Ω. Consider this homothety as the composition of two homotheties: one taking ω to ω_A (with a negative scale and center K), and another one taking ω_A to Ω (with a positive scale and center A). It is known that in such a case the three centers of homothety are collinear (this theorem is also referred to as the theorem on the three similitude centers). Hence, the center of h_1 lies on line AK. Analogously, it also lies on BL, so this center is F. Hence, F lies on the line of centers of ω and Ω, i. e. on IJ (if $I = J$, then $F = I$ as well, and the claim is obvious).

Consider quadrilateral $APCD$ and mark the equal segments of tangents to ω and ω_A (see the figure below to the left). Since circles ω and ω_A have a common point of tangency with PD, one can easily see that $AD + PC = AP + CD$. So, quadrilateral $APCD$ is circumscribed; analogously, circumscribed is also quadrilateral $BCDP$. Let Ω_A and Ω_B respectively be their incircles.





Consider the homothety h_2 with a positive scale taking ω to Ω. Consider h_2 as the composition of two homotheties: taking ω to Ω_A (with a positive scale and center C), and taking Ω_A to Ω (with a positive scale and center A), respectively. So the center of h_2 lies on line AC. By analogous reasons, it lies also on BD, hence this center is E. Thus, E also lies on the line of centers IJ, and the claim is proved.

Comment. In both main steps of the solution, there can be several different reasonings for the same claims. For instance, one can mostly use Desargues' theorem instead of the three homotheties theorem. Namely, if I_A and I_B are the centers of ω_A and ω_B, then lines I_AI_B, KL and AB are concurrent (by the theorem on three similitude centers applied to ω, ω_A and ω_B). Then Desargues' theorem, applied to triangles AI_AK and BI_BL, yields that the points $J = AI_A \cap BI_B$, $I = I_AK \cap I_BL$ and $F = AK \cap BL$ are collinear.

For the second step, let J_A and J_B be the centers of Ω_A and Ω_B. Then lines J_AJ_B, AB and CD are concurrent, since they appear to be the two common tangents and the line of centers of Ω_A and Ω_B. Applying Desargues' theorem to triangles AJ_AC and BJ_BD, we obtain that the points $J = AJ_A \cap BJ_B$, $I = CJ_A \cap DJ_B$ and $E = AC \cap BD$ are collinear.

Number Theory

N1. Find all pairs (k, n) of positive integers for which $7^k - 3^n$ divides $k^4 + n^2$.

(Austria)

Answer. $(2, 4)$.

Solution. Suppose that a pair (k, n) satisfies the condition of the problem. Since $7^k - 3^n$ is even, $k^4 + n^2$ is also even, hence k and n have the same parity. If k and n are odd, then $k^4 + n^2 \equiv 1 + 1 = 2 \pmod 4$, while $7^k - 3^n \equiv 7 - 3 \equiv 0 \pmod 4$, so $k^4 + n^2$ cannot be divisible by $7^k - 3^n$. Hence, both k and n must be even.

Write $k = 2a$, $n = 2b$. Then $7^k - 3^n = 7^{2a} - 3^{2b} = \dfrac{7^a - 3^b}{2} \cdot 2(7^a + 3^b)$, and both factors are integers. So $2(7^a + 3^b) \bigm| 7^k - 3^n$ and $7^k - 3^n \bigm| k^4 + n^2 = 2(8a^4 + 2b^2)$, hence

$$7^a + 3^b \leq 8a^4 + 2b^2. \tag{1}$$

We prove by induction that $8a^4 < 7^a$ for $a \geq 4$, $2b^2 < 3^b$ for $b \geq 1$ and $2b^2 + 9 \leq 3^b$ for $b \geq 3$. In the initial cases $a = 4$, $b = 1$, $b = 2$ and $b = 3$ we have $8 \cdot 4^4 = 2048 < 7^4 = 2401$, $2 < 3$, $2 \cdot 2^2 = 8 < 3^2 = 9$ and $2 \cdot 3^2 + 9 = 3^3 = 27$, respectively.

If $8a^4 < 7^a$ $(a \geq 4)$ and $2b^2 + 9 \leq 3^b$ $(b \geq 3)$, then

$$8(a+1)^4 = 8a^4 \left(\frac{a+1}{a}\right)^4 < 7^a \left(\frac{5}{4}\right)^4 = 7^a \frac{625}{256} < 7^{a+1} \qquad \text{and}$$

$$2(b+1)^2 + 9 < (2b^2 + 9)\left(\frac{b+1}{b}\right)^2 \leq 3^b \left(\frac{4}{3}\right)^2 = 3^b \frac{16}{9} < 3^{b+1},$$

as desired.

For $a \geq 4$ we obtain $7^a + 3^b > 8a^4 + 2b^2$ and inequality (1) cannot hold. Hence $a \leq 3$, and three cases are possible.

Case 1: $a = 1$. Then $k = 2$ and $8 + 2b^2 \geq 7 + 3^b$, thus $2b^2 + 1 \geq 3^b$. This is possible only if $b \leq 2$. If $b = 1$ then $n = 2$ and $\dfrac{k^4 + n^2}{7^k - 3^n} = \dfrac{2^4 + 2^2}{7^2 - 3^2} = \dfrac{1}{2}$, which is not an integer. If $b = 2$ then $n = 4$ and $\dfrac{k^4 + n^2}{7^k - 3^n} = \dfrac{2^4 + 4^2}{7^2 - 3^4} = -1$, so $(k, n) = (2, 4)$ is a solution.

Case 2: $a = 2$. Then $k = 4$ and $k^4 + n^2 = 256 + 4b^2 \geq |7^4 - 3^n| = |49 - 3^b| \cdot (49 + 3^b)$. The smallest value of the first factor is 22, attained at $b = 3$, so $128 + 2b^2 \geq 11(49 + 3^b)$, which is impossible since $3^b > 2b^2$.

Case 3: $a = 3$. Then $k = 6$ and $k^4 + n^2 = 1296 + 4b^2 \geq |7^6 - 3^n| = |343 - 3^b| \cdot (343 + 3^b)$. Analogously, $|343 - 3^b| \geq 100$ and we have $324 + b^2 \geq 25(343 + 3^b)$, which is impossible again.

We find that there exists a unique solution $(k, n) = (2, 4)$.

N2. Let $b, n > 1$ be integers. Suppose that for each $k > 1$ there exists an integer a_k such that $b - a_k^n$ is divisible by k. Prove that $b = A^n$ for some integer A.

(Canada)

Solution. Let the prime factorization of b be $b = p_1^{\alpha_1} \ldots p_s^{\alpha_s}$, where $p_1, \ldots, p_s$ are distinct primes. Our goal is to show that all exponents α_i are divisible by n, then we can set $A = p_1^{\alpha_1/n} \ldots p_s^{\alpha_s/n}$.

Apply the condition for $k = b^2$. The number $b - a_k^n$ is divisible by b^2 and hence, for each $1 \leq i \leq s$, it is divisible by $p_i^{2\alpha_i} > p_i^{\alpha_i}$ as well. Therefore

$$a_k^n \equiv b \equiv 0 \pmod{p_i^{\alpha_i}}$$

and

$$a_k^n \equiv b \not\equiv 0 \pmod{p_i^{\alpha_i+1}},$$

which implies that the largest power of p_i dividing a_k^n is $p_i^{\alpha_i}$. Since a_k^n is a complete nth power, this implies that α_i is divisible by n.

Comment. If $n = 8$ and $b = 16$, then for each *prime* p there exists an integer a_p such that $b - a_p^n$ is divisible by p. Actually, the congruency $x^8 - 16 \equiv 0 \pmod{p}$ expands as

$$(x^2 - 2)(x^2 + 2)(x^2 - 2x + 2)(x^2 + 2x + 2) \equiv 0 \pmod{p}.$$

Hence, if -1 is a quadratic residue modulo p, then congruency $x^2 + 2x + 2 = (x+1)^2 + 1 \equiv 0$ has a solution. Otherwise, one of congruencies $x^2 \equiv 2$ and $x^2 \equiv -2$ has a solution.

Thus, the solution cannot work using only prime values of k.

N3. Let X be a set of 10 000 integers, none of them is divisible by 47. Prove that there exists a 2007-element subset Y of X such that $a-b+c-d+e$ is not divisible by 47 for any $a,b,c,d,e \in Y$.

(Netherlands)

Solution. Call a set M of integers *good* if $47 \nmid a-b+c-d+e$ for any $a,b,c,d,e \in M$.

Consider the set $J=\{-9,-7,-5,-3,-1,1,3,5,7,9\}$. We claim that J is good. Actually, for any $a,b,c,d,e \in J$ the number $a-b+c-d+e$ is odd and

$$-45=(-9)-9+(-9)-9+(-9) \leq a-b+c-d+e \leq 9-(-9)+9-(-9)+9=45.$$

But there is no odd number divisible by 47 between -45 and 45.

For any $k=1,\ldots,46$ consider the set

$$A_k=\{x \in X \mid \exists j \in J: \quad kx \equiv j \pmod{47}\}.$$

If A_k is not good, then $47 \mid a-b+c-d+e$ for some $a,b,c,d,e \in A_k$, hence $47 \mid ka-kb+kc-kd+ke$. But set J contains numbers with the same residues modulo 47, so J also is not good. This is a contradiction; therefore each A_k is a good subset of X.

Then it suffices to prove that there exists a number k such that $|A_k| \geq 2007$. Note that each $x \in X$ is contained in exactly 10 sets A_k. Then

$$\sum_{k=1}^{46} |A_k| = 10|X| = 100\,000,$$

hence for some value of k we have

$$|A_k| \geq \frac{100\,000}{46} > 2173 > 2007.$$

This completes the proof.

Comment. For the solution, it is essential to find a good set consisting of 10 different residues. Actually, consider a set X containing almost uniform distribution of the nonzero residues (i. e. each residue occurs 217 or 218 times). Let $Y \subset X$ be a good subset containing 2007 elements. Then the set K of all residues appearing in Y contains not less than 10 residues, and obviously this set is good.

On the other hand, there is no good set K consisting of 11 different residues. The Cauchy–Davenport theorem claims that for any sets A, B of residues modulo a prime p,

$$|A+B| \geq \min\{p, |A|+|B|-1\}.$$

Hence, if $|K| \geq 11$, then $|K+K| \geq 21$, $|K+K+K| \geq 31 > 47-|K+K|$, hence $|K+K+K+(-K)+(-K)|=47$, and $0 \equiv a+c+e-b-d \pmod{47}$ for some $a,b,c,d,e \in K$.

From the same reasoning, one can see that a good set K containing 10 residues should satisfy equalities $|K+K|=19=2|K|-1$ and $|K+K+K|=28=|K+K|+|K|-1$. It can be proved that in this case set K consists of 10 residues forming an arithmetic progression. As an easy consequence, one obtains that set K has the form aJ for some nonzero residue a.

N4. For every integer $k \geq 2$, prove that 2^{3k} divides the number

$$\binom{2^{k+1}}{2^k} - \binom{2^k}{2^{k-1}} \tag{1}$$

but 2^{3k+1} does not.

(Poland)

Solution. We use the notation $(2n-1)!! = 1 \cdot 3 \cdots (2n-1)$ and $(2n)!! = 2 \cdot 4 \cdots (2n) = 2^n n!$ for any positive integer n. Observe that $(2n)! = (2n)!!\,(2n-1)!! = 2^n n!\,(2n-1)!!$.

For any positive integer n we have

$$\binom{4n}{2n} = \frac{(4n)!}{(2n)!^2} = \frac{2^{2n}(2n)!\,(4n-1)!!}{(2n)!^2} = \frac{2^{2n}}{(2n)!}(4n-1)!!,$$

$$\binom{2n}{n} = \frac{1}{(2n)!}\left(\frac{(2n)!}{n!}\right)^2 = \frac{1}{(2n)!}\left(2^n(2n-1)!!\right)^2 = \frac{2^{2n}}{(2n)!}(2n-1)!!^2.$$

Then expression (1) can be rewritten as follows:

$$\binom{2^{k+1}}{2^k} - \binom{2^k}{2^{k-1}} = \frac{2^{2^k}}{(2^k)!}(2^{k+1}-1)!! - \frac{2^{2^k}}{(2^k)!}(2^k-1)!!^2$$
$$= \frac{2^{2^k}(2^k-1)!!}{(2^k)!} \cdot \Big((2^k+1)(2^k+3)\ldots(2^k+2^k-1) - (2^k-1)(2^k-3)\ldots(2^k-2^k+1)\Big). \tag{2}$$

We compute the exponent of 2 in the prime decomposition of each factor (the first one is a rational number but not necessarily an integer; it is not important).

First, we show by induction on n that the exponent of 2 in $(2^n)!$ is $2^n - 1$. The base case $n = 1$ is trivial. Suppose that $(2^n)! = 2^{2^n-1}(2d+1)$ for some integer d. Then we have

$$(2^{n+1})! = 2^{2^n}(2^n)!\,(2^{n+1}-1)!! = 2^{2^n}2^{2^n-1} \cdot (2d+1)(2^{n+1}-1)!! = 2^{2^{n+1}-1} \cdot (2q+1)$$

for some integer q. This finishes the induction step.

Hence, the exponent of 2 in the first factor in (2) is $2^k - (2^k - 1) = 1$.

The second factor in (2) can be considered as the value of the polynomial

$$P(x) = (x+1)(x+3)\ldots(x+2^k-1) - (x-1)(x-3)\ldots(x-2^k+1). \tag{3}$$

at $x = 2^k$. Now we collect some information about $P(x)$.

Observe that $P(-x) = -P(x)$, since $k \geq 2$. So $P(x)$ is an odd function, and it has nonzero coefficients only at odd powers of x. Hence $P(x) = x^3 Q(x) + cx$, where $Q(x)$ is a polynomial with integer coefficients.

Compute the exponent of 2 in c. We have

$$c = 2(2^k-1)!!\sum_{i=1}^{2^{k-1}} \frac{1}{2i-1} = (2^k-1)!!\sum_{i=1}^{2^{k-1}}\left(\frac{1}{2i-1} + \frac{1}{2^k-2i+1}\right)$$
$$= (2^k-1)!!\sum_{i=1}^{2^{k-1}} \frac{2^k}{(2i-1)(2^k-2i+1)} = 2^k \sum_{i=1}^{2^{k-1}} \frac{(2^k-1)!!}{(2i-1)(2^k-2i+1)} = 2^k S.$$

For any integer $i = 1, \dots, 2^{k-1}$, denote by a_{2i-1} the residue inverse to $2i-1$ modulo 2^k. Clearly, when $2i-1$ runs through all odd residues, so does a_{2i-1}, hence

$$S = \sum_{i=1}^{2^{k-1}} \frac{(2^k-1)!!}{(2i-1)(2^k-2i+1)} \equiv -\sum_{i=1}^{2^{k-1}} \frac{(2^k-1)!!}{(2i-1)^2} \equiv -\sum_{i=1}^{2^{k-1}} (2^k-1)!!\, a_{2i-1}^2$$
$$= -(2^k-1)!! \sum_{i=1}^{2^{k-1}} (2i-1)^2 = -(2^k-1)!! \frac{2^{k-1}(2^{2k}-1)}{3} \pmod{2^k}.$$

Therefore, the exponent of 2 in S is $k-1$, so $c = 2^k S = 2^{2k-1}(2t+1)$ for some integer t.

Finally we obtain that

$$P(2^k) = 2^{3k} Q(2^k) + 2^k c = 2^{3k} Q(2^k) + 2^{3k-1}(2t+1),$$

which is divisible exactly by 2^{3k-1}. Thus, the exponent of 2 in (2) is $1 + (3k-1) = 3k$.

Comment. The fact that (1) is divisible by 2^{2k} is known; but it does not help in solving this problem.

N5. Find all surjective functions $f\colon \mathbb{N}\to\mathbb{N}$ such that for every $m,n\in\mathbb{N}$ and every prime p, the number $f(m+n)$ is divisible by p if and only if $f(m)+f(n)$ is divisible by p.

($\mathbb{N}$ is the set of all positive integers.)

(Iran)

Answer. $f(n)=n$.

Solution. Suppose that function $f\colon \mathbb{N}\to\mathbb{N}$ satisfies the problem conditions.

Lemma. For any prime p and any $x,y\in\mathbb{N}$, we have $x\equiv y \pmod p$ if and only if $f(x)\equiv f(y) \pmod p$. Moreover, $p \mid f(x)$ if and only if $p\mid x$.

Proof. Consider an arbitrary prime p. Since f is surjective, there exists some $x\in\mathbb{N}$ such that $p\mid f(x)$. Let

$$d=\min\bigl\{x\in\mathbb{N} : p \mid f(x)\bigr\}.$$

By induction on k, we obtain that $p\mid f(kd)$ for all $k\in\mathbb{N}$. The base is true since $p\mid f(d)$. Moreover, if $p\mid f(kd)$ and $p\mid f(d)$ then, by the problem condition, $p\mid f(kd+d)=f\bigl((k+1)d\bigr)$ as required.

Suppose that there exists an $x\in\mathbb{N}$ such that $d\nmid x$ but $p\mid f(x)$. Let

$$y=\min\bigl\{x\in\mathbb{N} : d\nmid x,\ p\mid f(x)\bigr\}.$$

By the choice of d, we have $y>d$, and $y-d$ is a positive integer not divisible by d. Then $p\nmid f(y-d)$, while $p\mid f(d)$ and $p\mid f\bigl(d+(y-d)\bigr)=f(y)$. This contradicts the problem condition. Hence, there is no such x, and

$$p\mid f(x) \iff d\mid x. \tag{1}$$

Take arbitrary $x,y\in\mathbb{N}$ such that $x\equiv y \pmod d$. We have $p\mid f\bigl(x+(2xd-x)\bigr)=f(2xd)$; moreover, since $d\mid 2xd+(y-x)=y+(2xd-x)$, we get $p\mid f\bigl(y+(2xd-x)\bigr)$. Then by the problem condition $p\mid f(x)+f(2xd-x)$, $p\mid f(y)+f(2xd-x)$, and hence $f(x)\equiv -f(2xd-x)\equiv f(y) \pmod p$.

On the other hand, assume that $f(x)\equiv f(y) \pmod p$. Again we have $p\mid f(x)+f(2xd-x)$ which by our assumption implies that $p\mid f(x)+f(2xd-x)+\bigl(f(y)-f(x)\bigr)=f(y)+f(2xd-x)$. Hence by the problem condition $p\mid f\bigl(y+(2xd-x)\bigr)$. Using (1) we get $0\equiv y+(2xd-x)\equiv y-x \pmod d$.

Thus, we have proved that

$$x\equiv y \pmod d \iff f(x)\equiv f(y) \pmod p. \tag{2}$$

We are left to show that $p=d$: in this case (1) and (2) provide the desired statements.

The numbers $1, 2, \dots, d$ have distinct residues modulo d. By (2), numbers $f(1), f(2), \dots, f(d)$ have distinct residues modulo p; hence there are at least d distinct residues, and $p\ge d$. On the other hand, by the surjectivity of f, there exist $x_1,\dots,x_p\in\mathbb{N}$ such that $f(x_i)=i$ for any $i=1,2,\dots,p$. By (2), all these x_i's have distinct residues modulo d. For the same reasons, $d\ge p$. Hence, $d=p$. $\square$

Now we prove that $f(n)=n$ by induction on n. If $n=1$ then, by the Lemma, $p\nmid f(1)$ for any prime p, so $f(1)=1$, and the base is established. Suppose that $n>1$ and denote $k=f(n)$. Note that there exists a prime $q\mid n$, so by the Lemma $q\mid k$ and $k>1$.

If $k>n$ then $k-n+1>1$, and there exists a prime $p\mid k-n+1$; we have $k\equiv n-1 \pmod p$. By the induction hypothesis we have $f(n-1)=n-1\equiv k=f(n) \pmod p$. Now, by the Lemma we obtain $n-1\equiv n \pmod p$ which cannot be true.

Analogously, if $k < n$, then $f(k-1) = k-1$ by induction hypothesis. Moreover, $n-k+1 > 1$, so there exists a prime $p \mid n-k+1$ and $n \equiv k-1 \pmod{p}$. By the Lemma again, $k = f(n) \equiv f(k-1) = k-1 \pmod{p}$, which is also false. The only remaining case is $k = n$, so $f(n) = n$.

Finally, the function $f(n) = n$ obviously satisfies the condition.

N6. Let k be a positive integer. Prove that the number $(4k^2-1)^2$ has a positive divisor of the form $8kn-1$ if and only if k is even.

(United Kingdom)

Solution. The statement follows from the following fact.

Lemma. For arbitrary positive integers x and y, the number $4xy-1$ divides $(4x^2-1)^2$ if and only if $x=y$.

Proof. If $x=y$ then $4xy-1=4x^2-1$ obviously divides $(4x^2-1)^2$ so it is sufficient to consider the opposite direction.

Call a pair (x,y) of positive integers *bad* if $4xy-1$ divides $(4x^2-1)^2$ but $x\neq y$. In order to prove that bad pairs do not exist, we present two properties of them which provide an infinite descent.

Property (i). *If (x,y) is a bad pair and $x<y$ then there exists a positive integer $z<x$ such that (x,z) is also bad.*

Let $r=\dfrac{(4x^2-1)^2}{4xy-1}$. Then

$$r=-r\cdot(-1)\equiv -r(4xy-1)=-(4x^2-1)^2\equiv -1 \pmod{4x}$$

and $r=4xz-1$ with some positive integer z. From $x<y$ we obtain that

$$4xz-1=\frac{(4x^2-1)^2}{4xy-1}<4x^2-1$$

and therefore $z<x$. By the construction, the number $4xz-1$ is a divisor of $(4x^2-1)^2$ so (x,z) is a bad pair.

Property (ii). *If (x,y) is a bad pair then (y,x) is also bad.*

Since $1=1^2\equiv(4xy)^2 \pmod{4xy-1}$, we have

$$(4y^2-1)^2\equiv\left(4y^2-(4xy)^2\right)^2=16y^4(4x^2-1)^2\equiv 0 \pmod{4xy-1}.$$

Hence, the number $4xy-1$ divides $(4y^2-1)^2$ as well.

Now suppose that there exists at least one bad pair. Take a bad pair (x,y) such that $2x+y$ attains its smallest possible value. If $x<y$ then property (i) provides a bad pair (x,z) with $z<y$ and thus $2x+z<2x+y$. Otherwise, if $y<x$, property (ii) yields that pair (y,x) is also bad while $2y+x<2x+y$. Both cases contradict the assumption that $2x+y$ is minimal; the Lemma is proved. $\square$

To prove the problem statement, apply the Lemma for $x=k$ and $y=2n$; the number $8kn-1$ divides $(4k^2-1)^2$ if and only if $k=2n$. Hence, there is no such n if k is odd and $n=k/2$ is the only solution if k is even.

Comment. The constant 4 in the Lemma can be replaced with an arbitrary integer greater than 1: if $a>1$ and $axy-1$ divides $(ax^2-1)^2$ then $x=y$.

N7. For a prime p and a positive integer n, denote by $\nu_p(n)$ the exponent of p in the prime factorization of $n!$. Given a positive integer d and a finite set $\{p_1, \ldots, p_k\}$ of primes. Show that there are infinitely many positive integers n such that $d \mid \nu_{p_i}(n)$ for all $1 \le i \le k$.

(India)

Solution 1. For arbitrary prime p and positive integer n, denote by $\operatorname{ord}_p(n)$ the exponent of p in n. Thus,

$$\nu_p(n) = \operatorname{ord}_p(n!) = \sum_{i=1}^{n} \operatorname{ord}_p(i).$$

Lemma. Let p be a prime number, q be a positive integer, k and r be positive integers such that $p^k > r$. Then $\nu_p(qp^k + r) = \nu_p(qp^k) + \nu_p(r)$.

Proof. We claim that $\operatorname{ord}_p(qp^k + i) = \operatorname{ord}_p(i)$ for all $0 < i < p^k$. Actually, if $d = \operatorname{ord}_p(i)$ then $d < k$, so $qp^k + i$ is divisible by p^d, but only the first term is divisible by p^{d+1}; hence the sum is not.

Using this claim, we obtain

$$\nu_p(qp^k + r) = \sum_{i=1}^{qp^k} \operatorname{ord}_p(i) + \sum_{i=qp^k+1}^{qp^k+r} \operatorname{ord}_p(i) = \sum_{i=1}^{qp^k} \operatorname{ord}_p(i) + \sum_{i=1}^{r} \operatorname{ord}_p(i) = \nu_p(qp^k) + \nu_p(r). \quad \square$$

For any integer a, denote by $\overline{a}$ its residue modulo d. The addition of residues will also be performed modulo d, i.e. $\overline{a} + \overline{b} = \overline{a+b}$. For any positive integer n, let $f(n) = \big(f_1(n), \ldots, f_k(n)\big)$, where $f_i(n) = \overline{\nu_{p_i}(n)}$.

Define the sequence $n_1 = 1$, $n_{\ell+1} = (p_1 p_2 \ldots p_k)^{n_\ell}$. We claim that

$$f(n_{\ell_1} + n_{\ell_2} + \ldots + n_{\ell_m}) = f(n_{\ell_1}) + f(n_{\ell_2}) + \ldots + f(n_{\ell_m})$$

for any $\ell_1 < \ell_2 < \ldots < \ell_m$. (The addition of k-tuples is componentwise.) The base case $m = 1$ is trivial.

Suppose that $m > 1$. By the construction of the sequence, $p_i^{n_{\ell_1}}$ divides $n_{\ell_2} + \ldots + n_{\ell_m}$; clearly, $p_i^{n_{\ell_1}} > n_{\ell_1}$ for all $1 \le i \le k$. Therefore the Lemma can be applied for $p = p_i$, $k = r = n_{\ell_1}$ and $qp^k = n_{\ell_2} + \ldots + n_{\ell_m}$ to obtain

$$f_i(n_{\ell_1} + n_{\ell_2} + \ldots + n_{\ell_m}) = f_i(n_{\ell_1}) + f_i(n_{\ell_2} + \ldots + n_{\ell_m}) \qquad \text{for all } 1 \le i \le k,$$

and hence

$$f(n_{\ell_1} + n_{\ell_2} + \ldots + n_{\ell_m}) = f(n_{\ell_1}) + f(n_{\ell_2} + \ldots + n_{\ell_m}) = f(n_{\ell_1}) + f(n_{\ell_2}) + \ldots + f(n_{\ell_m})$$

by the induction hypothesis.

Now consider the values $f(n_1)$, $f(n_2)$, ... There exist finitely many possible values of f. Hence, there exists an infinite sequence of indices $\ell_1 < \ell_2 < \ldots$ such that $f(n_{\ell_1}) = f(n_{\ell_2}) = \ldots$ and thus

$$f(n_{\ell_{m+1}} + n_{\ell_{m+2}} + \ldots + n_{\ell_{m+d}}) = f(n_{\ell_{m+1}}) + \ldots + f(n_{\ell_{m+d}}) = d \cdot f(n_{\ell_1}) = (\overline{0}, \ldots, \overline{0})$$

for all m. We have found infinitely many suitable numbers.

Solution 2. We use the same Lemma and definition of the function f.

Let $S=\{f(n): n\in\mathbb{N}\}$. Obviously, set S is finite. For every $s\in S$ choose the minimal n_s such that $f(n_s)=s$. Denote $N=\max\limits_{s\in S} n_s$. Moreover, let g be an integer such that $p_i^g>N$ for each $i=1,2,\ldots,k$. Let $P=(p_1p_2\ldots p_k)^g$.

We claim that

$$\big\{f(n)\mid n\in[mP,mP+N]\big\}=S \tag{1}$$

for every positive integer m. In particular, since $(\overline{0},\ldots,\overline{0})=f(1)\in S$, it follows that for an arbitrary m there exists $n\in[mP,mP+N]$ such that $f(n)=(\overline{0},\ldots,\overline{0})$. So there are infinitely many suitable numbers.

To prove (1), let $a_i=f_i(mP)$. Consider all numbers of the form $n_{m,s}=mP+n_s$ with $s=(s_1,\ldots,s_k)\in S$ (clearly, all $n_{m,s}$ belong to $[mP,mP+N]$). Since $n_s\le N<p_i^g$ and $p_i^g \bigm| mP$, we can apply the Lemma for the values $p=p_i$, $r=n_s$, $k=g$, $qp^k=mP$ to obtain

$$f_i(n_{m,s})=f_i(mP)+f_i(n_s)=a_i+s_i;$$

hence for distinct $s,t\in S$ we have $f(n_{m,s})\neq f(n_{m,t})$.

Thus, the function f attains at least $|S|$ distinct values in $[mP,mP+N]$. Since all these values belong to S, f should attain all possible values in $[mP,mP+N]$.

Comment. Both solutions can be extended to prove the following statements.

Claim 1. For any K there exist infinitely many n divisible by K, such that $d \bigm| \nu_{p_i}(n)$ for each i.

Claim 2. For any $s\in S$, there exist infinitely many $n\in\mathbb{N}$ such that $f(n)=s$.

49^{th} International Mathematical Olympiad

July, 10-22 2008

Madrid, Spain

SHORT LIST

WITH SOLUTIONS

49$^{\text{th}}$ International Mathematical Olympiad
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Contributing Countries

Australia, Austria, Belgium, Bulgaria, Canada, Colombia, Croatia, Czech Republic, Estonia, France, Germany, Greece, Hong Kong, India, Iran, Ireland, Japan, Korea (North), Korea (South), Lithuania, Luxembourg, Mexico, Moldova, Netherlands, Pakistan, Peru, Poland, Romania, Russia, Serbia, Slovakia, South Africa, Sweden, Ukraine, United Kingdom, United States of America

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Algebra

A1. Find all functions $f\colon (0,\infty) \to (0,\infty)$ such that

$$\frac{f(p)^2 + f(q)^2}{f(r^2) + f(s^2)} = \frac{p^2 + q^2}{r^2 + s^2}$$

for all $p, q, r, s > 0$ with $pq = rs$.

Solution. Let f satisfy the given condition. Setting $p = q = r = s = 1$ yields $f(1)^2 = f(1)$ and hence $f(1) = 1$. Now take any $x > 0$ and set $p = x$, $q = 1$, $r = s = \sqrt{x}$ to obtain

$$\frac{f(x)^2 + 1}{2f(x)} = \frac{x^2 + 1}{2x}\,.$$

This recasts into

$$\begin{gathered} xf(x)^2 + x = x^2 f(x) + f(x), \\ \big(xf(x) - 1\big)\big(f(x) - x\big) = 0. \end{gathered}$$

And thus,

$$\text{for every } \ x > 0, \quad \text{either } \ f(x) = x \ \text{ or } \ f(x) = \frac{1}{x}\,. \tag{1}$$

Obviously, if

$$f(x) = x \quad \text{for all } x > 0 \qquad \text{or} \qquad f(x) = \frac{1}{x} \quad \text{for all } x > 0 \tag{2}$$

then the condition of the problem is satisfied. We show that actually these two functions are the only solutions.

So let us assume that there exists a function f satisfying the requirement, other than those in (2). Then $f(a) \neq a$ and $f(b) \neq 1/b$ for some $a, b > 0$. By (1), these values must be $f(a) = 1/a$, $f(b) = b$. Applying now the equation with $p = a$, $q = b$, $r = s = \sqrt{ab}$ we obtain $(a^{-2} + b^2)/2f(ab) = (a^2 + b^2)/2ab$; equivalently,

$$f(ab) = \frac{ab(a^{-2} + b^2)}{a^2 + b^2}\,. \tag{3}$$

We know however (see (1)) that $f(ab)$ must be either ab or $1/ab$. If $f(ab) = ab$ then by (3) $a^{-2} + b^2 = a^2 + b^2$, so that $a = 1$. But, as $f(1) = 1$, this contradicts the relation $f(a) \neq a$. Likewise, if $f(ab) = 1/ab$ then (3) gives $a^2b^2(a^{-2} + b^2) = a^2 + b^2$, whence $b = 1$, in contradiction to $f(b) \neq 1/b$. Thus indeed the functions listed in (2) are the only two solutions.

Comment. The equation has as many as four variables with only one constraint $pq = rs$, leaving three degrees of freedom and providing a lot of information. Various substitutions force various useful properties of the function searched. We sketch one more method to reach conclusion (1); certainly there are many others.

Noticing that $f(1) = 1$ and setting, first, $p = q = 1$, $r = \sqrt{x}$, $s = 1/\sqrt{x}$, and then $p = x$, $q = 1/x$, $r = s = 1$, we obtain two relations, holding for every $x > 0$,

$$f(x) + f\left(\frac{1}{x}\right) = x + \frac{1}{x} \qquad \text{and} \qquad f(x)^2 + f\left(\frac{1}{x}\right)^2 = x^2 + \frac{1}{x^2}\,. \tag{4}$$

Squaring the first and subtracting the second gives $2f(x)f(1/x) = 2$. Subtracting this from the second relation of (4) leads to

$$\left(f(x) - f\left(\frac{1}{x}\right)\right)^2 = \left(x - \frac{1}{x}\right)^2 \qquad \text{or} \qquad f(x) - f\left(\frac{1}{x}\right) = \pm\left(x - \frac{1}{x}\right).$$

The last two alternatives combined with the first equation of (4) imply the two alternatives of (1).

A2. (a) Prove the inequality

$$\frac{x^2}{(x-1)^2}+\frac{y^2}{(y-1)^2}+\frac{z^2}{(z-1)^2}\geq 1$$

for real numbers $x, y, z \neq 1$ satisfying the condition $xyz = 1$.

(b) Show that there are infinitely many triples of rational numbers x, y, z for which this inequality turns into equality.

Solution 1. (a) We start with the substitution

$$\frac{x}{x-1}=a, \qquad \frac{y}{y-1}=b, \qquad \frac{z}{z-1}=c, \qquad \text{i.e.,} \qquad x=\frac{a}{a-1}, \qquad y=\frac{b}{b-1}, \qquad z=\frac{c}{c-1}.$$

The inequality to be proved reads $a^2+b^2+c^2\geq 1$. The new variables are subject to the constraints $a, b, c \neq 1$ and the following one coming from the condition $xyz = 1$,

$$(a-1)(b-1)(c-1)=abc.$$

This is successively equivalent to

$$\begin{aligned}
a+b+c-1&=ab+bc+ca,\\
2(a+b+c-1)&=(a+b+c)^2-(a^2+b^2+c^2),\\
a^2+b^2+c^2-2&=(a+b+c)^2-2(a+b+c),\\
a^2+b^2+c^2-1&=(a+b+c-1)^2.
\end{aligned}$$

Thus indeed $a^2+b^2+c^2\geq 1$, as desired.

(b) From the equation $a^2+b^2+c^2-1=(a+b+c-1)^2$ we see that the proposed inequality becomes an equality if and only if both sums $a^2+b^2+c^2$ and $a+b+c$ have value 1. The first of them is equal to $(a+b+c)^2-2(ab+bc+ca)$. So the instances of equality are described by the system of two equations

$$a+b+c=1, \qquad\qquad ab+bc+ca=0$$

plus the constraint $a, b, c \neq 1$. Elimination of c leads to $a^2+ab+b^2=a+b$, which we regard as a quadratic equation in b,

$$b^2+(a-1)b+a(a-1)=0,$$

with discriminant

$$\Delta=(a-1)^2-4a(a-1)=(1-a)(1+3a).$$

We are looking for rational triples (a, b, c); it will suffice to have a rational such that $1-a$ and $1+3a$ are both squares of rational numbers (then Δ will be so too). Set $a = k/m$. We want $m-k$ and $m+3k$ to be squares of integers. This is achieved for instance by taking $m=k^2-k+1$ (clearly nonzero); then $m-k=(k-1)^2$, $m+3k=(k+1)^2$. Note that distinct integers k yield distinct values of $a=k/m$.

And thus, if k is any integer and $m=k^2-k+1$, $a=k/m$ then $\Delta=(k^2-1)^2/m^2$ and the quadratic equation has rational roots $b=(m-k\pm k^2\mp 1)/(2m)$. Choose e.g. the larger root,

$$b=\frac{m-k+k^2-1}{2m}=\frac{m+(m-2)}{2m}=\frac{m-1}{m}.$$

Computing c from $a+b+c=1$ then gives $c=(1-k)/m$. The condition $a,b,c \neq 1$ eliminates only $k=0$ and $k=1$. Thus, as k varies over integers greater than 1, we obtain an infinite family of rational triples (a,b,c)—and coming back to the original variables ($x=a/(a-1)$ etc.)—an infinite family of rational triples (x,y,z) with the needed property. (A short calculation shows that the resulting triples are $x=-k/(k-1)^2$, $y=k-k^2$, $z=(k-1)/k^2$; but the proof was complete without listing them.)

Comment 1. There are many possible variations in handling the equation system $a^2+b^2+c^2=1$, $a+b+c=1$ $(a,b,c \neq 1)$ which of course describes a circle in the (a,b,c)-space (with three points excluded), and finding infinitely many rational points on it.

Also the initial substitution $x=a/(a-1)$ (etc.) can be successfully replaced by other similar substitutions, e.g. $x=1-1/\alpha$ (etc.); or $x=x'-1$ (etc.); or $1-yz=u$ (etc.)—eventually reducing the inequality to $(\cdots)^2 \geq 0$, the expression in the parentheses depending on the actual substitution.

Depending on the method chosen, one arrives at various sequences of rational triples (x,y,z) as needed; let us produce just one more such example: $x=(2r-2)/(r+1)^2$, $y=(2r+2)/(r-1)^2$, $z=(r^2-1)/4$ where r can be any rational number different from 1 or -1.

Solution 2 (an outline). (a) Without changing variables, just setting $z=1/xy$ and clearing fractions, the proposed inequality takes the form

$$(xy-1)^2\big(x^2(y-1)^2+y^2(x-1)^2\big)+(x-1)^2(y-1)^2 \geq (x-1)^2(y-1)^2(xy-1)^2.$$

With the notation $p=x+y$, $q=xy$ this becomes, after lengthy routine manipulation and a lot of cancellation

$$q^4-6q^3+2pq^2+9q^2-6pq+p^2 \geq 0.$$

It is not hard to notice that the expression on the left is just $(q^2-3q+p)^2$, hence nonnegative.

(Without introducing p and q, one is of course led with some more work to the same expression, just written in terms of x and y; but then it is not that easy to see that it is a square.)

(b) To have equality, one needs $q^2-3q+p=0$. Note that x and y are the roots of the quadratic trinomial (in a formal variable t): t^2-pt+q. When $q^2-3q+p=0$, the discriminant equals

$$\delta=p^2-4q=(3q-q^2)^2-4q=q(q-1)^2(q-4).$$

Now it suffices to have both q and $q-4$ squares of rational numbers (then $p=3q-q^2$ and $\sqrt{\delta}$ are also rational, and so are the roots of the trinomial). On setting $q=(n/m)^2=4+(l/m)^2$ the requirement becomes $4m^2+l^2=n^2$ (with l,m,n being integers). This is just the Pythagorean equation, known to have infinitely many integer solutions.

Comment 2. Part (a) alone might also be considered as a possible contest problem (in the category of easy problems).

A3. Let $S \subseteq \mathbb{R}$ be a set of real numbers. We say that a pair (f, g) of functions from S into S is a *Spanish Couple* on S, if they satisfy the following conditions:

(i) Both functions are strictly increasing, i.e. $f(x) < f(y)$ and $g(x) < g(y)$ for all $x, y \in S$ with $x < y$;

(ii) The inequality $f(g(g(x))) < g(f(x))$ holds for all $x \in S$.

Decide whether there exists a Spanish Couple

(a) on the set $S = \mathbb{N}$ of positive integers;

(b) on the set $S = \{a - 1/b : \ a, b \in \mathbb{N}\}$.

Solution. We show that the answer is NO for part (a), and YES for part (b).

(a) Throughout the solution, we will use the notation $g_k(x) = \overbrace{g(g(\ldots g}^{k}(x)\ldots))$, including $g_0(x) = x$ as well.

Suppose that there exists a Spanish Couple (f, g) on the set $\mathbb{N}$. From property (i) we have $f(x) \geq x$ and $g(x) \geq x$ for all $x \in \mathbb{N}$.

We claim that $g_k(x) \leq f(x)$ for all $k \geq 0$ and all positive integers x. The proof is done by induction on k. We already have the base case $k = 0$ since $x \leq f(x)$. For the induction step from k to $k + 1$, apply the induction hypothesis on $g_2(x)$ instead of x, then apply (ii):

$$g(g_{k+1}(x)) = g_k\big(g_2(x)\big) \leq f\big(g_2(x)\big) < g(f(x)).$$

Since g is increasing, it follows that $g_{k+1}(x) < f(x)$. The claim is proven.

If $g(x) = x$ for all $x \in \mathbb{N}$ then $f(g(g(x))) = f(x) = g(f(x))$, and we have a contradiction with (ii). Therefore one can choose an $x_0 \in S$ for which $x_0 < g(x_0)$. Now consider the sequence $x_0, x_1, \ldots$ where $x_k = g_k(x_0)$. The sequence is increasing. Indeed, we have $x_0 < g(x_0) = x_1$, and $x_k < x_{k+1}$ implies $x_{k+1} = g(x_k) < g(x_{k+1}) = x_{k+2}$.

Hence, we obtain a strictly increasing sequence $x_0 < x_1 < \ldots$ of positive integers which on the other hand has an upper bound, namely $f(x_0)$. This cannot happen in the set $\mathbb{N}$ of positive integers, thus no Spanish Couple exists on $\mathbb{N}$.

(b) We present a Spanish Couple on the set $S = \{a - 1/b : \ a, b \in \mathbb{N}\}$.

Let

$$\begin{aligned} f(a - 1/b) &= a + 1 - 1/b, \\ g(a - 1/b) &= a - 1/(b + 3^a). \end{aligned}$$

These functions are clearly increasing. Condition (ii) holds, since

$$f(g(g(a - 1/b))) = (a + 1) - 1/(b + 2 \cdot 3^a) < (a + 1) - 1/(b + 3^{a+1}) = g(f(a - 1/b)).$$

Comment. Another example of a Spanish couple is $f(a - 1/b) = 3a - 1/b$, $g(a - 1/b) = a - 1/(a+b)$. More generally, postulating $f(a - 1/b) = h(a) - 1/b$, $g(a - 1/b) = a - 1/G(a, b)$ with h increasing and G increasing in both variables, we get that $f \circ g \circ g < g \circ f$ holds if $G\big(a, G(a, b)\big) < G\big(h(a), b\big)$. A search just among linear functions $h(a) = Ca$, $G(a, b) = Aa + Bb$ results in finding that any integers $A > 0$, $C > 2$ and $B = 1$ produce a Spanish couple (in the example above, $A = 1$, $C = 3$). The proposer's example results from taking $h(a) = a + 1$, $G(a, b) = 3^a + b$.

A4. For an integer m, denote by $t(m)$ the unique number in $\{1,2,3\}$ such that $m+t(m)$ is a multiple of 3. A function $f\colon \mathbb{Z}\to\mathbb{Z}$ satisfies $f(-1)=0$, $f(0)=1$, $f(1)=-1$ and

$$f(2^n+m)=f(2^n-t(m))-f(m) \quad \text{for all integers } m,n\geq 0 \text{ with } 2^n>m.$$

Prove that $f(3p)\geq 0$ holds for all integers $p\geq 0$.

Solution. The given conditions determine f uniquely on the positive integers. The signs of $f(1), f(2),\ldots$ seem to change quite erratically. However values of the form $f(2^n-t(m))$ are sufficient to compute directly any functional value. Indeed, let $n>0$ have base 2 representation $n=2^{a_0}+2^{a_1}+\cdots+2^{a_k}$, $a_0>a_1>\cdots>a_k\geq 0$, and let $n_j=2^{a_j}+2^{a_{j-1}}+\cdots+2^{a_k}$, $j=0,\ldots,k$. Repeated applications of the recurrence show that $f(n)$ is an alternating sum of the quantities $f(2^{a_j}-t(n_{j+1}))$ plus $(-1)^{k+1}$. (The exact formula is not needed for our proof.)

So we focus attention on the values $f(2^n-1)$, $f(2^n-2)$ and $f(2^n-3)$. Six cases arise; more specifically,

$$t(2^{2k}-3)=2,\ t(2^{2k}-2)=1,\ t(2^{2k}-1)=3,\ t(2^{2k+1}-3)=1,\ t(2^{2k+1}-2)=3,\ t(2^{2k+1}-1)=2.$$

Claim. For all integers $k\geq 0$ the following equalities hold:

$$\begin{array}{lll} f(2^{2k+1}-3)=0, & f(2^{2k+1}-2)=3^k, & f(2^{2k+1}-1)=-3^k, \\ f(2^{2k+2}-3)=-3^k, & f(2^{2k+2}-2)=-3^k, & f(2^{2k+2}-1)=2\cdot 3^k. \end{array}$$

Proof. By induction on k. The base $k=0$ comes down to checking that $f(2)=-1$ and $f(3)=2$; the given values $f(-1)=0$, $f(0)=1$, $f(1)=-1$ are also needed. Suppose the claim holds for $k-1$. For $f(2^{2k+1}-t(m))$, the recurrence formula and the induction hypothesis yield

$$\begin{aligned} f(2^{2k+1}-3)&=f(2^{2k}+(2^{2k}-3))=f(2^{2k}-2)-f(2^{2k}-3)=-3^{k-1}+3^{k-1}=0,\\ f(2^{2k+1}-2)&=f(2^{2k}+(2^{2k}-2))=f(2^{2k}-1)-f(2^{2k}-2)=2\cdot 3^{k-1}+3^{k-1}=3^k,\\ f(2^{2k+1}-1)&=f(2^{2k}+(2^{2k}-1))=f(2^{2k}-3)-f(2^{2k}-1)=-3^{k-1}-2\cdot 3^{k-1}=-3^k. \end{aligned}$$

For $f(2^{2k+2}-t(m))$ we use the three equalities just established:

$$\begin{aligned} f(2^{2k+2}-3)&=f(2^{2k+1}+(2^{2k+1}-3))=f(2^{2k+1}-1)-f(2^{2k+1}-3)=-3^k-0=-3^k,\\ f(2^{2k+2}-2)&=f(2^{2k+1}+(2^{2k+1}-2))=f(2^{2k+1}-3)-f(2^{2k}-2)=0-3^k=-3^k,\\ f(2^{2k+2}-1)&=f(2^{2k+1}+(2^{2k+1}-1))=f(2^{2k+1}-2)-f(2^{2k+1}-1)=3^k+3^k=2\cdot 3^k. \end{aligned}$$

The claim follows.

A closer look at the six cases shows that $f(2^n-t(m))\geq 3^{(n-1)/2}$ if $2^n-t(m)$ is divisible by 3, and $f(2^n-t(m))\leq 0$ otherwise. On the other hand, note that $2^n-t(m)$ is divisible by 3 if and only if 2^n+m is. Therefore, for all nonnegative integers m and n,

(i) $f(2^n-t(m))\geq 3^{(n-1)/2}$ if 2^n+m is divisible by 3;

(ii) $f(2^n-t(m))\leq 0$ if 2^n+m is not divisible by 3.

One more (direct) consequence of the claim is that $|f(2^n-t(m))|\leq \frac{2}{3}\cdot 3^{n/2}$ for all $m,n\geq 0$.

The last inequality enables us to find an upper bound for $|f(m)|$ for m less than a given power of 2. We prove by induction on n that $|f(m)|\leq 3^{n/2}$ holds true for all integers $m,n\geq 0$ with $2^n>m$.

The base $n = 0$ is clear as $f(0) = 1$. For the inductive step from n to $n+1$, let m and n satisfy $2^{n+1} > m$. If $m < 2^n$, we are done by the inductive hypothesis. If $m \geq 2^n$ then $m = 2^n + k$ where $2^n > k \geq 0$. Now, by $|f(2^n - t(k))| \leq \frac{2}{3} \cdot 3^{n/2}$ and the inductive assumption,

$$|f(m)| = |f(2^n - t(k)) - f(k)| \leq |f(2^n - t(k))| + |f(k)| \leq \frac{2}{3} \cdot 3^{n/2} + 3^{n/2} < 3^{(n+1)/2}.$$

The induction is complete.

We proceed to prove that $f(3p) \geq 0$ for all integers $p \geq 0$. Since $3p$ is not a power of 2, its binary expansion contains at least two summands. Hence one can write $3p = 2^a + 2^b + c$ where $a > b$ and $2^b > c \geq 0$. Applying the recurrence formula twice yields

$$f(3p) = f(2^a + 2^b + c) = f(2^a - t(2^b + c)) - f(2^b - t(c)) + f(c).$$

Since $2^a + 2^b + c$ is divisible by 3, we have $f(2^a - t(2^b + c)) \geq 3^{(a-1)/2}$ by (i). Since $2^b + c$ is not divisible by 3, we have $f(2^b - t(c)) \leq 0$ by (ii). Finally $|f(c)| \leq 3^{b/2}$ as $2^b > c \geq 0$, so that $f(c) \geq -3^{b/2}$. Therefore $f(3p) \geq 3^{(a-1)/2} - 3^{b/2}$ which is nonnegative because $a > b$.

A5. Let a, b, c, d be positive real numbers such that

$$abcd = 1 \quad \text{and} \quad a+b+c+d > \frac{a}{b}+\frac{b}{c}+\frac{c}{d}+\frac{d}{a}.$$

Prove that

$$a+b+c+d < \frac{b}{a}+\frac{c}{b}+\frac{d}{c}+\frac{a}{d}.$$

Solution. We show that if $abcd = 1$, the sum $a+b+c+d$ cannot exceed a certain weighted mean of the expressions $\frac{a}{b}+\frac{b}{c}+\frac{c}{d}+\frac{d}{a}$ and $\frac{b}{a}+\frac{c}{b}+\frac{d}{c}+\frac{a}{d}$.

By applying the AM-GM inequality to the numbers $\frac{a}{b}$, $\frac{a}{b}$, $\frac{b}{c}$ and $\frac{a}{d}$, we obtain

$$a = \sqrt[4]{\frac{a^4}{abcd}} = \sqrt[4]{\frac{a}{b}\cdot\frac{a}{b}\cdot\frac{b}{c}\cdot\frac{a}{d}} \le \frac{1}{4}\left(\frac{a}{b}+\frac{a}{b}+\frac{b}{c}+\frac{a}{d}\right).$$

Analogously,

$$b \le \frac{1}{4}\left(\frac{b}{c}+\frac{b}{c}+\frac{c}{d}+\frac{b}{a}\right), \quad c \le \frac{1}{4}\left(\frac{c}{d}+\frac{c}{d}+\frac{d}{a}+\frac{c}{b}\right) \quad \text{and} \quad d \le \frac{1}{4}\left(\frac{d}{a}+\frac{d}{a}+\frac{a}{b}+\frac{d}{c}\right).$$

Summing up these estimates yields

$$a+b+c+d \le \frac{3}{4}\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{d}+\frac{d}{a}\right)+\frac{1}{4}\left(\frac{b}{a}+\frac{c}{b}+\frac{d}{c}+\frac{a}{d}\right).$$

In particular, if $a+b+c+d > \frac{a}{b}+\frac{b}{c}+\frac{c}{d}+\frac{d}{a}$ then $a+b+c+d < \frac{b}{a}+\frac{c}{b}+\frac{d}{c}+\frac{a}{d}$.

Comment. The estimate in the above solution was obtained by applying the AM-GM inequality to each column of the 4×4 array

$$\begin{array}{cccc} a/b & b/c & c/d & d/a \\ a/b & b/c & c/d & d/a \\ b/c & c/d & d/a & a/b \\ a/d & b/a & c/b & d/c \end{array}$$

and adding up the resulting inequalities. The same table yields a stronger bound: If $a, b, c, d > 0$ and $abcd = 1$ then

$$\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{d}+\frac{d}{a}\right)^3\left(\frac{b}{a}+\frac{c}{b}+\frac{d}{c}+\frac{a}{d}\right) \ge (a+b+c+d)^4.$$

It suffices to apply Hölder's inequality to the sequences in the four rows, with weights $1/4$:

$$\begin{gathered}\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{d}+\frac{d}{a}\right)^{1/4}\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{d}+\frac{d}{a}\right)^{1/4}\left(\frac{b}{c}+\frac{c}{d}+\frac{d}{a}+\frac{a}{b}\right)^{1/4}\left(\frac{a}{d}+\frac{b}{a}+\frac{c}{b}+\frac{d}{c}\right)^{1/4} \\ \ge \left(\frac{aaba}{bbcd}\right)^{1/4}+\left(\frac{bbcb}{ccda}\right)^{1/4}+\left(\frac{ccdc}{ddab}\right)^{1/4}+\left(\frac{ddad}{aabc}\right)^{1/4} = a+b+c+d.\end{gathered}$$

A6. Let $f\colon \mathbb{R} \to \mathbb{N}$ be a function which satisfies

$$f\left(x + \frac{1}{f(y)}\right) = f\left(y + \frac{1}{f(x)}\right) \qquad \text{for all } x, y \in \mathbb{R}. \tag{1}$$

Prove that there is a positive integer which is not a value of f.

Solution. Suppose that the statement is false and $f(\mathbb{R}) = \mathbb{N}$. We prove several properties of the function f in order to reach a contradiction.

To start with, observe that one can assume $f(0) = 1$. Indeed, let $a \in \mathbb{R}$ be such that $f(a) = 1$, and consider the function $g(x) = f(x + a)$. By substituting $x + a$ and $y + a$ for x and y in (1), we have

$$g\left(x + \frac{1}{g(y)}\right) = f\left(x + a + \frac{1}{f(y + a)}\right) = f\left(y + a + \frac{1}{f(x + a)}\right) = g\left(y + \frac{1}{g(x)}\right).$$

So g satisfies the functional equation (1), with the additional property $g(0) = 1$. Also, g and f have the same set of values: $g(\mathbb{R}) = f(\mathbb{R}) = \mathbb{N}$. Henceforth we assume $f(0) = 1$.

Claim 1. For an arbitrary fixed $c \in \mathbb{R}$ we have $\left\{f\left(c + \frac{1}{n}\right) : n \in \mathbb{N}\right\} = \mathbb{N}$.

Proof. Equation (1) and $f(\mathbb{R}) = \mathbb{N}$ imply

$$f(\mathbb{R}) = \left\{f\left(x + \frac{1}{f(c)}\right) : x \in \mathbb{R}\right\} = \left\{f\left(c + \frac{1}{f(x)}\right) : x \in \mathbb{R}\right\} \subset \left\{f\left(c + \frac{1}{n}\right) : n \in \mathbb{N}\right\} \subset f(\mathbb{R}).$$

The claim follows.

We will use Claim 1 in the special cases $c = 0$ and $c = 1/3$:

$$\left\{f\left(\frac{1}{n}\right) : n \in \mathbb{N}\right\} = \left\{f\left(\frac{1}{3} + \frac{1}{n}\right) : n \in \mathbb{N}\right\} = \mathbb{N}. \tag{2}$$

Claim 2. If $f(u) = f(v)$ for some $u, v \in \mathbb{R}$ then $f(u+q) = f(v+q)$ for all nonnegative rational q. Furthermore, if $f(q) = 1$ for some nonnegative rational q then $f(kq) = 1$ for all $k \in \mathbb{N}$.

Proof. For all $x \in \mathbb{R}$ we have by (1)

$$f\left(u + \frac{1}{f(x)}\right) = f\left(x + \frac{1}{f(u)}\right) = f\left(x + \frac{1}{f(v)}\right) = f\left(v + \frac{1}{f(x)}\right).$$

Since $f(x)$ attains all positive integer values, this yields $f(u + 1/n) = f(v + 1/n)$ for all $n \in \mathbb{N}$. Let $q = k/n$ be a positive rational number. Then k repetitions of the last step yield

$$f(u + q) = f\left(u + \frac{k}{n}\right) = f\left(v + \frac{k}{n}\right) = f(v + q).$$

Now let $f(q) = 1$ for some nonnegative rational q, and let $k \in \mathbb{N}$. As $f(0) = 1$, the previous conclusion yields successively $f(q) = f(2q), f(2q) = f(3q), \ldots, f\left((k-1)q\right) = f(kq)$, as needed.

Claim 3. The equality $f(q) = f(q + 1)$ holds for all nonnegative rational q.

Proof. Let m be a positive integer such that $f(1/m) = 1$. Such an m exists by (2). Applying the second statement of Claim 2 with $q = 1/m$ and $k = m$ yields $f(1) = 1$.

Given that $f(0) = f(1) = 1$, the first statement of Claim 2 implies $f(q) = f(q + 1)$ for all nonnegative rational q.

Claim 4. The equality $f\left(\dfrac{1}{n}\right) = n$ holds for every $n \in \mathbb{N}$.

Proof. For a nonnegative rational q we set $x = q$, $y = 0$ in (1) and use Claim 3 to obtain

$$f\left(\frac{1}{f(q)}\right) = f\left(q + \frac{1}{f(0)}\right) = f(q+1) = f(q).$$

By (2), for each $n \in \mathbb{N}$ there exists a $k \in \mathbb{N}$ such that $f(1/k) = n$. Applying the last equation with $q = 1/k$, we have

$$n = f\left(\frac{1}{k}\right) = f\left(\frac{1}{f(1/k)}\right) = f\left(\frac{1}{n}\right).$$

Now we are ready to obtain a contradiction. Let $n \in \mathbb{N}$ be such that $f(1/3 + 1/n) = 1$. Such an n exists by (2). Let $1/3 + 1/n = s/t$, where $s, t \in \mathbb{N}$ are coprime. Observe that $t > 1$ as $1/3 + 1/n$ is not an integer. Choose $k, l \in \mathbb{N}$ so that that $ks - lt = 1$.

Because $f(0) = f(s/t) = 1$, Claim 2 implies $f(ks/t) = 1$. Now $f(ks/t) = f(1/t + l)$; on the other hand $f(1/t + l) = f(1/t)$ by l successive applications of Claim 3. Finally, $f(1/t) = t$ by Claim 4, leading to the impossible $t = 1$. The solution is complete.

A7. Prove that for any four positive real numbers a, b, c, d the inequality

$$\frac{(a-b)(a-c)}{a+b+c}+\frac{(b-c)(b-d)}{b+c+d}+\frac{(c-d)(c-a)}{c+d+a}+\frac{(d-a)(d-b)}{d+a+b} \geq 0$$

holds. Determine all cases of equality.

Solution 1. Denote the four terms by

$$A=\frac{(a-b)(a-c)}{a+b+c}, \quad B=\frac{(b-c)(b-d)}{b+c+d}, \quad C=\frac{(c-d)(c-a)}{c+d+a}, \quad D=\frac{(d-a)(d-b)}{d+a+b}.$$

The expression $2A$ splits into two summands as follows,

$$2A=A'+A'' \qquad \text{where} \qquad A'=\frac{(a-c)^2}{a+b+c}, \qquad A''=\frac{(a-c)(a-2b+c)}{a+b+c};$$

this is easily verified. We analogously represent $2B=B'+B''$, $2C=C'+C''$, $2B=D'+D''$ and examine each of the sums $A'+B'+C'+D'$ and $A''+B''+C''+D''$ separately.

Write $s=a+b+c+d$; the denominators become $s-d$, $s-a$, $s-b$, $s-c$. By the Cauchy-Schwarz inequality,

$$\begin{aligned}&\left(\frac{|a-c|}{\sqrt{s-d}}\cdot\sqrt{s-d}+\frac{|b-d|}{\sqrt{s-a}}\cdot\sqrt{s-a}+\frac{|c-a|}{\sqrt{s-b}}\cdot\sqrt{s-b}+\frac{|d-b|}{\sqrt{s-c}}\cdot\sqrt{s-c}\right)^2\\ &\qquad\leq\left(\frac{(a-c)^2}{s-d}+\frac{(b-d)^2}{s-a}+\frac{(c-a)^2}{s-b}+\frac{(d-b)^2}{s-c}\right)\big(4s-s\big)=3s\big(A'+B'+C'+D'\big).\end{aligned}$$

Hence

$$A'+B'+C'+D' \geq \frac{\big(2|a-c|+2|b-d|\big)^2}{3s} \geq \frac{16\cdot|a-c|\cdot|b-d|}{3s}. \tag{1}$$

Next we estimate the absolute value of the other sum. We couple A'' with C'' to obtain

$$\begin{aligned}A''+C'' &= \frac{(a-c)(a+c-2b)}{s-d}+\frac{(c-a)(c+a-2d)}{s-b}\\ &= \frac{(a-c)(a+c-2b)(s-b)+(c-a)(c+a-2d)(s-d)}{(s-d)(s-b)}\\ &= \frac{(a-c)\big(-2b(s-b)-b(a+c)+2d(s-d)+d(a+c)\big)}{s(a+c)+bd}\\ &= \frac{3(a-c)(d-b)(a+c)}{M}, \qquad \text{with} \qquad M=s(a+c)+bd.\end{aligned}$$

Hence by cyclic shift

$$B''+D''=\frac{3(b-d)(a-c)(b+d)}{N}, \qquad \text{with} \qquad N=s(b+d)+ca.$$

Thus

$$A''+B''+C''+D''=3(a-c)(b-d)\left(\frac{b+d}{N}-\frac{a+c}{M}\right)=\frac{3(a-c)(b-d)W}{MN} \tag{2}$$

where

$$W=(b+d)M-(a+c)N=bd(b+d)-ac(a+c). \tag{3}$$

Note that

$$MN > \big(ac(a+c) + bd(b+d)\big)s \geq |W| \cdot s. \tag{4}$$

Now (2) and (4) yield

$$|A'' + B'' + C'' + D''| \leq \frac{3 \cdot |a-c| \cdot |b-d|}{s}. \tag{5}$$

Combined with (1) this results in

$$\begin{aligned} 2(A+B+C+D) &= (A'+B'+C'+D') + (A''+B''+C''+D'') \\ &\geq \frac{16 \cdot |a-c| \cdot |b-d|}{3s} - \frac{3 \cdot |a-c| \cdot |b-d|}{s} = \frac{7 \cdot |a-c| \cdot |b-d|}{3(a+b+c+d)} \geq 0. \end{aligned}$$

This is the required inequality. From the last line we see that equality can be achieved only if either $a = c$ or $b = d$. Since we also need equality in (1), this implies that actually $a = c$ and $b = d$ must hold simultaneously, which is obviously also a sufficient condition.

Solution 2. We keep the notations A, B, C, D, s, and also M, N, W from the preceding solution; the definitions of M, N, W and relations (3), (4) in that solution did not depend on the foregoing considerations. Starting from

$$2A = \frac{(a-c)^2 + 3(a+c)(a-c)}{a+b+c} - 2a + 2c,$$

we get

$$\begin{aligned} 2(A+C) &= (a-c)^2 \left(\frac{1}{s-d} + \frac{1}{s-b} \right) + 3(a+c)(a-c) \left(\frac{1}{s-d} - \frac{1}{s-b} \right) \\ &= (a-c)^2 \, \frac{2s-b-d}{M} + 3(a+c)(a-c) \cdot \frac{d-b}{M} = \frac{p(a-c)^2 - 3(a+c)(a-c)(b-d)}{M} \end{aligned}$$

where $p = 2s - b - d = s + a + c$. Similarly, writing $q = s + b + d$ we have

$$2(B+D) = \frac{q(b-d)^2 - 3(b+d)(b-d)(c-a)}{N};$$

specific grouping of terms in the numerators has its aim. Note that $pq > 2s^2$. By adding the fractions expressing $2(A+C)$ and $2(B+D)$,

$$2(A+B+C+D) = \frac{p(a-c)^2}{M} + \frac{3(a-c)(b-d)W}{MN} + \frac{q(b-d)^2}{N}$$

with W defined by (3).

Substitution $x = (a-c)/M$, $y = (b-d)/N$ brings the required inequality to the form

$$2(A+B+C+D) = Mpx^2 + 3Wxy + Nqy^2 \geq 0. \tag{6}$$

It will be enough to verify that the discriminant $\Delta = 9W^2 - 4MNpq$ of the quadratic trinomial $Mpt^2 + 3Wt + Nq$ is negative; on setting $t = x/y$ one then gets (6). The first inequality in (4) together with $pq > 2s^2$ imply $4MNpq > 8s^3\big(ac(a+c) + bd(b+d)\big)$. Since

$$(a+c)s^3 > (a+c)^4 \geq 4ac(a+c)^2 \quad \text{and likewise} \quad (b+d)s^3 > 4bd(b+d)^2,$$

the estimate continues as follows,

$$4MNpq > 8\big(4(ac)^2(a+c)^2 + 4(bd)^2(b+d)^2\big) > 32\big(bd(b+d) - ac(a+c)\big)^2 = 32W^2 \geq 9W^2.$$

Thus indeed $\Delta < 0$. The desired inequality (6) hence results. It becomes an equality if and only if $x = y = 0$; equivalently, if and only if $a = c$ and simultaneously $b = d$.

Comment. The two solutions presented above do not differ significantly; large portions overlap. The properties of the number W turn out to be crucial in both approaches. The Cauchy-Schwarz inequality, applied in the first solution, is avoided in the second, which requires no knowledge beyond quadratic trinomials.

The estimates in the proof of $\Delta < 0$ in the second solution seem to be very wasteful. However, they come close to sharp when the terms in one of the pairs (a, c), (b, d) are equal and much bigger than those in the other pair.

In attempts to prove the inequality by just considering the six cases of arrangement of the numbers a, b, c, d on the real line, one soon discovers that the cases which create real trouble are precisely those in which a and c are both greater or both smaller than b and d.

Solution 3.

$$(a-b)(a-c)(a+b+d)(a+c+d)(b+c+d) =$$
$$= \Big((a-b)(a+b+d)\Big)\Big((a-c)(a+c+d)\Big)(b+c+d) =$$
$$= (a^2+ad-b^2-bd)(a^2+ad-c^2-cd)(b+c+d) =$$
$$= \big(a^4+2a^3d-a^2b^2-a^2bd-a^2c^2-a^2cd+a^2d^2-ab^2d-abd^2-ac^2d-acd^2+b^2c^2+b^2cd+bc^2d+bcd^2\big)(b+c+d) =$$
$$= a^4b+a^4c+a^4d+(b^3c^2+a^2d^3)-a^2c^3+(2a^3d^2-b^3a^2+c^3b^2)+$$
$$+(b^3cd-c^3da-d^3ab)+(2a^3bd+c^3db-d^3ac)+(2a^3cd-b^3da+d^3bc)$$
$$+(-a^2b^2c+3b^2c^2d-2ac^2d^2)+(-2a^2b^2d+2bc^2d^2)+(-a^2bc^2-2a^2c^2d-2ab^2d^2+2b^2cd^2)+$$
$$+(-2a^2bcd-ab^2cd-abc^2d-2abcd^2)$$

Introducing the notation $S_{xyzw} = \sum_{cyc} a^x b^y c^z d^w$, one can write

$$\sum_{cyc}(a-b)(a-c)(a+b+d)(a+c+d)(b+c+d) =$$
$$= S_{4100}+S_{4010}+S_{4001}+2S_{3200}-S_{3020}+2S_{3002}-S_{3110}+2S_{3101}+2S_{3011}-3S_{2120}-6S_{2111} =$$
$$+\left(S_{4100}+S_{4001}+\frac{1}{2}S_{3110}+\frac{1}{2}S_{3011}-3S_{2120}\right)+$$
$$+\left(S_{4010}-S_{3020}-\frac{3}{2}S_{3110}+\frac{3}{2}S_{3011}+\frac{9}{16}S_{2210}+\frac{9}{16}S_{2201}-\frac{9}{8}S_{2111}\right)+$$
$$+\frac{9}{16}\Big(S_{3200}-S_{2210}-S_{2201}+S_{3002}\Big)+\frac{23}{16}\Big(S_{3200}-2S_{3101}+S_{3002}\Big)+\frac{39}{8}\Big(S_{3101}-S_{2111}\Big),$$

where the expressions

$$S_{4100}+S_{4001}+\frac{1}{2}S_{3110}+\frac{1}{2}S_{3011}-3S_{2120} = \sum_{cyc}\left(a^4b+bc^4+\frac{1}{2}a^3bc+\frac{1}{2}abc^3-3a^2bc^2\right),$$

$$S_{4010}-S_{3020}-\frac{3}{2}S_{3110}+\frac{3}{2}S_{3011}+\frac{9}{16}S_{2210}+\frac{9}{16}S_{2201}-\frac{9}{8}S_{2111} = \sum_{cyc}a^2c\left(a-c-\frac{3}{4}b+\frac{3}{4}d\right)^2,$$

$$S_{3200}-S_{2210}-S_{2201}+S_{3002} = \sum_{cyc}b^2(a^3-a^2c-ac^2+c^3) = \sum_{cyc}b^2(a+c)(a-c)^2,$$

$$S_{3200}-2S_{3101}+S_{3002} = \sum_{cyc}a^3(b-d)^2 \qquad \text{and} \qquad S_{3101}-S_{2111} = \frac{1}{3}\sum_{cyc}bd(2a^3+c^3-3a^2c)$$

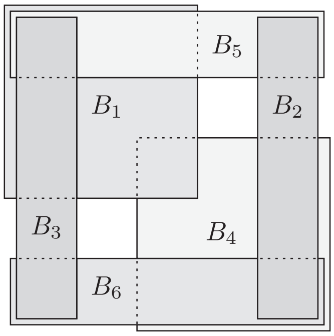
are all nonnegative.

Combinatorics

C1. In the plane we consider rectangles whose sides are parallel to the coordinate axes and have positive length. Such a rectangle will be called a *box*. Two boxes *intersect* if they have a common point in their interior or on their boundary.

Find the largest n for which there exist n boxes $B_1, \ldots, B_n$ such that B_i and B_j intersect if and only if $i \not\equiv j \pm 1 \pmod{n}$.

Solution. The maximum number of such boxes is 6. One example is shown in the figure.



Now we show that 6 is the maximum. Suppose that boxes $B_1, \ldots, B_n$ satisfy the condition. Let the closed intervals I_k and J_k be the projections of B_k onto the x- and y-axis, for $1 \leq k \leq n$.

If B_i and B_j intersect, with a common point (x, y), then $x \in I_i \cap I_j$ and $y \in J_i \cap J_j$. So the intersections $I_i \cap I_j$ and $J_i \cap J_j$ are nonempty. Conversely, if $x \in I_i \cap I_j$ and $y \in J_i \cap J_j$ for some real numbers x, y, then (x, y) is a common point of B_i and B_j. Putting it around, B_i and B_j are disjoint if and only if their projections on at least one coordinate axis are disjoint.

For brevity we call two boxes or intervals adjacent if their indices differ by 1 modulo n, and nonadjacent otherwise.

The adjacent boxes B_k and B_{k+1} do not intersect for each $k = 1, \ldots, n$. Hence (I_k, I_{k+1}) or (J_k, J_{k+1}) is a pair of disjoint intervals, $1 \leq k \leq n$. So there are at least n pairs of disjoint intervals among $(I_1, I_2), \ldots, (I_{n-1}, I_n), (I_n, I_1); (J_1, J_2), \ldots, (J_{n-1}, J_n), (J_n, J_1)$.

Next, every two nonadjacent boxes intersect, hence their projections on both axes intersect, too. Then the claim below shows that at most 3 pairs among $(I_1, I_2), \ldots, (I_{n-1}, I_n), (I_n, I_1)$ are disjoint, and the same holds for $(J_1, J_2), \ldots, (J_{n-1}, J_n), (J_n, J_1)$. Consequently $n \leq 3 + 3 = 6$, as stated. Thus we are left with the claim and its justification.

Claim. Let $\Delta_1, \Delta_2, \ldots, \Delta_n$ be intervals on a straight line such that every two nonadjacent intervals intersect. Then Δ_k and Δ_{k+1} are disjoint for at most three values of $k = 1, \ldots, n$.

Proof. Denote $\Delta_k = [a_k, b_k]$, $1 \leq k \leq n$. Let $\alpha = \max(a_1, \ldots, a_n)$ be the rightmost among the left endpoints of $\Delta_1, \ldots, \Delta_n$, and let $\beta = \min(b_1, \ldots, b_n)$ be the leftmost among their right endpoints. Assume that $\alpha = a_2$ without loss of generality.

If $\alpha \leq \beta$ then $a_i \leq \alpha \leq \beta \leq b_i$ for all i. Every Δ_i contains α, and thus no disjoint pair (Δ_i, Δ_{i+1}) exists.

If $\beta < \alpha$ then $\beta = b_i$ for some i such that $a_i < b_i = \beta < \alpha = a_2 < b_2$, hence Δ_2 and Δ_i are disjoint. Now Δ_2 intersects all remaining intervals except possibly Δ_1 and Δ_3, so Δ_2 and Δ_i can be disjoint only if $i = 1$ or $i = 3$. Suppose by symmetry that $i = 3$; then $\beta = b_3$. Since each of the intervals $\Delta_4, \ldots, \Delta_n$ intersects Δ_2, we have $a_i \leq \alpha \leq b_i$ for $i = 4, \ldots, n$. Therefore $\alpha \in \Delta_4 \cap \ldots \cap \Delta_n$, in particular $\Delta_4 \cap \ldots \cap \Delta_n \neq \emptyset$. Similarly, $\Delta_5, \ldots, \Delta_n, \Delta_1$ all intersect Δ_3, so that $\Delta_5 \cap \ldots \cap \Delta_n \cap \Delta_1 \neq \emptyset$ as $\beta \in \Delta_5 \cap \ldots \cap \Delta_n \cap \Delta_1$. This leaves (Δ_1, Δ_2), (Δ_2, Δ_3) and (Δ_3, Δ_4) as the only candidates for disjoint interval pairs, as desired.

Comment. The problem is a two-dimensional version of the original proposal which is included below. The extreme shortage of easy and appropriate submissions forced the Problem Selection Committee to shortlist a simplified variant. The same one-dimensional Claim is used in both versions.

Original proposal. We consider parallelepipeds in three-dimensional space, with edges parallel to the coordinate axes and of positive length. Such a parallelepiped will be called a *box*. Two boxes *intersect* if they have a common point in their interior or on their boundary.

Find the largest n for which there exist n boxes $B_1, \ldots, B_n$ such that B_i and B_j intersect if and only if $i \not\equiv j \pm 1 \pmod{n}$.

The maximum number of such boxes is 9. Suppose that boxes $B_1, \ldots, B_n$ satisfy the condition. Let the closed intervals I_k, J_k and K_k be the projections of box B_k onto the x-, y- and z-axis, respectively, for $1 \leq k \leq n$. As before, B_i and B_j are disjoint if and only if their projections on at least one coordinate axis are disjoint.

We call again two boxes or intervals adjacent if their indices differ by 1 modulo n, and nonadjacent otherwise.

The adjacent boxes B_i and B_{i+1} do not intersect for each $i = 1, \ldots, n$. Hence at least one of the pairs (I_i, I_{i+1}), (J_i, J_{i+1}) and (K_i, K_{i+1}) is a pair of disjoint intervals. So there are at least n pairs of disjoint intervals among $(I_i, I_{i+1}), (J_i, J_{i+1}), (K_i, K_{i+1})$, $1 \leq i \leq n$.

Next, every two nonadjacent boxes intersect, hence their projections on the three axes intersect, too. Referring to the Claim in the solution of the two-dimensional version, we cocnclude that at most 3 pairs among $(I_1, I_2), \ldots, (I_{n-1}, I_n), (I_n, I_1)$ are disjoint; the same holds for $(J_1, J_2), \ldots, (J_{n-1}, J_n), (J_n, J_1)$ and $(K_1, K_2), \ldots, (K_{n-1}, K_n), (K_n, K_1)$. Consequently $n \leq 3 + 3 + 3 = 9$, as stated.

For $n = 9$, the desired system of boxes exists. Consider the intervals in the following table:

i	I_i	J_i	K_i
1	$[1,4]$	$[1,6]$	$[3,6]$
2	$[5,6]$	$[1,6]$	$[1,6]$
3	$[1,2]$	$[1,6]$	$[1,6]$
4	$[3,6]$	$[1,4]$	$[1,6]$
5	$[1,6]$	$[5,6]$	$[1,6]$
6	$[1,6]$	$[1,2]$	$[1,6]$
7	$[1,6]$	$[3,6]$	$[1,4]$
8	$[1,6]$	$[1,6]$	$[5,6]$
9	$[1,6]$	$[1,6]$	$[1,2]$

We have $I_1 \cap I_2 = I_2 \cap I_3 = I_3 \cap I_4 = \emptyset$, $J_4 \cap J_5 = J_5 \cap J_6 = J_6 \cap J_7 = \emptyset$, and finally $K_7 \cap K_8 = K_8 \cap K_9 = K_9 \cap K_1 = \emptyset$. The intervals in each column intersect in all other cases. It follows that the boxes $B_i = I_i \times J_i \times K_i$, $i = 1, \ldots, 9$, have the stated property.

C2. For every positive integer n determine the number of permutations $(a_1, a_2, \dots, a_n)$ of the set $\{1, 2, \dots, n\}$ with the following property:

$$2(a_1 + a_2 + \cdots + a_k) \quad \text{is divisible by } k \quad \text{for } k = 1, 2, \dots, n.$$

Solution. For each n let F_n be the number of permutations of $\{1, 2, \dots, n\}$ with the required property; call them *nice*. For $n = 1, 2, 3$ every permutation is nice, so $F_1 = 1$, $F_2 = 2$, $F_3 = 6$.

Take an $n > 3$ and consider any nice permutation $(a_1, a_2, \dots, a_n)$ of $\{1, 2, \dots, n\}$. Then $n - 1$ must be a divisor of the number

$$\begin{aligned} 2(a_1 + a_2 + \cdots + a_{n-1}) &= 2\big((1 + 2 + \cdots + n) - a_n\big) \\ &= n(n+1) - 2a_n = (n+2)(n-1) + (2 - 2a_n). \end{aligned}$$

So $2a_n - 2$ must be divisible by $n - 1$, hence equal to 0 or $n - 1$ or $2n - 2$. This means that

$$a_n = 1 \qquad \text{or} \qquad a_n = \frac{n+1}{2} \qquad \text{or} \qquad a_n = n.$$

Suppose that $a_n = (n + 1)/2$. Since the permutation is nice, taking $k = n - 2$ we get that $n - 2$ has to be a divisor of

$$\begin{aligned} 2(a_1 + a_2 + \cdots + a_{n-2}) &= 2\big((1 + 2 + \cdots + n) - a_n - a_{n-1}\big) \\ &= n(n+1) - (n+1) - 2a_{n-1} = (n+2)(n-2) + (3 - 2a_{n-1}). \end{aligned}$$

So $2a_{n-1} - 3$ should be divisible by $n - 2$, hence equal to 0 or $n - 2$ or $2n - 4$. Obviously 0 and $2n - 4$ are excluded because $2a_{n-1} - 3$ is odd. The remaining possibility ($2a_{n-1} - 3 = n - 2$) leads to $a_{n-1} = (n + 1)/2 = a_n$, which also cannot hold. This eliminates $(n + 1)/2$ as a possible value of a_n. Consequently $a_n = 1$ or $a_n = n$.

If $a_n = n$ then $(a_1, a_2, \dots, a_{n-1})$ is a nice permutation of $\{1, 2, \dots, n-1\}$. There are F_{n-1} such permutations. Attaching n to any one of them at the end creates a nice permutation of $\{1, 2, \dots, n\}$.

If $a_n = 1$ then $(a_1-1, a_2-1, \dots, a_{n-1}-1)$ is a permutation of $\{1, 2, \dots, n-1\}$. It is also nice because the number

$$2\big((a_1-1) + \cdots + (a_k-1)\big) = 2(a_1 + \cdots + a_k) - 2k$$

is divisible by k, for any $k \le n - 1$. And again, any one of the F_{n-1} nice permutations $(b_1, b_2, \dots, b_{n-1})$ of $\{1, 2, \dots, n-1\}$ gives rise to a nice permutation of $\{1, 2, \dots, n\}$ whose last term is 1, namely $(b_1+1, b_2+1, \dots, b_{n-1}+1, 1)$.

The bijective correspondences established in both cases show that there are F_{n-1} nice permutations of $\{1, 2, \dots, n\}$ with the last term 1 and also F_{n-1} nice permutations of $\{1, 2, \dots, n\}$ with the last term n. Hence follows the recurrence $F_n = 2F_{n-1}$. With the base value $F_3 = 6$ this gives the outcome formula $F_n = 3 \cdot 2^{n-2}$ for $n \ge 3$.

C3. In the coordinate plane consider the set S of all points with integer coordinates. For a positive integer k, two distinct points $A, B \in S$ will be called k-*friends* if there is a point $C \in S$ such that the area of the triangle ABC is equal to k. A set $T \subset S$ will be called a k-*clique* if every two points in T are k-friends. Find the least positive integer k for which there exists a k-clique with more than 200 elements.

Solution. To begin, let us describe those points $B \in S$ which are k-friends of the point $(0, 0)$. By definition, $B = (u, v)$ satisfies this condition if and only if there is a point $C = (x, y) \in S$ such that $\frac{1}{2}|uy - vx| = k$. (This is a well-known formula expressing the area of triangle ABC when A is the origin.)

To say that there exist integers x, y for which $|uy - vx| = 2k$, is equivalent to saying that the greatest common divisor of u and v is also a divisor of $2k$. Summing up, a point $B = (u, v) \in S$ is a k-friend of $(0, 0)$ if and only if $\gcd(u, v)$ divides $2k$.

Translation by a vector with integer coordinates does not affect k-friendship; if two points are k-friends, so are their translates. It follows that two points $A, B \in S$, $A = (s, t)$, $B = (u, v)$, are k-friends if and only if the point $(u - s, v - t)$ is a k-friend of $(0, 0)$; i.e., if $\gcd(u - s, v - t) | 2k$.

Let n be a positive integer which does not divide $2k$. We claim that a k-clique cannot have more than n^2 elements.

Indeed, all points $(x, y) \in S$ can be divided into n^2 classes determined by the remainders that x and y leave in division by n. If a set T has more than n^2 elements, some two points $A, B \in T$, $A = (s, t)$, $B = (u, v)$, necessarily fall into the same class. This means that $n | u - s$ and $n | v - t$. Hence $n | d$ where $d = \gcd(u - s, v - t)$. And since n does not divide $2k$, also d does not divide $2k$. Thus A and B are not k-friends and the set T is not a k-clique.

Now let $M(k)$ be the *least* positive integer which does not divide $2k$. Write $M(k) = m$ for the moment and consider the set T of all points (x, y) with $0 \le x, y < m$. There are m^2 of them. If $A = (s, t)$, $B = (u, v)$ are two distinct points in T then both differences $|u - s|$, $|v - t|$ are integers less than m and at least one of them is positive. By the definition of m, every positive integer less than m divides $2k$. Therefore $u - s$ (if nonzero) divides $2k$, and the same is true of $v - t$. So $2k$ is divisible by $\gcd(u - s, v - t)$, meaning that A, B are k-friends. Thus T is a k-clique.

It follows that the maximum size of a k-clique is $M(k)^2$, with $M(k)$ defined as above. We are looking for the minimum k such that $M(k)^2 > 200$.

By the definition of $M(k)$, $2k$ is divisible by the numbers $1, 2, \ldots, M(k)-1$, but not by $M(k)$ itself. If $M(k)^2 > 200$ then $M(k) \ge 15$. Trying to hit $M(k) = 15$ we get a contradiction immediately ($2k$ would have to be divisible by 3 and 5, but not by 15).

So let us try $M(k) = 16$. Then $2k$ is divisible by the numbers $1, 2, \ldots, 15$, hence also by their least common multiple L, but not by 16. And since L is not a multiple of 16, we infer that $k = L/2$ is the least k with $M(k) = 16$.

Finally, observe that if $M(k) \ge 17$ then $2k$ must be divisible by the least common multiple of $1, 2, \ldots, 16$, which is equal to $2L$. Then $2k \ge 2L$, yielding $k > L/2$.

In conclusion, the least k with the required property is equal to $L/2 = 180180$.

C4. Let n and k be fixed positive integers of the same parity, $k \geq n$. We are given $2n$ lamps numbered 1 through $2n$; each of them can be *on* or *off*. At the beginning all lamps are *off*. We consider sequences of k steps. At each step one of the lamps is switched (from *off* to *on* or from *on* to *off*).

Let N be the number of k-step sequences ending in the state: lamps $1, \ldots, n$ *on*, lamps $n+1, \ldots, 2n$ *off*.

Let M be the number of k-step sequences leading to the same state and not touching lamps $n+1, \ldots, 2n$ at all.

Find the ratio N/M.

Solution. A sequence of k switches ending in the state as described in the problem statement (lamps $1, \ldots, n$ *on*, lamps $n+1, \ldots, 2n$ *off*) will be called an *admissible process*. If, moreover, the process does not touch the lamps $n+1, \ldots, 2n$, it will be called *restricted*. So there are N admissible processes, among which M are restricted.

In every admissible process, restricted or not, each one of the lamps $1, \ldots, n$ goes from *off* to *on*, so it is switched an odd number of times; and each one of the lamps $n+1, \ldots, 2n$ goes from *off* to *off*, so it is switched an even number of times.

Notice that $M > 0$; i.e., restricted admissible processes do exist (it suffices to switch each one of the lamps $1, \ldots, n$ just once and then choose one of them and switch it $k - n$ times, which by hypothesis is an even number).

Consider any restricted admissible process $\mathbf{p}$. Take any lamp ℓ, $1 \leq \ell \leq n$, and suppose that it was switched k_ℓ times. As noticed, k_ℓ must be odd. Select arbitrarily an even number of these k_ℓ switches and replace each of them by the switch of lamp $n+\ell$. This can be done in $2^{k_\ell - 1}$ ways (because a k_ℓ-element set has $2^{k_\ell - 1}$ subsets of even cardinality). Notice that $k_1 + \cdots + k_n = k$.

These actions are independent, in the sense that the action involving lamp ℓ does not affect the action involving any other lamp. So there are $2^{k_1 - 1} \cdot 2^{k_2 - 1} \cdots 2^{k_n - 1} = 2^{k-n}$ ways of combining these actions. In any of these combinations, each one of the lamps $n+1, \ldots, 2n$ gets switched an even number of times and each one of the lamps $1, \ldots, n$ remains switched an odd number of times, so the final state is the same as that resulting from the original process $\mathbf{p}$.

This shows that every restricted admissible process $\mathbf{p}$ can be modified in 2^{k-n} ways, giving rise to 2^{k-n} distinct admissible processes (with all lamps allowed).

Now we show that every admissible process $\mathbf{q}$ can be achieved in that way. Indeed, it is enough to replace every switch of a lamp with a label $\ell > n$ that occurs in $\mathbf{q}$ by the switch of the corresponding lamp $\ell - n$; in the resulting process $\mathbf{p}$ the lamps $n+1, \ldots, 2n$ are not involved.

Switches of each lamp with a label $\ell > n$ had occurred in $\mathbf{q}$ an even number of times. So the performed replacements have affected each lamp with a label $\ell \leq n$ also an even number of times; hence in the overall effect the final state of each lamp has remained the same. This means that the resulting process $\mathbf{p}$ is admissible—and clearly restricted, as the lamps $n+1, \ldots, 2n$ are not involved in it any more.

If we now take process $\mathbf{p}$ and reverse all these replacements, then we obtain process $\mathbf{q}$. These reversed replacements are nothing else than the modifications described in the foregoing paragraphs.

Thus there is a one–to–(2^{k-n}) correspondence between the M restricted admissible processes and the total of N admissible processes. Therefore $N/M = 2^{k-n}$.

C5. Let $S = \{x_1, x_2, \ldots, x_{k+\ell}\}$ be a $(k+\ell)$-element set of real numbers contained in the interval $[0,1]$; k and ℓ are positive integers. A k-element subset $A \subset S$ is called *nice* if

$$\left| \frac{1}{k} \sum_{x_i \in A} x_i - \frac{1}{\ell} \sum_{x_j \in S \setminus A} x_j \right| \le \frac{k+\ell}{2k\ell}.$$

Prove that the number of nice subsets is at least $\dfrac{2}{k+\ell}\dbinom{k+\ell}{k}$.

Solution. For a k-element subset $A \subset S$, let $f(A) = \dfrac{1}{k} \displaystyle\sum_{x_i \in A} x_i - \dfrac{1}{\ell} \sum_{x_j \in S \setminus A} x_j$. Denote $\dfrac{k+\ell}{2k\ell} = d$. By definition a subset A is nice if $|f(A)| \le d$.

To each permutation $(y_1, y_2, \ldots, y_{k+\ell})$ of the set $S = \{x_1, x_2, \ldots, x_{k+\ell}\}$ we assign $k+\ell$ subsets of S with k elements each, namely $A_i = \{y_i, y_{i+1}, \ldots, y_{i+k-1}\}$, $i = 1, 2, \ldots, k+\ell$. Indices are taken modulo $k+\ell$ here and henceforth. In other words, if $y_1, y_2, \ldots, y_{k+\ell}$ are arranged around a circle in this order, the sets in question are all possible blocks of k consecutive elements.

Claim. At least two nice sets are assigned to every permutation of S.

Proof. Adjacent sets A_i and A_{i+1} differ only by the elements y_i and y_{i+k}, $i = 1, \ldots, k+\ell$. By the definition of f, and because $y_i, y_{i+k} \in [0,1]$,

$$|f(A_{i+1}) - f(A_i)| = \left| \left(\frac{1}{k} + \frac{1}{\ell} \right) (y_{i+k} - y_i) \right| \le \frac{1}{k} + \frac{1}{\ell} = 2d.$$

Each element $y_i \in S$ belongs to exactly k of the sets $A_1, \ldots, A_{k+\ell}$. Hence in k of the expressions $f(A_1), \ldots, f(A_{k+\ell})$ the coefficient of y_i is $1/k$; in the remaining ℓ expressions, its coefficient is $-1/\ell$. So the contribution of y_i to the sum of all $f(A_i)$ equals $k \cdot 1/k - \ell \cdot 1/\ell = 0$. Since this holds for all i, it follows that $f(A_1) + \cdots + f(A_{k+\ell}) = 0$.

If $f(A_p) = \min f(A_i)$, $f(A_q) = \max f(A_i)$, we obtain in particular $f(A_p) \le 0$, $f(A_q) \ge 0$. Let $p < q$ (the case $p > q$ is analogous; and the claim is true for $p = q$ as $f(A_i) = 0$ for all i).

We are ready to prove that at least two of the sets $A_1, \ldots, A_{k+\ell}$ are nice. The interval $[-d, d]$ has length $2d$, and we saw that adjacent numbers in the circular arrangement $f(A_1), \ldots, f(A_{k+\ell})$ differ by at most $2d$. Suppose that $f(A_p) < -d$ and $f(A_q) > d$. Then one of the numbers $f(A_{p+1}), \ldots, f(A_{q-1})$ lies in $[-d, d]$, and also one of the numbers $f(A_{q+1}), \ldots, f(A_{p-1})$ lies there. Consequently, one of the sets $A_{p+1}, \ldots, A_{q-1}$ is nice, as well as one of the sets $A_{q+1}, \ldots, A_{p-1}$. If $-d \le f(A_p)$ and $f(A_q) \le d$ then A_p and A_q are nice.

Let now $f(A_p) < -d$ and $f(A_q) \le d$. Then $f(A_p) + f(A_q) < 0$, and since $\sum f(A_i) = 0$, there is an $r \ne q$ such that $f(A_r) > 0$. We have $0 < f(A_r) \le f(A_q) \le d$, so the sets $f(A_r)$ and $f(A_q)$ are nice. The only case remaining, $-d \le f(A_p)$ and $d < f(A_q)$, is analogous.

Apply the claim to each of the $(k+\ell)!$ permutations of $S = \{x_1, x_2, \ldots, x_{k+\ell}\}$. This gives at least $2(k+\ell)!$ nice sets, counted with repetitions: each nice set is counted as many times as there are permutations to which it is assigned.

On the other hand, each k-element set $A \subset S$ is assigned to exactly $(k+\ell)\, k!\, \ell!$ permutations. Indeed, such a permutation $(y_1, y_2, \ldots, y_{k+\ell})$ is determined by three independent choices: an index $i \in \{1, 2, \ldots, k+\ell\}$ such that $A = \{y_i, y_{i+1}, \ldots, y_{i+k-1}\}$, a permutation $(y_i, y_{i+1}, \ldots, y_{i+k-1})$ of the set A, and a permutation $(y_{i+k}, y_{i+k+1}, \ldots, y_{i-1})$ of the set $S \setminus A$.

In summary, there are at least $\dfrac{2(k+\ell)!}{(k+\ell)\, k!\, \ell!} = \dfrac{2}{k+\ell}\dbinom{k+\ell}{k}$ nice sets.

C6. For $n \geq 2$, let $S_1, S_2, \ldots, S_{2^n}$ be 2^n subsets of $A = \{1, 2, 3, \ldots, 2^{n+1}\}$ that satisfy the following property: There do not exist indices a and b with $a < b$ and elements $x, y, z \in A$ with $x < y < z$ such that $y, z \in S_a$ and $x, z \in S_b$. Prove that at least one of the sets $S_1, S_2, \ldots, S_{2^n}$ contains no more than $4n$ elements.

Solution 1. We prove that there exists a set S_a with at most $3n + 1$ elements.

Given a $k \in \{1, \ldots, n\}$, we say that an element $z \in A$ is k-*good* to a set S_a if $z \in S_a$ and S_a contains two other elements x and y with $x < y < z$ such that $z - y < 2^k$ and $z - x \geq 2^k$. Also, $z \in A$ will be called *good* to S_a if z is k-good to S_a for some $k = 1, \ldots, n$.

We claim that each $z \in A$ can be k-good to at most one set S_a. Indeed, suppose on the contrary that z is k-good simultaneously to S_a and S_b, with $a < b$. Then there exist $y_a \in S_a$, $y_a < z$, and $x_b \in S_b$, $x_b < z$, such that $z - y_a < 2^k$ and $z - x_b \geq 2^k$. On the other hand, since $z \in S_a \cap S_b$, by the condition of the problem there is no element of S_a strictly between x_b and z. Hence $y_a \leq x_b$, implying $z - y_a \geq z - x_b$. However this contradicts $z - y_a < 2^k$ and $z - x_b \geq 2^k$. The claim follows.

As a consequence, a fixed $z \in A$ can be good to at most n of the given sets (no more than one of them for each $k = 1, \ldots, n$).

Furthermore, let $u_1 < u_2 < \cdots < u_m < \cdots < u_p$ be all elements of a fixed set S_a that are not good to S_a. We prove that $u_m - u_1 > 2(u_{m-1} - u_1)$ for all $m \geq 3$.

Indeed, assume that $u_m - u_1 \leq 2(u_{m-1} - u_1)$ holds for some $m \geq 3$. This inequality can be written as $2(u_m - u_{m-1}) \leq u_m - u_1$. Take the unique k such that $2^k \leq u_m - u_1 < 2^{k+1}$. Then $2(u_m - u_{m-1}) \leq u_m - u_1 < 2^{k+1}$ yields $u_m - u_{m-1} < 2^k$. However the elements $z = u_m$, $x = u_1$, $y = u_{m-1}$ of S_a then satisfy $z - y < 2^k$ and $z - x \geq 2^k$, so that $z = u_m$ is k-good to S_a.

Thus each term of the sequence $u_2 - u_1, u_3 - u_1, \ldots, u_p - u_1$ is more than twice the previous one. Hence $u_p - u_1 > 2^{p-1}(u_2 - u_1) \geq 2^{p-1}$. But $u_p \in \{1, 2, 3, \ldots, 2^{n+1}\}$, so that $u_p \leq 2^{n+1}$. This yields $p - 1 \leq n$, i. e. $p \leq n + 1$.

In other words, each set S_a contains at most $n + 1$ elements that are not good to it.

To summarize the conclusions, mark with red all elements in the sets S_a that are good to the respective set, and with blue the ones that are not good. Then the total number of red elements, counting multiplicities, is at most $n \cdot 2^{n+1}$ (each $z \in A$ can be marked red in at most n sets). The total number of blue elements is at most $(n + 1)2^n$ (each set S_a contains at most $n + 1$ blue elements). Therefore the sum of cardinalities of $S_1, S_2, \ldots, S_{2^n}$ does not exceed $(3n + 1)2^n$. By averaging, the smallest set has at most $3n + 1$ elements.

Solution 2. We show that one of the sets S_a has at most $2n + 1$ elements. In the sequel $|\cdot|$ denotes the cardinality of a (finite) set.

Claim. For $n \geq 2$, suppose that k subsets $S_1, \ldots, S_k$ of $\{1, 2, \ldots, 2^n\}$ (not necessarily different) satisfy the condition of the problem. Then

$$\sum_{i=1}^{k} (|S_i| - n) \leq (2n - 1)2^{n-2}.$$

Proof. Observe that if the sets S_i $(1 \leq i \leq k)$ satisfy the condition then so do their arbitrary subsets T_i $(1 \leq i \leq k)$. The condition also holds for the sets $t + S_i = \{t + x \mid x \in S_i\}$ where t is arbitrary.

Note also that a set may occur more than once among $S_1, \ldots, S_k$ only if its cardinality is less than 3, in which case its contribution to the sum $\sum_{i=1}^{k}(|S_i| - n)$ is nonpositive (as $n \geq 2$).

The proof is by induction on n. In the base case $n = 2$ we have subsets S_i of $\{1, 2, 3, 4\}$. Only the ones of cardinality 3 and 4 need to be considered by the remark above; each one of them occurs at most once among $S_1, \ldots, S_k$. If $S_i = \{1,2,3,4\}$ for some i then no S_j is a 3-element subset in view of the condition, hence $\sum_{i=1}^{k}(|S_i|-2) \leq 2$. By the condition again, it is impossible that $S_i = \{1,3,4\}$ and $S_j = \{2,3,4\}$ for some i, j. So if $|S_i| \leq 3$ for all i then at most 3 summands $|S_i| - 2$ are positive, corresponding to 3-element subsets. This implies $\sum_{i=1}^{k}(|S_i|-2) \leq 3$, therefore the conclusion is true for $n = 2$.

Suppose that the claim holds for some $n \geq 2$, and let the sets $S_1, \ldots, S_k \subseteq \{1, 2, \ldots, 2^{n+1}\}$ satisfy the given property. Denote $U_i = S_i \cap \{1, 2, \ldots, 2^n\}$, $V_i = S_i \cap \{2^n + 1, \ldots, 2^{n+1}\}$. Let

$$I = \{i \mid 1 \leq i \leq k,\ |U_i| \neq 0\}, \qquad J = \{1, \ldots, k\} \setminus I.$$

The sets S_j with $j \in J$ are all contained in $\{2^n + 1, \ldots, 2^{n+1}\}$, so the induction hypothesis applies to their translates $-2^n + S_j$ which have the same cardinalities. Consequently, this gives $\sum_{j \in J}(|S_j| - n) \leq (2n-1)2^{n-2}$, so that

$$\sum_{j \in J}(|S_j| - (n+1)) \leq \sum_{j \in J}(|S_j| - n) \leq (2n-1)2^{n-2}. \tag{1}$$

For $i \in I$, denote by v_i the least element of V_i. Observe that if V_a and V_b intersect, with $a < b$, $a, b \in I$, then v_a is their unique common element. Indeed, let $z \in V_a \cap V_b \subseteq S_a \cap S_b$ and let m be the least element of S_b. Since $b \in I$, we have $m \leq 2^n$. By the condition, there is no element of S_a strictly between $m \leq 2^n$ and $z > 2^n$, which implies $z = v_a$.

It follows that if the element v_i is removed from each V_i, a family of pairwise disjoint sets $W_i = V_i \setminus \{v_i\}$ is obtained, $i \in I$ (we assume $W_i = \emptyset$ if $V_i = \emptyset$). As $W_i \subseteq \{2^n + 1, \ldots, 2^{n+1}\}$ for all i, we infer that $\sum_{i \in I}|W_i| \leq 2^n$. Therefore $\sum_{i \in I}(|V_i| - 1) \leq \sum_{i \in I}|W_i| \leq 2^n$.

On the other hand, the induction hypothesis applies directly to the sets U_i, $i \in I$, so that $\sum_{i \in \mathcal{I}}(|U_i| - n) \leq (2n-1)2^{n-2}$. In summary,

$$\sum_{i \in I}(|S_i| - (n+1)) = \sum_{i \in I}(|U_i| - n) + \sum_{i \in I}(|V_i| - 1) \leq (2n-1)2^{n-2} + 2^n. \tag{2}$$

The estimates (1) and (2) are sufficient to complete the inductive step:

$$\begin{aligned} \sum_{i=1}^{k}(|S_i| - (n+1)) &= \sum_{i \in I}(|S_i| - (n+1)) + \sum_{j \in J}(|S_j| - (n+1)) \\ &\leq (2n-1)2^{n-2} + 2^n + (2n-1)2^{n-2} = (2n+1)2^{n-1}. \end{aligned}$$

Returning to the problem, consider $k = 2^n$ subsets $S_1, S_2, \ldots, S_{2^n}$ of $\{1, 2, 3, \ldots, 2^{n+1}\}$. If they satisfy the given condition, the claim implies $\sum_{i=1}^{2^n}(|S_i| - (n+1)) \leq (2n+1)2^{n-1}$. By averaging again, we see that the smallest set has at most $2n + 1$ elements.

Comment. It can happen that each set S_i has cardinality at least $n + 1$. Here is an example by the proposer.

For $i = 1, \ldots, 2^n$, let $S_i = \{i + 2^k \mid 0 \leq k \leq n\}$. Then $|S_i| = n + 1$ for all i. Suppose that there exist $a < b$ and $x < y < z$ such that $y, z \in S_a$ and $x, z \in S_b$. Hence $z = a + 2^k = b + 2^l$ for some $k > l$. Since $y \in S_a$ and $y < z$, we have $y \leq a + 2^{k-1}$. So the element $x \in S_b$ satisfies

$$x < y \leq a + 2^{k-1} = z - 2^{k-1} \leq z - 2^l = b.$$

However the least element of S_b is $b + 1$, a contradiction.

Geometry

G1. In an acute-angled triangle ABC, point H is the orthocentre and A_0, B_0, C_0 are the midpoints of the sides BC, CA, AB, respectively. Consider three circles passing through H: ω_a around A_0, ω_b around B_0 and ω_c around C_0. The circle ω_a intersects the line BC at A_1 and A_2; ω_b intersects CA at B_1 and B_2; ω_c intersects AB at C_1 and C_2. Show that the points A_1, A_2, B_1, B_2, C_1, C_2 lie on a circle.

Solution 1. The perpendicular bisectors of the segments A_1A_2, B_1B_2, C_1C_2 are also the perpendicular bisectors of BC, CA, AB. So they meet at O, the circumcentre of ABC. Thus O is the only point that can possibly be the centre of the desired circle.

From the right triangle OA_0A_1 we get

$$OA_1^2 = OA_0^2 + A_0A_1^2 = OA_0^2 + A_0H^2. \tag{1}$$

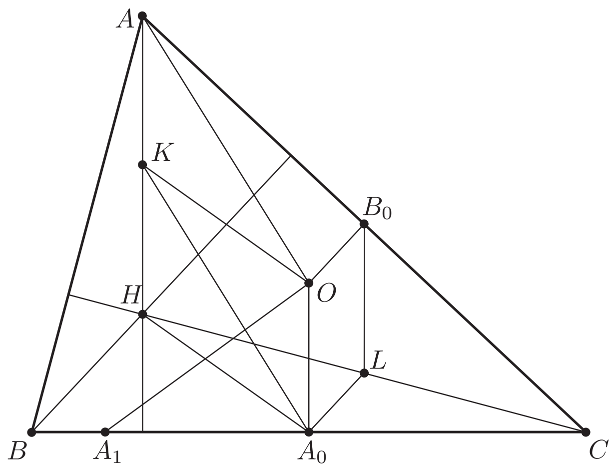
Let K be the midpoint of AH and let L be the midpoint of CH. Since A_0 and B_0 are the midpoints of BC and CA, we see that $A_0L \| BH$ and $B_0L \| AH$. Thus the segments A_0L and B_0L are perpendicular to AC and BC, hence parallel to OB_0 and OA_0, respectively. Consequently OA_0LB_0 is a parallelogram, so that OA_0 and B_0L are equal and parallel. Also, the midline B_0L of triangle AHC is equal and parallel to AK and KH.

It follows that AKA_0O and HA_0OK are parallelograms. The first one gives $A_0K = OA = R$, where R is the circumradius of ABC. From the second one we obtain

$$2(OA_0^2 + A_0H^2) = OH^2 + A_0K^2 = OH^2 + R^2. \tag{2}$$

(In a parallelogram, the sum of squares of the diagonals equals the sum of squares of the sides).

From (1) and (2) we get $OA_1^2 = (OH^2 + R^2)/2$. By symmetry, the same holds for the distances OA_2, OB_1, OB_2, OC_1 and OC_2. Thus A_1, A_2, B_1, B_2, C_1, C_2 all lie on a circle with centre at O and radius $(OH^2 + R^2)/2$.

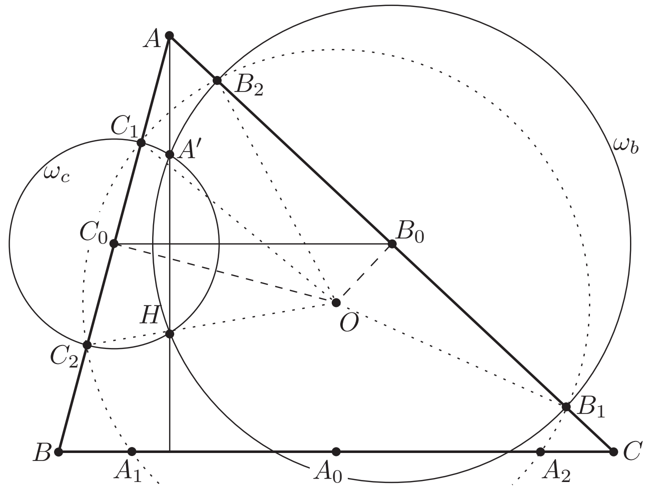


Solution 2. We are going to show again that the circumcentre O is equidistant from the six points in question.

Let A' be the second intersection point of ω_b and ω_c. The line B_0C_0, which is the line of centers of circles ω_b and ω_c, is a midline in triangle ABC, parallel to BC and perpendicular to the altitude AH. The points A' and H are symmetric with respect to the line of centers. Therefore A' lies on the line AH.

From the two circles ω_b and ω_c we obtain $AC_1 \cdot AC_2 = AA' \cdot AH = AB_1 \cdot AB_2$. So the quadrilateral $B_1B_2C_1C_2$ is cyclic. The perpendicular bisectors of the sides B_1B_2 and C_1C_2 meet at O. Hence O is the circumcentre of $B_1B_2C_1C_2$ and so $OB_1 = OB_2 = OC_1 = OC_2$.

Analogous arguments yield $OA_1 = OA_2 = OB_1 = OB_2$ and $OA_1 = OA_2 = OC_1 = OC_2$. Thus A_1, A_2, B_1, B_2, C_1, C_2 lie on a circle centred at O.



Comment. The problem can be solved without much difficulty in many ways by calculation, using trigonometry, coordinate geometry or complex numbers. As an example we present a short proof using vectors.

Solution 3. Let again O and R be the circumcentre and circumradius. Consider the vectors

$$\overrightarrow{OA} = \mathbf{a}, \quad \overrightarrow{OB} = \mathbf{b}, \quad \overrightarrow{OC} = \mathbf{c}, \quad \text{where} \quad \mathbf{a}^2 = \mathbf{b}^2 = \mathbf{c}^2 = R^2.$$

It is well known that $\overrightarrow{OH} = \mathbf{a} + \mathbf{b} + \mathbf{c}$. Accordingly,

$$\overrightarrow{A_0H} = \overrightarrow{OH} - \overrightarrow{OA_0} = (\mathbf{a} + \mathbf{b} + \mathbf{c}) - \frac{\mathbf{b} + \mathbf{c}}{2} = \frac{2\mathbf{a} + \mathbf{b} + \mathbf{c}}{2},$$

and

$$OA_1^2 = OA_0^2 + A_0A_1^2 = OA_0^2 + A_0H^2 = \left(\frac{\mathbf{b} + \mathbf{c}}{2}\right)^2 + \left(\frac{2\mathbf{a} + \mathbf{b} + \mathbf{c}}{2}\right)^2$$

$$= \frac{1}{4}(\mathbf{b}^2 + 2\mathbf{bc} + \mathbf{c}^2) + \frac{1}{4}(4\mathbf{a}^2 + 4\mathbf{ab} + 4\mathbf{ac} + \mathbf{b}^2 + 2\mathbf{bc} + \mathbf{c}^2) = 2R^2 + (\mathbf{ab} + \mathbf{ac} + \mathbf{bc});$$

here $\mathbf{ab}$, $\mathbf{bc}$, etc. denote dot products of vectors. We get the same for the distances OA_2, OB_1, OB_2, OC_1 and OC_2.

G2. Given trapezoid $ABCD$ with parallel sides AB and CD, assume that there exist points E on line BC outside segment BC, and F inside segment AD, such that $\angle DAE = \angle CBF$. Denote by I the point of intersection of CD and EF, and by J the point of intersection of AB and EF. Let K be the midpoint of segment EF; assume it does not lie on line AB.

Prove that I belongs to the circumcircle of ABK if and only if K belongs to the circumcircle of CDJ.

Solution. Assume that the disposition of points is as in the diagram.

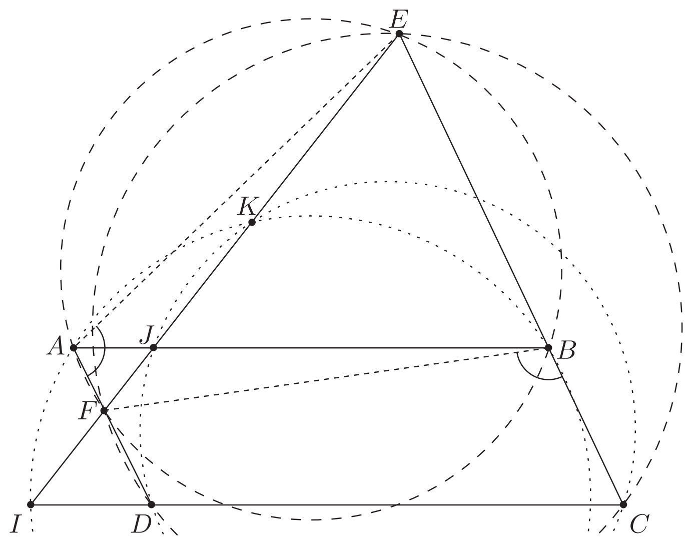
Since $\angle EBF = 180° - \angle CBF = 180° - \angle EAF$ by hypothesis, the quadrilateral $AEBF$ is cyclic. Hence $AJ \cdot JB = FJ \cdot JE$. In view of this equality, I belongs to the circumcircle of ABK if and only if $IJ \cdot JK = FJ \cdot JE$. Expressing $IJ = IF + FJ$, $JE = FE - FJ$, and $JK = \frac{1}{2}FE - FJ$, we find that I belongs to the circumcircle of ABK if and only if

$$FJ = \frac{IF \cdot FE}{2IF + FE}.$$

Since $AEBF$ is cyclic and AB, CD are parallel, $\angle FEC = \angle FAB = 180° - \angle CDF$. Then $CDFE$ is also cyclic, yielding $ID \cdot IC = IF \cdot IE$. It follows that K belongs to the circumcircle of CDJ if and only if $IJ \cdot IK = IF \cdot IE$. Expressing $IJ = IF + FJ$, $IK = IF + \frac{1}{2}FE$, and $IE = IF + FE$, we find that K is on the circumcircle of CDJ if and only if

$$FJ = \frac{IF \cdot FE}{2IF + FE}.$$

The conclusion follows.



Comment. While the figure shows B inside segment CE, it is possible that C is inside segment BE. Consequently, I would be inside segment EF and J outside segment EF. The position of point K on line EF with respect to points I, J may also vary.

Some case may require that an angle φ be replaced by $180° - \varphi$, and in computing distances, a sum may need to become a difference. All these cases can be covered by the proposed solution if it is clearly stated that *signed* distances and angles are used.

G3. Let $ABCD$ be a convex quadrilateral and let P and Q be points in $ABCD$ such that $PQDA$ and $QPBC$ are cyclic quadrilaterals. Suppose that there exists a point E on the line segment PQ such that $\angle PAE = \angle QDE$ and $\angle PBE = \angle QCE$. Show that the quadrilateral $ABCD$ is cyclic.

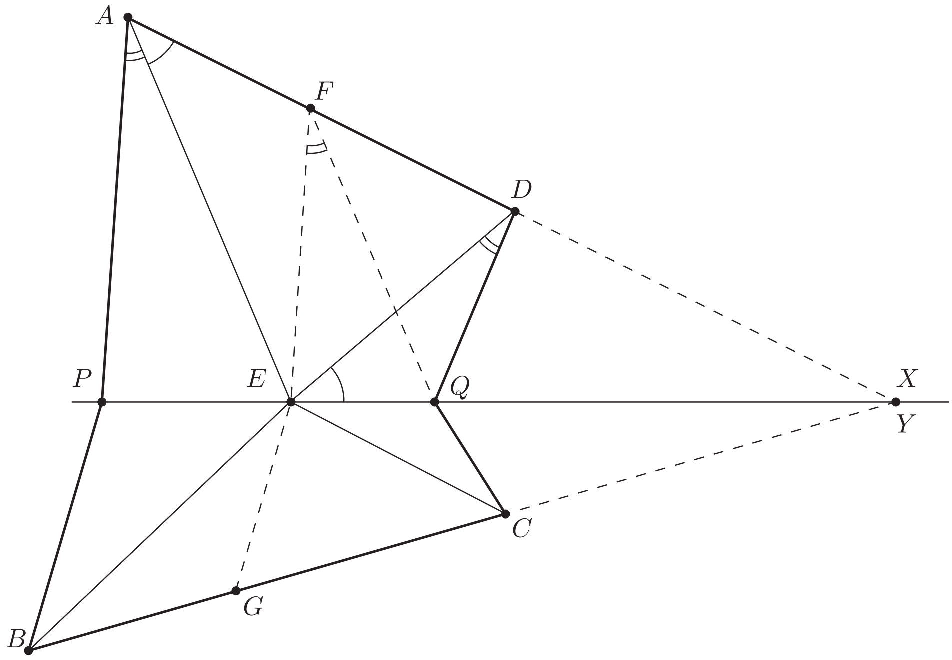
Solution 1. Let F be the point on the line AD such that $EF \| PA$. By hypothesis, the quadrilateral $PQDA$ is cyclic. So if F lies between A and D then $\angle EFD = \angle PAD = 180^\circ - \angle EQD$; the points F and Q are on distinct sides of the line DE and we infer that $EFDQ$ is a cyclic quadrilateral. And if D lies between A and F then a similar argument shows that $\angle EFD = \angle EQD$; but now the points F and Q lie on the same side of DE, so that $EDFQ$ is a cyclic quadrilateral.

In either case we obtain the equality $\angle EFQ = \angle EDQ = \angle PAE$ which implies that $FQ \| AE$. So the triangles EFQ and PAE are either homothetic or parallel-congruent. More specifically, triangle EFQ is the image of PAE under the mapping f which carries the points P, E respectively to E, Q and is either a homothety or translation by a vector. Note that f is uniquely determined by these conditions and the position of the points P, E, Q alone.

Let now G be the point on the line BC such that $EG \| PB$. The same reasoning as above applies to points B, C in place of A, D, implying that the triangle EGQ is the image of PBE under the same mapping f. So f sends the four points A, P, B, E respectively to F, E, G, Q.

If $PE \neq QE$, so that f is a homothety with a centre X, then the lines AF, PE, BG—i.e. the lines AD, PQ, BC—are concurrent at X. And since $PQDA$ and $QPBC$ are cyclic quadrilaterals, the equalities $XA \cdot XD = XP \cdot XQ = XB \cdot XC$ hold, showing that the quadrilateral $ABCD$ is cyclic.

Finally, if $PE = QE$, so that f is a translation, then $AD \| PQ \| BC$. Thus $PQDA$ and $QPBC$ are isosceles trapezoids. Then also $ABCD$ is an isosceles trapezoid, hence a cyclic quadrilateral.



Solution 2. Here is another way to reach the conclusion that the lines AD, BC and PQ are either concurrent or parallel. From the cyclic quadrilateral $PQDA$ we get

$$\angle PAD = 180^\circ - \angle PQD = \angle QDE + \angle QED = \angle PAE + \angle QED.$$

Hence $\angle QED = \angle PAD - \angle PAE = \angle EAD$. This in view of the tangent-chord theorem means that the circumcircle of triangle EAD is tangent to the line PQ at E. Analogously, the circumcircle of triangle EBC is tangent to PQ at E.

Suppose that the line AD intersects PQ at X. Since XE is tangent to the circle (EAD), $XE^2 = XA \cdot XD$. Also, $XA \cdot XD = XP \cdot XQ$ because P, Q, D, A lie on a circle. Therefore $XE^2 = XP \cdot XQ$.

It is not hard to see that this equation determines the position of the point X on the line PQ uniquely. Thus, if BC also cuts PQ, say at Y, then the analogous equation for Y yields $X = Y$, meaning that the three lines indeed concur. In this case, as well as in the case where $AD \| PQ \| BC$, the concluding argument is the same as in the first solution.

It remains to eliminate the possibility that e.g. AD meets PQ at X while $BC \| PQ$. Indeed, $QPBC$ would then be an isosceles trapezoid and the angle equality $\angle PBE = \angle QCE$ would force that E is the midpoint of PQ. So the length of XE, which is the geometric mean of the lengths of XP and XQ, should also be their arithmetic mean—impossible, as $XP \neq XQ$. The proof is now complete.

Comment. After reaching the conclusion that the circles (EDA) and (EBC) are tangent to PQ one may continue as follows. Denote the circles $(PQDA)$, (EDA), (EBC), $(QPBC)$ by ω_1, ω_2, ω_3, ω_4 respectively. Let ℓ_{ij} be the radical axis of the pair (ω_i, ω_j) for $i < j$. As is well-known, the lines ℓ_{12}, ℓ_{13}, ℓ_{23} concur, possibly at infinity (let this be the meaning of the word *concur* in this comment). So do the lines ℓ_{12}, ℓ_{14}, ℓ_{24}. Note however that ℓ_{23} and ℓ_{14} both coincide with the line PQ. Hence the pair ℓ_{12}, PQ is in both triples; thus the four lines ℓ_{12}, ℓ_{13}, ℓ_{24} and PQ are concurrent.

Similarly, ℓ_{13}, ℓ_{14}, ℓ_{34} concur, ℓ_{23}, ℓ_{24}, ℓ_{34} concur, and since $\ell_{14} = \ell_{23} = PQ$, the four lines ℓ_{13}, ℓ_{24}, ℓ_{34} and PQ are concurrent. The lines ℓ_{13} and ℓ_{24} are present in both quadruples, therefore all the lines ℓ_{ij} are concurrent. Hence the result.

G4. In an acute triangle ABC segments BE and CF are altitudes. Two circles passing through the points A and F are tangent to the line BC at the points P and Q so that B lies between C and Q. Prove that the lines PE and QF intersect on the circumcircle of triangle AEF.

Solution 1. To approach the desired result we need some information about the slopes of the lines PE and QF; this information is provided by formulas (1) and (2) which we derive below.

The tangents BP and BQ to the two circles passing through A and F are equal, as $BP^2 = BA \cdot BF = BQ^2$. Consider the altitude AD of triangle ABC and its orthocentre H. From the cyclic quadrilaterals $CDFA$ and $CDHE$ we get $BA \cdot BF = BC \cdot BD = BE \cdot BH$. Thus $BP^2 = BE \cdot BH$, or $BP/BH = BE/BP$, implying that the triangles BPH and BEP are similar. Hence

$$\angle BPE = \angle BHP. \tag{1}$$

The point P lies between D and C; this follows from the equality $BP^2 = BC \cdot BD$. In view of this equality, and because $BP = BQ$,

$$DP \cdot DQ = (BP - BD) \cdot (BP + BD) = BP^2 - BD^2 = BD \cdot (BC - BD) = BD \cdot DC.$$

Also $AD \cdot DH = BD \cdot DC$, as is seen from the similar triangles BDH and ADC. Combining these equalities we obtain $AD \cdot DH = DP \cdot DQ$. Therefore $DH/DP = DQ/DA$, showing that the triangles HDP and QDA are similar. Hence $\angle HPD = \angle QAD$, which can be rewritten as $\angle BPH = \angle BAD + \angle BAQ$. And since BQ is tangent to the circumcircle of triangle FAQ,

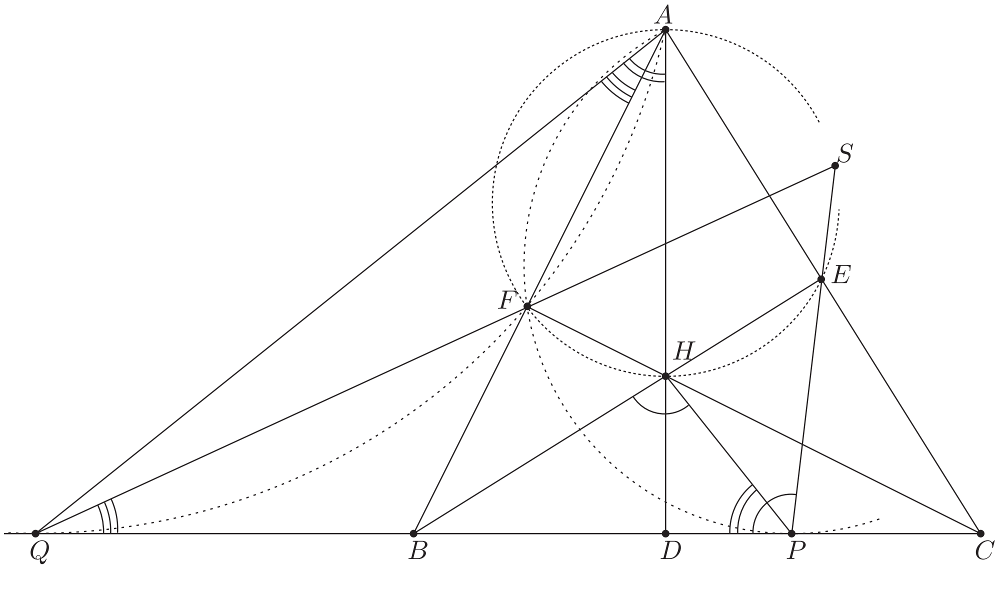
$$\angle BQF = \angle BAQ = \angle BPH - \angle BAD. \tag{2}$$

From (1) and (2) we deduce

$$\begin{aligned}\angle BPE + \angle BQF &= (\angle BHP + \angle BPH) - \angle BAD = (180^\circ - \angle PBH) - \angle BAD \\ &= (90^\circ + \angle BCA) - (90^\circ - \angle ABC) = \angle BCA + \angle ABC = 180^\circ - \angle CAB.\end{aligned}$$

Thus $\angle BPE + \angle BQF < 180^\circ$, which means that the rays PE and QF meet. Let S be the point of intersection. Then $\angle PSQ = 180^\circ - (\angle BPE + \angle BQF) = \angle CAB = \angle EAF$.

If S lies between P and E then $\angle PSQ = 180^\circ - \angle ESF$; and if E lies between P and S then $\angle PSQ = \angle ESF$. In either case the equality $\angle PSQ = \angle EAF$ which we have obtained means that S lies on the circumcircle of triangle AEF.



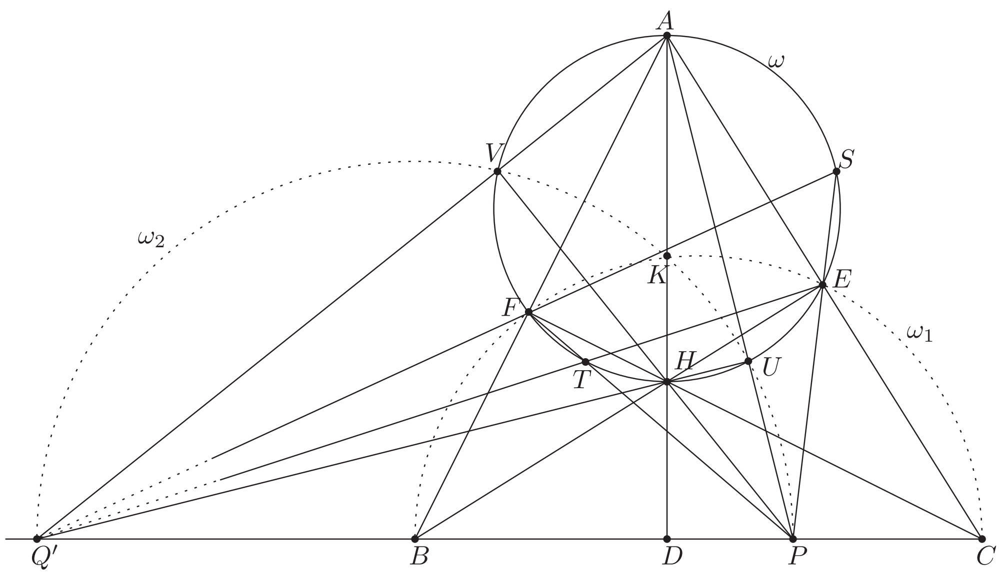
Solution 2. Let H be the orthocentre of triangle ABC and let ω be the circle with diameter AH, passing through E and F. Introduce the points of intersection of ω with the following lines emanating from P: $PA \cap \omega = \{A, U\}$, $PH \cap \omega = \{H, V\}$, $PE \cap \omega = \{E, S\}$. The altitudes of triangle AHP are contained in the lines AV, HU, BC, meeting at its orthocentre Q'.

By Pascal's theorem applied to the (tied) hexagon $AESFHV$, the points $AE \cap FH = C$, $ES \cap HV = P$ and $SF \cap VA$ are collinear, so FS passes through Q'.

Denote by ω_1 and ω_2 the circles with diameters BC and PQ', respectively. Let D be the foot of the altitude from A in triangle ABC. Suppose that AD meets the circles ω_1 and ω_2 at the respective points K and L.

Since H is the orthocentre of ABC, the triangles BDH and ADC are similar, and so $DA \cdot DH = DB \cdot DC = DK^2$; the last equality holds because BKC is a right triangle. Since H is the orthocentre also in triangle $AQ'P$, we analogously have $DL^2 = DA \cdot DH$. Therefore $DK = DL$ and $K = L$.

Also, $BD \cdot BC = BA \cdot BF$, from the similar triangles ABD, CBF. In the right triangle BKC we have $BK^2 = BD \cdot BC$. Hence, and because $BA \cdot BF = BP^2 = BQ^2$ (by the definition of P and Q in the problem statement), we obtain $BK = BP = BQ$. It follows that B is the centre of ω_2 and hence $Q' = Q$. So the lines PE and QF meet at the point S lying on the circumcircle of triangle AEF.



Comment 1. If T is the point defined by $PF \cap \omega = \{F, T\}$, Pascal's theorem for the hexagon $AFTEHV$ will analogously lead to the conclusion that the line ET goes through Q'. In other words, the lines PF and QE also concur on ω.

Comment 2. As is known from algebraic geometry, the points of the circle ω form a commutative groups with the operation defined as follows. Choose any point $0 \in \omega$ (to be the neutral element of the group) and a line ℓ exterior to the circle. For $X, Y \in \omega$, draw the line from the point $XY \cap \ell$ through 0 to its second intersection with ω and define this point to be $X + Y$.

In our solution we have chosen H to be the neutral element in this group and line BC to be ℓ. The fact that the lines AV, HU, ET, FS are concurrent can be deduced from the identities $A + A = 0$, $F = E + A$, $V = U + A = S + E = T + F$.

Comment 3. The problem was submitted in the following equivalent formulation:

Let BE and CF be altitudes of an acute triangle ABC. We choose P on the side BC and Q on the extension of CB beyond B such that $BQ^2 = BP^2 = BF \cdot AB$. If QF and PE intersect at S, prove that $ESAF$ is cyclic.

G5. Let k and n be integers with $0 \leq k \leq n-2$. Consider a set L of n lines in the plane such that no two of them are parallel and no three have a common point. Denote by I the set of intersection points of lines in L. Let O be a point in the plane not lying on any line of L.

A point $X \in I$ is colored red if the open line segment OX intersects at most k lines in L. Prove that I contains at least $\frac{1}{2}(k+1)(k+2)$ red points.

Solution. There are at least $\frac{1}{2}(k+1)(k+2)$ points in the intersection set I in view of the condition $n \geq k+2$.

For each point $P \in I$, define its *order* as the number of lines that intersect the open line segment OP. By definition, P is red if its order is at most k. Note that there is always at least one point $X \in I$ of order 0. Indeed, the lines in L divide the plane into regions, bounded or not, and O belongs to one of them. Clearly any corner of this region is a point of I with order 0.

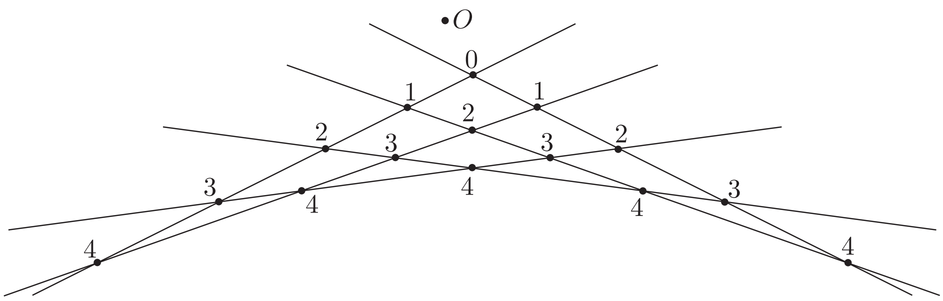
Claim. Suppose that two points $P, Q \in I$ lie on the same line of L, and no other line of L intersects the open line segment PQ. Then the orders of P and Q differ by at most 1.

Proof. Let P and Q have orders p and q, respectively, with $p \geq q$. Consider triangle OPQ. Now p equals the number of lines in L that intersect the interior of side OP. None of these lines intersects the interior of side PQ, and at most one can pass through Q. All remaining lines must intersect the interior of side OQ, implying that $q \geq p-1$. The conclusion follows.

We prove the main result by induction on k. The base $k=0$ is clear since there is a point of order 0 which is red. Assuming the statement true for $k-1$, we pass on to the inductive step. Select a point $P \in I$ of order 0, and consider one of the lines $\ell \in L$ that pass through P. There are $n-1$ intersection points on ℓ, one of which is P. Out of the remaining $n-2$ points, the k closest to P have orders not exceeding k by the Claim. It follows that there are at least $k+1$ red points on ℓ.

Let us now consider the situation with ℓ removed (together with all intersection points it contains). By hypothesis of induction, there are at least $\frac{1}{2}k(k+1)$ points of order not exceeding $k-1$ in the resulting configuration. Restoring ℓ back produces at most one new intersection point on each line segment joining any of these points to O, so their order is at most k in the original configuration. The total number of points with order not exceeding k is therefore at least $(k+1)+\frac{1}{2}k(k+1)=\frac{1}{2}(k+1)(k+2)$. This completes the proof.

Comment. The steps of the proof can be performed in reverse order to obtain a configuration of n lines such that equality holds simultaneously for all $0 \leq k \leq n-2$. Such a set of lines is illustrated in the Figure.



G6. There is given a convex quadrilateral $ABCD$. Prove that there exists a point P inside the quadrilateral such that

$$\angle PAB + \angle PDC = \angle PBC + \angle PAD = \angle PCD + \angle PBA = \angle PDA + \angle PCB = 90^\circ \qquad (1)$$

if and only if the diagonals AC and BD are perpendicular.

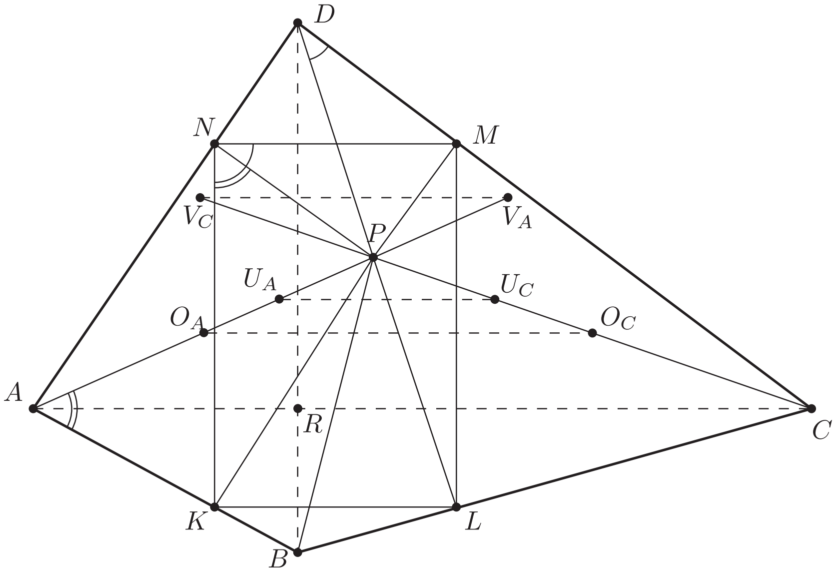
Solution 1. For a point P in $ABCD$ which satisfies (1), let K, L, M, N be the feet of perpendiculars from P to lines AB, BC, CD, DA, respectively. Note that K, L, M, N are interior to the sides as all angles in (1) are acute. The cyclic quadrilaterals $AKPN$ and $DNPM$ give

$$\angle PAB + \angle PDC = \angle PNK + \angle PNM = \angle KNM.$$

Analogously, $\angle PBC + \angle PAD = \angle LKN$ and $\angle PCD + \angle PBA = \angle MLK$. Hence the equalities (1) imply $\angle KNM = \angle LKN = \angle MLK = 90^\circ$, so that $KLMN$ is a rectangle. The converse also holds true, provided that K, L, M, N are interior to sides AB, BC, CD, DA.

(i) Suppose that there exists a point P in $ABCD$ such that $KLMN$ is a rectangle. We show that AC and BD are parallel to the respective sides of $KLMN$.

Let O_A and O_C be the circumcentres of the cyclic quadrilaterals $AKPN$ and $CMPL$. Line O_AO_C is the common perpendicular bisector of LM and KN, therefore O_AO_C is parallel to KL and MN. On the other hand, O_AO_C is the midline in the triangle ACP that is parallel to AC. Therefore the diagonal AC is parallel to the sides KL and MN of the rectangle. Likewise, BD is parallel to KN and LM. Hence AC and BD are perpendicular.



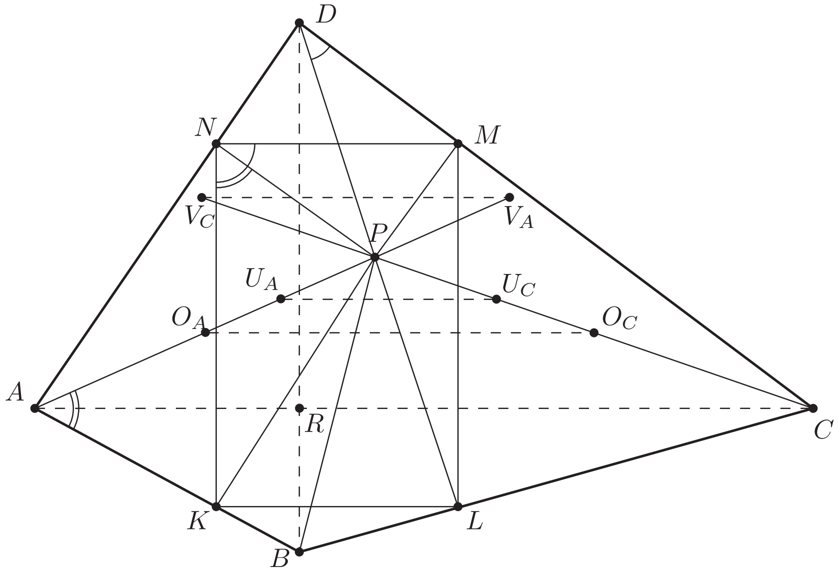
(ii) Suppose that AC and BD are perpendicular and meet at R. If $ABCD$ is a rhombus, P can be chosen to be its centre. So assume that $ABCD$ is not a rhombus, and let $BR < DR$ without loss of generality.

Denote by U_A and U_C the circumcentres of the triangles ABD and CDB, respectively. Let AV_A and CV_C be the diameters through A and C of the two circumcircles. Since AR is an altitude in triangle ADB, lines AC and AV_A are isogonal conjugates, i. e. $\angle DAV_A = \angle BAC$. Now $BR < DR$ implies that ray AU_A lies in $\angle DAC$. Similarly, ray CU_C lies in $\angle DCA$. Both diameters AV_A and CV_C intersect BD as the angles at B and D of both triangles are acute. Also U_AU_C is parallel to AC as it is the perpendicular bisector of BD. Hence V_AV_C is parallel to AC, too. We infer that AV_A and CV_C intersect at a point P inside triangle ACD, hence inside $ABCD$.

Construct points K, L, M, N, O_A and O_C in the same way as in the introduction. It follows from the previous paragraph that K, L, M, N are interior to the respective sides. Now O_AO_C is a midline in triangle ACP again. Therefore lines AC, O_AO_C and U_AU_C are parallel.

The cyclic quadrilateral $AKPN$ yields $\angle NKP = \angle NAP$. Since $\angle NAP = \angle DAU_A = \angle BAC$, as specified above, we obtain $\angle NKP = \angle BAC$. Because PK is perpendicular to AB, it follows that NK is perpendicular to AC, hence parallel to BD. Likewise, LM is parallel to BD.

Consider the two homotheties with centres A and C which transform triangles ABD and CDB into triangles AKN and CML, respectively. The images of points U_A and U_C are O_A and O_C, respectively. Since U_AU_C and O_AO_C are parallel to AC, the two ratios of homothety are the same, equal to $\lambda = AN/AD = AK/AB = AO_A/AU_A = CO_C/CU_C = CM/CD = CL/CB$. It is now straightforward that $DN/DA = DM/DC = BK/BA = BL/BC = 1-\lambda$. Hence KL and MN are parallel to AC, implying that $KLMN$ is a rectangle and completing the proof.



Solution 2. For a point P distinct from A, B, C, D, let circles (APD) and (BPC) intersect again at Q ($Q = P$ if the circles are tangent). Next, let circles (AQB) and (CQD) intersect again at R. We show that if P lies in $ABCD$ and satisfies (1) then AC and BD intersect at R and are perpendicular; the converse is also true. It is convenient to use directed angles. Let $\measuredangle(UV, XY)$ denote the angle of counterclockwise rotation that makes line UV parallel to line XY. Recall that four noncollinear points U, V, X, Y are concyclic if and only if $\measuredangle(UX, VX) = \measuredangle(UY, VY)$.

The definitions of points P, Q and R imply

$$\begin{aligned}
\measuredangle(AR, BR) &= \measuredangle(AQ, BQ) = \measuredangle(AQ, PQ) + \measuredangle(PQ, BQ) = \measuredangle(AD, PD) + \measuredangle(PC, BC),\\
\measuredangle(CR, DR) &= \measuredangle(CQ, DQ) = \measuredangle(CQ, PQ) + \measuredangle(PQ, DQ) = \measuredangle(CB, PB) + \measuredangle(PA, DA),\\
\measuredangle(BR, CR) &= \measuredangle(BR, RQ) + \measuredangle(RQ, CR) = \measuredangle(BA, AQ) + \measuredangle(DQ, CD)\\
&= \measuredangle(BA, AP) + \measuredangle(AP, AQ) + \measuredangle(DQ, DP) + \measuredangle(DP, CD)\\
&= \measuredangle(BA, AP) + \measuredangle(DP, CD).
\end{aligned}$$

Observe that the whole construction is reversible. One may start with point R, define Q as the second intersection of circles (ARB) and (CRD), and then define P as the second intersection of circles (AQD) and (BQC). The equalities above will still hold true.

Assume in addition that P is interior to $ABCD$. Then

$$\measuredangle(AD,PD) = \angle PDA,\ \measuredangle(PC,BC) = \angle PCB,\ \measuredangle(CB,PB) = \angle PBC,\ \measuredangle(PA,DA) = \angle PAD,$$
$$\measuredangle(BA,AP) = \angle PAB,\ \measuredangle(DP,CD) = \angle PDC.$$

(i) Suppose that P lies in $ABCD$ and satisfies (1). Then $\measuredangle(AR,BR) = \angle PDA+\angle PCB = 90°$ and similarly $\measuredangle(BR,CR) = \measuredangle(CR,DR) = 90°$. It follows that R is the common point of lines AC and BD, and that these lines are perpendicular.
(ii) Suppose that AC and BD are perpendicular and intersect at R. We show that the point P defined by the reverse construction (starting with R and ending with P) lies in $ABCD$. This is enough to finish the solution, because then the angle equalities above will imply (1).

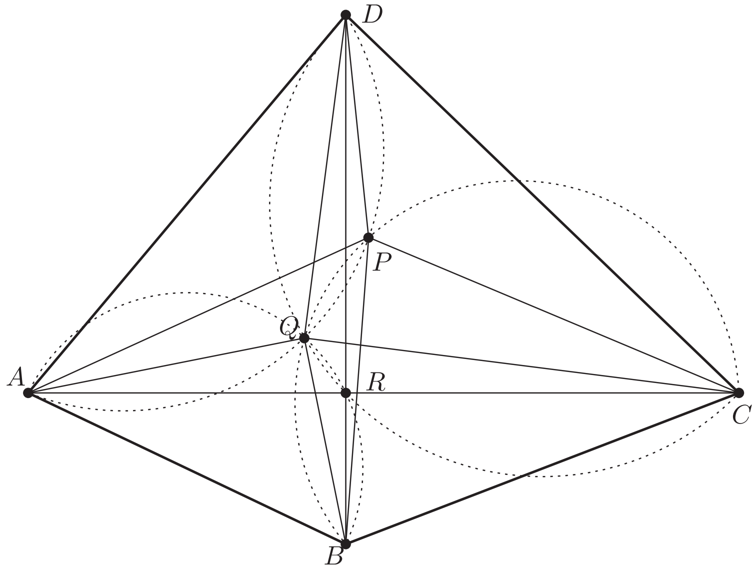
One can assume that Q, the second common point of circles (ABR) and (CDR), lies in $\angle ARD$. Then in fact Q lies in triangle ADR as angles AQR and DQR are obtuse. Hence $\angle AQD$ is obtuse, too, so that B and C are outside circle (ADQ) ($\angle ABD$ and $\angle ACD$ are acute).

Now $\angle CAB+\angle CDB = \angle BQR+\angle CQR = \angle CQB$ implies $\angle CAB < \angle CQB$ and $\angle CDB < \angle CQB$. Hence A and D are outside circle (BCQ). In conclusion, the second common point P of circles (ADQ) and (BCQ) lies on their arcs ADQ and BCQ.

We can assume that P lies in $\angle CQD$. Since

$$\angle QPC + \angle QPD = (180° - \angle QBC) + (180° - \angle QAD) =$$
$$= 360° - (\angle RBC + \angle QBR) - (\angle RAD - \angle QAR) = 360° - \angle RBC - \angle RAD > 180°,$$

point P lies in triangle CDQ, and hence in $ABCD$. The proof is complete.



G7. Let $ABCD$ be a convex quadrilateral with $AB \neq BC$. Denote by ω_1 and ω_2 the incircles of triangles ABC and ADC. Suppose that there exists a circle ω inscribed in angle ABC, tangent to the extensions of line segments AD and CD. Prove that the common external tangents of ω_1 and ω_2 intersect on ω.

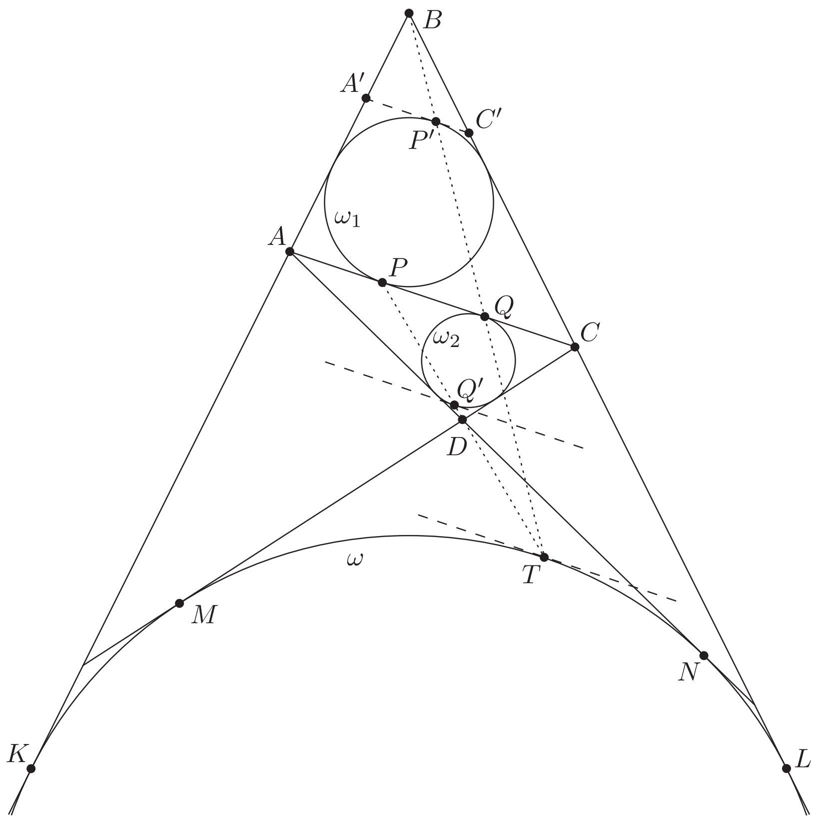
Solution. The proof below is based on two known facts.

Lemma 1. Given a convex quadrilateral $ABCD$, suppose that there exists a circle which is inscribed in angle ABC and tangent to the extensions of line segments AD and CD. Then $AB + AD = CB + CD$.

Proof. The circle in question is tangent to each of the lines AB, BC, CD, DA, and the respective points of tangency K, L, M, N are located as with circle ω in the figure. Then

$$AB + AD = (BK - AK) + (AN - DN), \quad CB + CD = (BL - CL) + (CM - DM).$$

Also $BK = BL$, $DN = DM$, $AK = AN$, $CL = CM$ by equalities of tangents. It follows that $AB + AD = CB + CD$.



For brevity, in the sequel we write "excircle AC" for the excircle of a triangle with side AC which is tangent to line segment AC and the extensions of the other two sides.

Lemma 2. The incircle of triangle ABC is tangent to its side AC at P. Let PP' be the diameter of the incircle through P, and let line BP' intersect AC at Q. Then Q is the point of tangency of side AC and excircle AC.

Proof. Let the tangent at P' to the incircle ω_1 meet BA and BC at A' and C'. Now ω_1 is the excircle $A'C'$ of triangle $A'BC'$, and it touches side $A'C'$ at P'. Since $A'C' \parallel AC$, the homothety with centre B and ratio BQ/BP' takes ω_1 to the excircle AC of triangle ABC. Because this homothety takes P' to Q, the lemma follows.

Recall also that if the incircle of a triangle touches its side AC at P, then the tangency point Q of the same side and excircle AC is the unique point on line segment AC such that $AP = CQ$.

We pass on to the main proof. Let ω_1 and ω_2 touch AC at P and Q, respectively; then $AP = (AC + AB - BC)/2$, $CQ = (CA + CD - AD)/2$. Since $AB - BC = CD - AD$ by Lemma 1, we obtain $AP = CQ$. It follows that in triangle ABC side AC and excircle AC are tangent at Q. Likewise, in triangle ADC side AC and excircle AC are tangent at P. Note that $P \neq Q$ as $AB \neq BC$.

Let PP' and QQ' be the diameters perpendicular to AC of ω_1 and ω_2, respectively. Then Lemma 2 shows that points B, P' and Q are collinear, and so are points D, Q' and P.

Consider the diameter of ω perpendicular to AC and denote by T its endpoint that is closer to AC. The homothety with centre B and ratio BT/BP' takes ω_1 to ω. Hence B, P' and T are collinear. Similarly, D, Q' and T are collinear since the homothety with centre D and ratio $-DT/DQ'$ takes ω_2 to ω.

We infer that points T, P' and Q are collinear, as well as T, Q' and P. Since $PP' \parallel QQ'$, line segments PP' and QQ' are then homothetic with centre T. The same holds true for circles ω_1 and ω_2 because they have PP' and QQ' as diameters. Moreover, it is immediate that T lies on the same side of line PP' as Q and Q', hence the ratio of homothety is positive. In particular ω_1 and ω_2 are not congruent.

In summary, T is the centre of a homothety with positive ratio that takes circle ω_1 to circle ω_2. This completes the solution, since the only point with the mentioned property is the intersection of the the common external tangents of ω_1 and ω_2.

Number Theory

N1. Let n be a positive integer and let p be a prime number. Prove that if a, b, c are integers (not necessarily positive) satisfying the equations

$$a^n + pb = b^n + pc = c^n + pa,$$

then $a = b = c$.

Solution 1. If two of a, b, c are equal, it is immediate that all the three are equal. So we may assume that $a \neq b \neq c \neq a$. Subtracting the equations we get $a^n - b^n = -p(b-c)$ and two cyclic copies of this equation, which upon multiplication yield

$$\frac{a^n - b^n}{a - b} \cdot \frac{b^n - c^n}{b - c} \cdot \frac{c^n - a^n}{c - a} = -p^3. \tag{1}$$

If n is odd then the differences $a^n - b^n$ and $a - b$ have the same sign and the product on the left is positive, while $-p^3$ is negative. So n must be even.

Let d be the greatest common divisor of the three differences $a-b$, $b-c$, $c-a$, so that $a - b = du$, $b - c = dv$, $c - a = dw$; $\gcd(u, v, w) = 1$, $u + v + w = 0$.

From $a^n - b^n = -p(b-c)$ we see that $(a-b)|p(b-c)$, i.e., $u|pv$; and cyclically $v|pw$, $w|pu$. As $\gcd(u, v, w) = 1$ and $u + v + w = 0$, at most one of u, v, w can be divisible by p. Supposing that the prime p does not divide any one of them, we get $u|v$, $v|w$, $w|u$, whence $|u| = |v| = |w| = 1$; but this quarrels with $u + v + w = 0$.

Thus p must divide exactly one of these numbers. Let e.g. $p|u$ and write $u = pu_1$. Now we obtain, similarly as before, $u_1|v$, $v|w$, $w|u_1$ so that $|u_1| = |v| = |w| = 1$. The equation $pu_1 + v + w = 0$ forces that the prime p must be even; i.e. $p = 2$. Hence $v + w = -2u_1 = \pm 2$, implying $v = w \, (= \pm 1)$ and $u = -2v$. Consequently $a - b = -2(b - c)$.

Knowing that n is even, say $n = 2k$, we rewrite the equation $a^n - b^n = -p(b-c)$ with $p = 2$ in the form

$$(a^k + b^k)(a^k - b^k) = -2(b - c) = a - b.$$

The second factor on the left is divisible by $a - b$, so the first factor $(a^k + b^k)$ must be ± 1. Then exactly one of a and b must be odd; yet $a - b = -2(b - c)$ is even. Contradiction ends the proof.

Solution 2. The beginning is as in the first solution. Assuming that a, b, c are not all equal, hence are all distinct, we derive equation (1) with the conclusion that n is even. Write $n = 2k$.

Suppose that p is odd. Then the integer

$$\frac{a^n - b^n}{a - b} = a^{n-1} + a^{n-2}b + \cdots + b^{n-1},$$

which is a factor in (1), must be odd as well. This sum of $n = 2k$ summands is odd only if a and b have different parities. The same conclusion holding for b, c and for c, a, we get that a, b, c, a alternate in their parities, which is clearly impossible.

Thus $p = 2$. The original system shows that a, b, c must be of the same parity. So we may divide (1) by p^3, i.e. 2^3, to obtain the following product of six integer factors:

$$\frac{a^k + b^k}{2} \cdot \frac{a^k - b^k}{a - b} \cdot \frac{b^k + c^k}{2} \cdot \frac{b^k - c^k}{b - c} \cdot \frac{c^k + a^k}{2} \cdot \frac{c^k - a^k}{c - a} = -1. \tag{2}$$

Each one of the factors must be equal to ± 1. In particular, $a^k + b^k = \pm 2$. If k is even, this becomes $a^k + b^k = 2$ and yields $|a| = |b| = 1$, whence $a^k - b^k = 0$, contradicting (2).

Let now k be odd. Then the sum $a^k + b^k$, with value ± 2, has $a + b$ as a factor. Since a and b are of the same parity, this means that $a + b = \pm 2$; and cyclically, $b + c = \pm 2$, $c + a = \pm 2$. In some two of these equations the signs must coincide, hence some two of a, b, c are equal. This is the desired contradiction.

Comment. Having arrived at the equation (1) one is tempted to write down all possible decompositions of $-p^3$ (cube of a prime) into a product of three integers. This leads to cumbersome examination of many cases, some of which are unpleasant to handle. One may do that just for $p = 2$, having earlier in some way eliminated odd primes from consideration.

However, the second solution shows that the condition of p being a prime is far too strong. What is actually being used in that solution, is that p is either a positive odd integer or $p = 2$.

N2. Let $a_1, a_2, \ldots, a_n$ be distinct positive integers, $n \geq 3$. Prove that there exist distinct indices i and j such that $a_i + a_j$ does not divide any of the numbers $3a_1, 3a_2, \ldots, 3a_n$.

Solution. Without loss of generality, let $0 < a_1 < a_2 < \cdots < a_n$. One can also assume that $a_1, a_2, \ldots, a_n$ are coprime. Otherwise division by their greatest common divisor reduces the question to the new sequence whose terms are coprime integers.

Suppose that the claim is false. Then for each $i < n$ there exists a j such that $a_n + a_i$ divides $3a_j$. If $a_n + a_i$ is not divisible by 3 then $a_n + a_i$ divides a_j which is impossible as $0 < a_j \leq a_n < a_n + a_i$. Thus $a_n + a_i$ is a multiple of 3 for $i = 1, \ldots, n-1$, so that $a_1, a_2, \ldots, a_{n-1}$ are all congruent (to $-a_n$) modulo 3.

Now a_n is not divisible by 3 or else so would be all remaining a_i's, meaning that $a_1, a_2, \ldots, a_n$ are not coprime. Hence $a_n \equiv r \pmod 3$ where $r \in \{1, 2\}$, and $a_i \equiv 3 - r \pmod 3$ for all $i = 1, \ldots, n-1$.

Consider a sum $a_{n-1} + a_i$ where $1 \leq i \leq n-2$. There is at least one such sum as $n \geq 3$. Let j be an index such that $a_{n-1} + a_i$ divides $3a_j$. Observe that $a_{n-1} + a_i$ is not divisible by 3 since $a_{n-1} + a_i \equiv 2a_i \not\equiv 0 \pmod 3$. It follows that $a_{n-1} + a_i$ divides a_j, in particular $a_{n-1} + a_i \leq a_j$. Hence $a_{n-1} < a_j \leq a_n$, implying $j = n$. So a_n is divisible by all sums $a_{n-1} + a_i$, $1 \leq i \leq n-2$. In particular $a_{n-1} + a_i \leq a_n$ for $i = 1, \ldots, n-2$.

Let j be such that $a_n + a_{n-1}$ divides $3a_j$. If $j \leq n-2$ then $a_n + a_{n-1} \leq 3a_j < a_j + 2a_{n-1}$. This yields $a_n < a_{n-1} + a_j$; however $a_{n-1} + a_j \leq a_n$ for $j \leq n-2$. Therefore $j = n-1$ or $j = n$.

For $j = n-1$ we obtain $3a_{n-1} = k(a_n + a_{n-1})$ with k an integer, and it is straightforward that $k = 1$ ($k \leq 0$ and $k \geq 3$ contradict $0 < a_{n-1} < a_n$; $k = 2$ leads to $a_{n-1} = 2a_n > a_{n-1}$). Thus $3a_{n-1} = a_n + a_{n-1}$, i. e. $a_n = 2a_{n-1}$.

Similarly, if $j = n$ then $3a_n = k(a_n + a_{n-1})$ for some integer k, and only $k = 2$ is possible. Hence $a_n = 2a_{n-1}$ holds true in both cases remaining, $j = n-1$ and $j = n$.

Now $a_n = 2a_{n-1}$ implies that the sum $a_{n-1} + a_1$ is strictly between $a_n/2$ and a_n. But a_{n-1} and a_1 are distinct as $n \geq 3$, so it follows from the above that $a_{n-1} + a_1$ divides a_n. This provides the desired contradiction.

N3. Let $a_0, a_1, a_2, \ldots$ be a sequence of positive integers such that the greatest common divisor of any two consecutive terms is greater than the preceding term; in symbols, $\gcd(a_i, a_{i+1}) > a_{i-1}$. Prove that $a_n \geq 2^n$ for all $n \geq 0$.

Solution. Since $a_i \geq \gcd(a_i, a_{i+1}) > a_{i-1}$, the sequence is strictly increasing. In particular $a_0 \geq 1$, $a_1 \geq 2$. For each $i \geq 1$ we also have $a_{i+1} - a_i \geq \gcd(a_i, a_{i+1}) > a_{i-1}$, and consequently $a_{i+1} \geq a_i + a_{i-1} + 1$. Hence $a_2 \geq 4$ and $a_3 \geq 7$. The equality $a_3 = 7$ would force equalities in the previous estimates, leading to $\gcd(a_2, a_3) = \gcd(4, 7) > a_1 = 2$, which is false. Thus $a_3 \geq 8$; the result is valid for $n = 0, 1, 2, 3$. These are the base cases for a proof by induction.

Take an $n \geq 3$ and assume that $a_i \geq 2^i$ for $i = 0, 1, \ldots, n$. We must show that $a_{n+1} \geq 2^{n+1}$. Let $\gcd(a_n, a_{n+1}) = d$. We know that $d > a_{n-1}$. The induction claim is reached immediately in the following cases:

$$\begin{array}{lll} \text{if } a_{n+1} \geq 4d & \text{then} & a_{n+1} > 4a_{n-1} \geq 4 \cdot 2^{n-1} = 2^{n+1}; \\ \text{if } a_n \geq 3d & \text{then} & a_{n+1} \geq a_n + d \geq 4d > 4a_{n-1} \geq 4 \cdot 2^{n-1} = 2^{n+1}; \\ \text{if } a_n = d & \text{then} & a_{n+1} \geq a_n + d = 2a_n \geq 2 \cdot 2^n = 2^{n+1}. \end{array}$$

The only remaining possibility is that $a_n = 2d$ and $a_{n+1} = 3d$, which we assume for the sequel. So $a_{n+1} = \frac{3}{2}a_n$.

Let now $\gcd(a_{n-1}, a_n) = d'$; then $d' > a_{n-2}$. Write $a_n = md'$ (m an integer). Keeping in mind that $d' \leq a_{n-1} < d$ and $a_n = 2d$, we get that $m \geq 3$. Also $a_{n-1} < d = \frac{1}{2}md'$, $a_{n+1} = \frac{3}{2}md'$. Again we single out the cases which imply the induction claim immediately:

$$\begin{array}{lll} \text{if } m \geq 6 & \text{then} & a_{n+1} = \frac{3}{2}md' \geq 9d' > 9a_{n-2} \geq 9 \cdot 2^{n-2} > 2^{n+1}; \\ \text{if } 3 \leq m \leq 4 & \text{then} & a_{n-1} < \frac{1}{2} \cdot 4d', \text{ and hence } a_{n-1} = d', \\ & & \qquad a_{n+1} = \frac{3}{2}ma_{n-1} \geq \frac{3}{2} \cdot 3a_{n-1} \geq \frac{9}{2} \cdot 2^{n-1} > 2^{n+1}. \end{array}$$

So we are left with the case $m = 5$, which means that $a_n = 5d'$, $a_{n+1} = \frac{15}{2}d'$, $a_{n-1} < d = \frac{5}{2}d'$. The last relation implies that a_{n-1} is either d' or $2d'$. Anyway, $a_{n-1} | 2d'$.

The same pattern repeats once more. We denote $\gcd(a_{n-2}, a_{n-1}) = d''$; then $d'' > a_{n-3}$. Because d'' is a divisor of a_{n-1}, hence also of $2d'$, we may write $2d' = m'd''$ (m' an integer). Since $d'' \leq a_{n-2} < d'$, we get $m' \geq 3$. Also, $a_{n-2} < d' = \frac{1}{2}m'd''$, $a_{n+1} = \frac{15}{2}d' = \frac{15}{4}m'd''$. As before, we consider the cases:

$$\begin{array}{lll} \text{if } m' \geq 5 & \text{then} & a_{n+1} = \frac{15}{4}m'd'' \geq \frac{75}{4}d'' > \frac{75}{4}a_{n-3} \geq \frac{75}{4} \cdot 2^{n-3} > 2^{n+1}; \\ \text{if } 3 \leq m' \leq 4 & \text{then} & a_{n-2} < \frac{1}{2} \cdot 4d'', \text{ and hence } a_{n-2} = d'', \\ & & \qquad a_{n+1} = \frac{15}{4}m'a_{n-2} \geq \frac{15}{4} \cdot 3a_{n-2} \geq \frac{45}{4} \cdot 2^{n-2} > 2^{n+1}. \end{array}$$

Both of them have produced the induction claim. But now there are no cases left. Induction is complete; the inequality $a_n \geq 2^n$ holds for all n.

N4. Let n be a positive integer. Show that the numbers

$$\binom{2^n-1}{0}, \quad \binom{2^n-1}{1}, \quad \binom{2^n-1}{2}, \quad \ldots, \quad \binom{2^n-1}{2^{n-1}-1}$$

are congruent modulo 2^n to $1, 3, 5, \ldots, 2^n-1$ in some order.

Solution 1. It is well-known that all these numbers are odd. So the assertion that their remainders $\pmod{2^n}$ make up a permutation of $\{1, 3, \ldots, 2^n-1\}$ is equivalent just to saying that these remainders are all distinct. We begin by showing that

$$\binom{2^n-1}{2k} + \binom{2^n-1}{2k+1} \equiv 0 \pmod{2^n} \quad \text{and} \quad \binom{2^n-1}{2k} \equiv (-1)^k \binom{2^{n-1}-1}{k} \pmod{2^n}. \quad (1)$$

The first relation is immediate, as the sum on the left is equal to $\binom{2^n}{2k+1} = \frac{2^n}{2k+1}\binom{2^n-1}{2k}$, hence is divisible by 2^n. The second relation:

$$\binom{2^n-1}{2k} = \prod_{j=1}^{2k} \frac{2^n-j}{j} = \prod_{i=1}^{k} \frac{2^n-(2i-1)}{2i-1} \cdot \prod_{i=1}^{k} \frac{2^{n-1}-i}{i} \equiv (-1)^k \binom{2^{n-1}-1}{k} \pmod{2^n}.$$

This prepares ground for a proof of the required result by induction on n. The base case $n=1$ is obvious. Assume the assertion is true for $n-1$ and pass to n, denoting $a_k = \binom{2^{n-1}-1}{k}$, $b_m = \binom{2^n-1}{m}$. The induction hypothesis is that all the numbers a_k $(0 \le k < 2^{n-2})$ are distinct $\pmod{2^{n-1}}$; the claim is that all the numbers b_m $(0 \le m < 2^{n-1})$ are distinct $\pmod{2^n}$.

The congruence relations (1) are restated as

$$b_{2k} \equiv (-1)^k a_k \equiv -b_{2k+1} \pmod{2^n}. \quad (2)$$

Shifting the exponent in the first relation of (1) from n to $n-1$ we also have the congruence $a_{2i+1} \equiv -a_{2i} \pmod{2^{n-1}}$. We hence conclude:

If, for some $j, k < 2^{n-2}$, $a_k \equiv -a_j \pmod{2^{n-1}}$, then $\{j,k\} = \{2i, 2i+1\}$ for some i. (3)

This is so because in the sequence $(a_k : k < 2^{n-2})$ each term a_j is complemented to $0 \pmod{2^{n-1}}$ by only one other term a_k, according to the induction hypothesis.

From (2) we see that $b_{4i} \equiv a_{2i}$ and $b_{4i+3} \equiv a_{2i+1} \pmod{2^n}$. Let

$$M = \{m : 0 \le m < 2^{n-1},\ m \equiv 0 \text{ or } 3 \pmod 4\}, \quad L = \{l : 0 \le l < 2^{n-1},\ l \equiv 1 \text{ or } 2 \pmod 4\}.$$

The last two congruences take on the unified form

$$b_m \equiv a_{\lfloor m/2 \rfloor} \pmod{2^n} \quad \text{for all} \quad m \in M. \quad (4)$$

Thus all the numbers b_m for $m \in M$ are distinct $\pmod{2^n}$ because so are the numbers a_k (they are distinct $\pmod{2^{n-1}}$, hence also $\pmod{2^n}$).

Every $l \in L$ is paired with a unique $m \in M$ into a pair of the form $\{2k, 2k+1\}$. So (2) implies that also all the b_l for $l \in L$ are distinct $\pmod{2^n}$. It remains to eliminate the possibility that $b_m \equiv b_l \pmod{2^n}$ for some $m \in M$, $l \in L$.

Suppose that such a situation occurs. Let $m' \in M$ be such that $\{m', l\}$ is a pair of the form $\{2k, 2k+1\}$, so that (see (2)) $b_{m'} \equiv -b_l \pmod{2^n}$. Hence $b_{m'} \equiv -b_m \pmod{2^n}$. Since both m' and m are in M, we have by (4) $b_{m'} \equiv a_j$, $b_m \equiv a_k \pmod{2^n}$ for $j = \lfloor m'/2 \rfloor$, $k = \lfloor m/2 \rfloor$.

Then $a_j \equiv -a_k \pmod{2^n}$. Thus, according to (3), $j = 2i$, $k = 2i+1$ for some i (or vice versa). The equality $a_{2i+1} \equiv -a_{2i} \pmod{2^n}$ now means that $\binom{2^{n-1}-1}{2i} + \binom{2^{n-1}-1}{2i+1} \equiv 0 \pmod{2^n}$. However, the sum on the left is equal to $\binom{2^{n-1}}{2i+1}$. A number of this form cannot be divisible by 2^n. This is a contradiction which concludes the induction step and proves the result.

Solution 2. We again proceed by induction, writing for brevity $N = 2^{n-1}$ and keeping notation $a_k = \binom{N-1}{k}$, $b_m = \binom{2N-1}{m}$. Assume that the result holds for the sequence $(a_0, a_1, a_2, \ldots, a_{N/2-1})$. In view of the symmetry $a_{N-1-k} = a_k$ this sequence is a permutation of $(a_0, a_2, a_4, \ldots, a_{N-2})$. So the induction hypothesis says that this latter sequence, taken (mod N), is a permutation of $(1, 3, 5, \ldots, N-1)$. Similarly, the induction claim is that $(b_0, b_2, b_4, \ldots, b_{2N-2})$, taken (mod $2N$), is a permutation of $(1, 3, 5, \ldots, 2N-1)$.

In place of the congruence relations (2) we now use the following ones,

$$b_{4i} \equiv a_{2i} \pmod{N} \qquad \text{and} \qquad b_{4i+2} \equiv b_{4i} + N \pmod{2N}. \tag{5}$$

Given this, the conclusion is immediate: the first formula of (5) together with the induction hypothesis tells us that $(b_0, b_4, b_8, \ldots, b_{2N-4})$ (mod N) is a permutation of $(1, 3, 5, \ldots, N-1)$. Then the second formula of (5) shows that $(b_2, b_6, b_{10}, \ldots, b_{2N-2})$ (mod N) is exactly the same permutation; moreover, this formula distinguishes (mod $2N$) each b_{4i} from b_{4i+2}.

Consequently, these two sequences combined represent (mod $2N$) a permutation of the sequence $(1, 3, 5, \ldots, N-1,\ N+1, N+3, N+5, \ldots, N+N-1)$, and this is precisely the induction claim.

Now we prove formulas (5); we begin with the second one. Since $b_{m+1} = b_m \cdot \frac{2N-m-1}{m+1}$,

$$b_{4i+2} = b_{4i} \cdot \frac{2N-4i-1}{4i+1} \cdot \frac{2N-4i-2}{4i+2} = b_{4i} \cdot \frac{2N-4i-1}{4i+1} \cdot \frac{N-2i-1}{2i+1}\,.$$

The desired congruence $b_{4i+2} \equiv b_{4i} + N$ may be multiplied by the odd number $(4i+1)(2i+1)$, giving rise to a chain of successively equivalent congruences:

$$\begin{aligned} b_{4i}(2N-4i-1)(N-2i-1) &\equiv (b_{4i}+N)(4i+1)(2i+1) && \pmod{2N}, \\ b_{4i}(2i+1-N) &\equiv (b_{4i}+N)(2i+1) && \pmod{2N}, \\ (b_{4i}+2i+1)N &\equiv 0 && \pmod{2N}; \end{aligned}$$

and the last one is satisfied, as b_{4i} is odd. This settles the second relation in (5).

The first one is proved by induction on i. It holds for $i = 0$. Assume $b_{4i} \equiv a_{2i} \pmod{2N}$ and consider $i+1$:

$$b_{4i+4} = b_{4i+2} \cdot \frac{2N-4i-3}{4i+3} \cdot \frac{2N-4i-4}{4i+4}\,; \qquad a_{2i+2} = a_{2i} \cdot \frac{N-2i-1}{2i+1} \cdot \frac{N-2i-2}{2i+2}\,.$$

Both expressions have the fraction $\frac{N-2i-2}{2i+2}$ as the last factor. Since $2i+2 < N = 2^{n-1}$, this fraction reduces to ℓ/m with ℓ and m odd. In showing that $b_{4i+4} \equiv a_{2i+2} \pmod{2N}$, we may ignore this common factor ℓ/m. Clearing other odd denominators reduces the claim to

$$b_{4i+2}(2N-4i-3)(2i+1) \equiv a_{2i}(N-2i-1)(4i+3) \pmod{2N}.$$

By the inductive assumption (saying that $b_{4i} \equiv a_{2i} \pmod{2N}$) and by the second relation of (5), this is equivalent to

$$(b_{4i}+N)(2i+1) \equiv b_{4i}(2i+1-N) \pmod{2N},$$

a congruence which we have already met in the preceding proof a few lines above. This completes induction (on i) and the proof of (5), hence also the whole solution.

Comment. One can avoid the words *congruent modulo* in the problem statement by rephrasing the assertion into: *Show that these numbers leave distinct remainders in division by* 2^n.

N5. For every $n \in \mathbb{N}$ let $d(n)$ denote the number of (positive) divisors of n. Find all functions $f\colon \mathbb{N} \to \mathbb{N}$ with the following properties:

(i) $d(f(x)) = x$ for all $x \in \mathbb{N}$;

(ii) $f(xy)$ divides $(x-1)y^{xy-1}f(x)$ for all $x, y \in \mathbb{N}$.

Solution. There is a unique solution: the function $f\colon \mathbb{N} \to \mathbb{N}$ defined by $f(1) = 1$ and

$$f(n) = p_1^{p_1^{a_1}-1} p_2^{p_2^{a_2}-1} \cdots p_k^{p_k^{a_k}-1} \quad \text{where } n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k} \text{ is the prime factorization of } n > 1. \tag{1}$$

Direct verification shows that this function meets the requirements.

Conversely, let $f\colon \mathbb{N} \to \mathbb{N}$ satisfy (i) and (ii). Applying (i) for $x = 1$ gives $d(f(1)) = 1$, so $f(1) = 1$. In the sequel we prove that (1) holds for all $n > 1$. Notice that $f(m) = f(n)$ implies $m = n$ in view of (i). The formula $d\left(p_1^{b_1} \cdots p_k^{b_k}\right) = (b_1+1)\cdots(b_k+1)$ will be used throughout.

Let p be a prime. Since $d(f(p)) = p$, the formula just mentioned yields $f(p) = q^{p-1}$ for some prime q; in particular $f(2) = q^{2-1} = q$ is a prime. We prove that $f(p) = p^{p-1}$ for all primes p.

Suppose that p is odd and $f(p) = q^{p-1}$ for a prime q. Applying (ii) first with $x = 2$, $y = p$ and then with $x = p$, $y = 2$ shows that $f(2p)$ divides both $(2-1)p^{2p-1}f(2) = p^{2p-1}f(2)$ and $(p-1)2^{2p-1}f(p) = (p-1)2^{2p-1}q^{p-1}$. If $q \neq p$ then the odd prime p does not divide $(p-1)2^{2p-1}q^{p-1}$, hence the greatest common divisor of $p^{2p-1}f(2)$ and $(p-1)2^{2p-1}q^{p-1}$ is a divisor of $f(2)$. Thus $f(2p)$ divides $f(2)$ which is a prime. As $f(2p) > 1$, we obtain $f(2p) = f(2)$ which is impossible. So $q = p$, i. e. $f(p) = p^{p-1}$.

For $p = 2$ the same argument with $x = 2$, $y = 3$ and $x = 3$, $y = 2$ shows that $f(6)$ divides both $3^5 f(2)$ and $2^6 f(3) = 2^6 3^2$. If the prime $f(2)$ is odd then $f(6)$ divides $3^2 = 9$, so $f(6) \in \{1, 3, 9\}$. However then $6 = d(f(6)) \in \{d(1), d(3), d(9)\} = \{1, 2, 3\}$ which is false. In conclusion $f(2) = 2$.

Next, for each $n > 1$ the prime divisors of $f(n)$ are among the ones of n. Indeed, let p be the least prime divisor of n. Apply (ii) with $x = p$ and $y = n/p$ to obtain that $f(n)$ divides $(p-1)y^{n-1}f(p) = (p-1)y^{n-1}p^{p-1}$. Write $f(n) = \ell P$ where ℓ is coprime to n and P is a product of primes dividing n. Since ℓ divides $(p-1)y^{n-1}p^{p-1}$ and is coprime to $y^{n-1}p^{p-1}$, it divides $p-1$; hence $d(\ell) \le \ell < p$. But (i) gives $n = d(f(n)) = d(\ell P)$, and $d(\ell P) = d(\ell)d(P)$ as ℓ and P are coprime. Therefore $d(\ell)$ is a divisor of n less than p, meaning that $\ell = 1$ and proving the claim.

Now (1) is immediate for prime powers. If p is a prime and $a \ge 1$, by the above the only prime factor of $f(p^a)$ is p (a prime factor does exist as $f(p^a) > 1$). So $f(p^a) = p^b$ for some $b \ge 1$, and (i) yields $p^a = d(f(p^a)) = d\left(p^b\right) = b + 1$. Hence $f(p^a) = p^{p^a-1}$, as needed.

Let us finally show that (1) is true for a general $n > 1$ with prime factorization $n = p_1^{a_1} \cdots p_k^{a_k}$. We saw that the prime factorization of $f(n)$ has the form $f(n) = p_1^{b_1} \cdots p_k^{b_k}$. For $i = 1, \dots, k$, set $x = p_i^{a_i}$ and $y = n/x$ in (ii) to infer that $f(n)$ divides $(p_i^{a_i} - 1)\, y^{n-1} f(p_i^{a_i})$. Hence $p_i^{b_i}$ divides $(p_i^{a_i} - 1)\, y^{n-1} f(p_i^{a_i})$, and because $p_i^{b_i}$ is coprime to $(p_i^{a_i} - 1)\, y^{n-1}$, it follows that $p_i^{b_i}$ divides $f(p_i^{a_i}) = p_i^{p_i^{a_i}-1}$. So $b_i \le p_i^{a_i} - 1$ for all $i = 1, \dots, k$. Combined with (i), these conclusions imply

$$p_1^{a_1} \cdots p_k^{a_k} = n = d(f(n)) = d\left(p_1^{b_1} \cdots p_k^{b_k}\right) = (b_1+1)\cdots(b_k+1) \le p_1^{a_1} \cdots p_k^{a_k}.$$

Hence all inequalities $b_i \le p_i^{a_i} - 1$ must be equalities, $i = 1, \dots, k$, implying that (1) holds true. The proof is complete.

N6. Prove that there exist infinitely many positive integers n such that n^2+1 has a prime divisor greater than $2n+\sqrt{2n}$.

Solution. Let $p \equiv 1 \pmod 8$ be a prime. The congruence $x^2 \equiv -1 \pmod p$ has two solutions in $[1, p-1]$ whose sum is p. If n is the smaller one of them then p divides n^2+1 and $n \le (p-1)/2$. We show that $p > 2n + \sqrt{10n}$.

Let $n = (p-1)/2 - \ell$ where $\ell \ge 0$. Then $n^2 \equiv -1 \pmod p$ gives

$$\left(\frac{p-1}{2} - \ell\right)^2 \equiv -1 \pmod p \qquad \text{or} \qquad (2\ell+1)^2 + 4 \equiv 0 \pmod p.$$

Thus $(2\ell+1)^2+4 = rp$ for some $r \ge 0$. As $(2\ell+1)^2 \equiv 1 \equiv p \pmod 8$, we have $r \equiv 5 \pmod 8$, so that $r \ge 5$. Hence $(2\ell+1)^2 + 4 \ge 5p$, implying $\ell \ge \left(\sqrt{5p-4}-1\right)/2$. Set $\sqrt{5p-4} = u$ for clarity; then $\ell \ge (u-1)/2$. Therefore

$$n = \frac{p-1}{2} - \ell \le \frac{1}{2}(p-u).$$

Combined with $p = (u^2+4)/5$, this leads to $u^2 - 5u - 10n + 4 \ge 0$. Solving this quadratic inequality with respect to $u \ge 0$ gives $u \ge \left(5 + \sqrt{40n+9}\right)/2$. So the estimate $n \le \left(p-u\right)/2$ leads to

$$p \ge 2n + u \ge 2n + \frac{1}{2}\left(5 + \sqrt{40n+9}\right) > 2n + \sqrt{10n}.$$

Since there are infinitely many primes of the form $8k+1$, it follows easily that there are also infinitely many n with the stated property.

Comment. By considering the prime factorization of the product $\prod_{n=1}^{N}(n^2+1)$, it can be obtained that its greatest prime divisor is at least $cN\log N$. This could improve the statement as $p > n\log n$.

However, the proof applies some advanced information about the distribution of the primes of the form $4k+1$, which is inappropriate for high schools contests.

50th International Mathematical Olympiad

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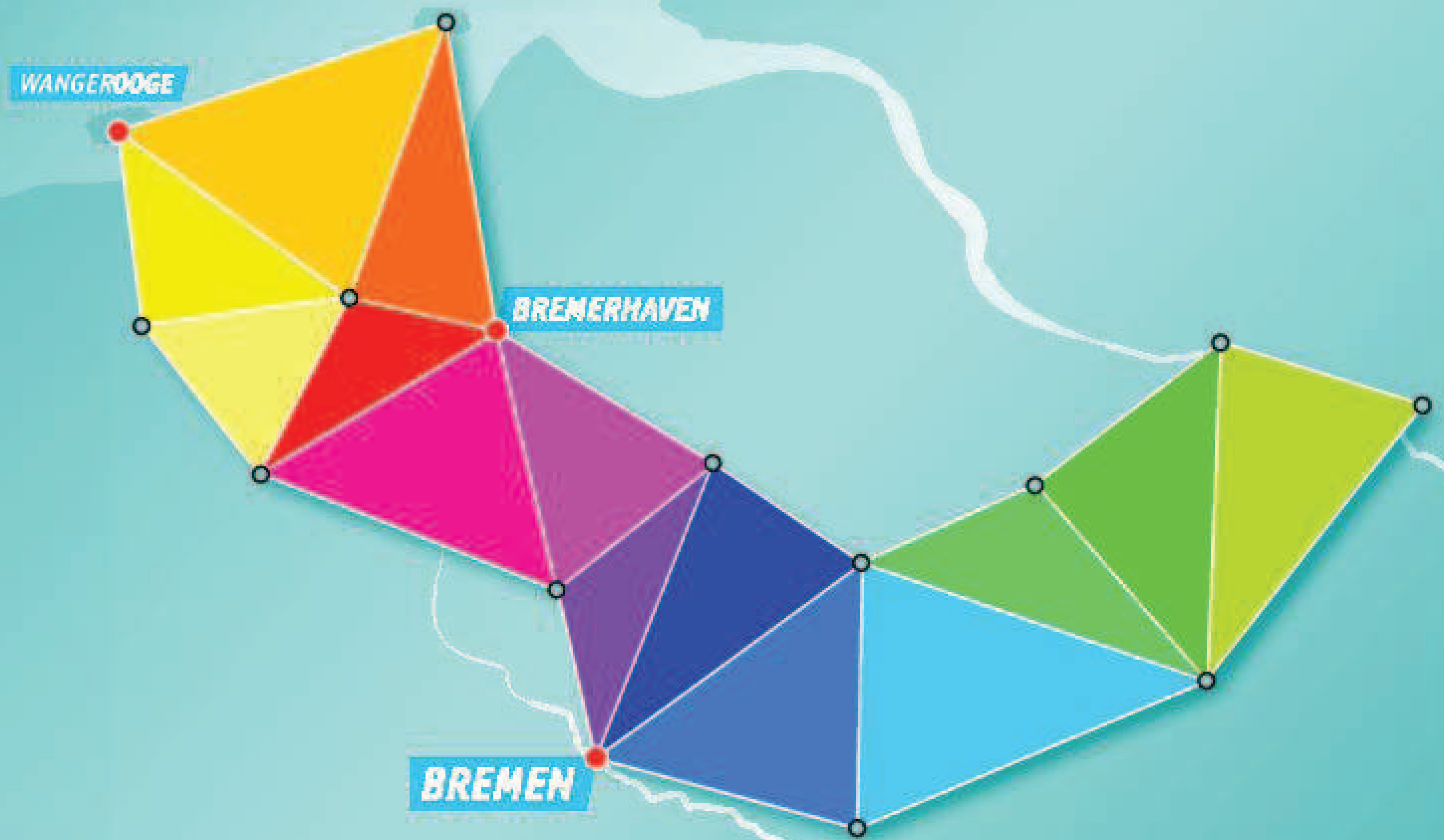
Problem Shortlist with solutions



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Problem Shortlist with Solutions

The Problem Selection Committee

The Note of Confidentiality **IMPORTANT**

We insistently ask everybody to consider the following IMO Regulations rule:

These Shortlist Problems have to be kept strictly confidential until IMO 2010.

The Problem Selection Committee

Konrad Engel, Karl Fegert, Andreas Felgenhauer, Hans-Dietrich Gronau, Roger Labahn, Bernd Mulansky, Jürgen Prestin, Christian Reiher, Peter Scholze, Eckard Specht, Robert Strich, Martin Welk

gratefully received

132 problem proposals submitted by 39 countries:

Algeria, Australia, Austria, Belarus, Belgium, Bulgaria, Colombia, Croatia, Czech Republic, El Salvador, Estonia, Finland, France, Greece, Hong Kong, Hungary, India, Ireland, Islamic Republic of Iran, Japan, Democratic People's Republic of Korea, Lithuania, Luxembourg, The former Yugoslav Republic of Macedonia, Mongolia, Netherlands, New Zealand, Pakistan, Peru, Poland, Romania, Russian Federation, Slovenia, South Africa, Taiwan, Turkey, Ukraine, United Kingdom, United States of America .

Layout: Roger Labahn with LaTeX & TeX
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: The Problem Selection Committee

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Algebra

A1 CZE (Czech Republic)

Find the largest possible integer k, such that the following statement is true:

Let 2009 arbitrary non-degenerated triangles be given. In every triangle the three sides are colored, such that one is blue, one is red and one is white. Now, for every color separately, let us sort the lengths of the sides. We obtain

$$\begin{array}{rll} & b_1 \leq b_2 \leq \ldots \leq b_{2009} & \text{the lengths of the blue sides,} \\ & r_1 \leq r_2 \leq \ldots \leq r_{2009} & \text{the lengths of the red sides,} \\ \text{and} & w_1 \leq w_2 \leq \ldots \leq w_{2009} & \text{the lengths of the white sides.} \end{array}$$

Then there exist k indices j such that we can form a non-degenerated triangle with side lengths b_j, r_j, w_j.

A2 EST (Estonia)

Let a, b, c be positive real numbers such that $\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}=a+b+c$. Prove that

$$\frac{1}{(2a+b+c)^2}+\frac{1}{(2b+c+a)^2}+\frac{1}{(2c+a+b)^2} \leq \frac{3}{16}.$$

A3 FRA (France)

Determine all functions f from the set of positive integers into the set of positive integers such that for all x and y there exists a non degenerated triangle with sides of lengths

$$x, \quad f(y) \quad \text{and} \quad f(y+f(x)-1).$$

A4 BLR (Belarus)

Let a, b, c be positive real numbers such that $ab+bc+ca \leq 3abc$. Prove that

$$\sqrt{\frac{a^2+b^2}{a+b}}+\sqrt{\frac{b^2+c^2}{b+c}}+\sqrt{\frac{c^2+a^2}{c+a}}+3 \leq \sqrt{2}\left(\sqrt{a+b}+\sqrt{b+c}+\sqrt{c+a}\right).$$

A5 BLR (Belarus)

Let f be any function that maps the set of real numbers into the set of real numbers. Prove that there exist real numbers x and y such that

$$f\left(x-f(y)\right) > yf(x)+x.$$

A6 USA (United States of America)

Suppose that $s_1, s_2, s_3, \ldots$ is a strictly increasing sequence of positive integers such that the subsequences

$$s_{s_1},\ s_{s_2},\ s_{s_3},\ \ldots \qquad \text{and} \qquad s_{s_1+1},\ s_{s_2+1},\ s_{s_3+1},\ \ldots$$

are both arithmetic progressions. Prove that $s_1, s_2, s_3, \ldots$ is itself an arithmetic progression.

A7 JPN (Japan)

Find all functions f from the set of real numbers into the set of real numbers which satisfy for all real x, y the identity

$$f(xf(x+y)) = f(yf(x)) + x^2.$$

Combinatorics

C1 NZL (New Zealand)

Consider 2009 cards, each having one gold side and one black side, lying in parallel on a long table. Initially all cards show their gold sides. Two players, standing by the same long side of the table, play a game with alternating moves. Each move consists of choosing a block of 50 consecutive cards, the leftmost of which is showing gold, and turning them all over, so those which showed gold now show black and vice versa. The last player who can make a legal move wins.

(a) Does the game necessarily end?

(b) Does there exist a winning strategy for the starting player?

C2 ROU (Romania)

For any integer $n \geq 2$, let $N(n)$ be the maximal number of triples (a_i, b_i, c_i), $i = 1, \ldots, N(n)$, consisting of nonnegative integers a_i, b_i and c_i such that the following two conditions are satisfied:

(1) $a_i + b_i + c_i = n$ for all $i = 1, \ldots, N(n)$,

(2) If $i \neq j$, then $a_i \neq a_j, b_i \neq b_j$ and $c_i \neq c_j$.

Determine $N(n)$ for all $n \geq 2$.

Comment. The original problem was formulated for m-tuples instead for triples. The numbers $N(m, n)$ are then defined similarly to $N(n)$ in the case $m = 3$. The numbers $N(3, n)$ and $N(n, n)$ should be determined. The case $m = 3$ is the same as in the present problem. The upper bound for $N(n, n)$ can be proved by a simple generalization. The construction of a set of triples attaining the bound can be easily done by induction from n to $n + 2$.

C3 RUS (Russian Federation)

Let n be a positive integer. Given a sequence $\varepsilon_1, \ldots, \varepsilon_{n-1}$ with $\varepsilon_i = 0$ or $\varepsilon_i = 1$ for each $i = 1, \ldots, n-1$, the sequences $a_0, \ldots, a_n$ and $b_0, \ldots, b_n$ are constructed by the following rules:

$$a_0 = b_0 = 1, \quad a_1 = b_1 = 7,$$

$$a_{i+1} = \begin{cases} 2a_{i-1} + 3a_i, & \text{if } \varepsilon_i = 0, \\ 3a_{i-1} + a_i, & \text{if } \varepsilon_i = 1, \end{cases} \quad \text{for each } i = 1, \ldots, n-1,$$

$$b_{i+1} = \begin{cases} 2b_{i-1} + 3b_i, & \text{if } \varepsilon_{n-i} = 0, \\ 3b_{i-1} + b_i, & \text{if } \varepsilon_{n-i} = 1, \end{cases} \quad \text{for each } i = 1, \ldots, n-1.$$

Prove that $a_n = b_n$.

C4 NLD (Netherlands)

For an integer $m \geq 1$, we consider partitions of a $2^m \times 2^m$ chessboard into rectangles consisting of cells of the chessboard, in which each of the 2^m cells along one diagonal forms a separate rectangle of side length 1. Determine the smallest possible sum of rectangle perimeters in such a partition.

C5 NLD (Netherlands)

Five identical empty buckets of 2-liter capacity stand at the vertices of a regular pentagon. Cinderella and her wicked Stepmother go through a sequence of rounds: At the beginning of every round, the Stepmother takes one liter of water from the nearby river and distributes it arbitrarily over the five buckets. Then Cinderella chooses a pair of neighboring buckets, empties them into the river, and puts them back. Then the next round begins. The Stepmother's goal is to make one of these buckets overflow. Cinderella's goal is to prevent this. Can the wicked Stepmother enforce a bucket overflow?

C6 BGR (Bulgaria)

On a 999×999 board a *limp rook* can move in the following way: From any square it can move to any of its adjacent squares, i.e. a square having a common side with it, and every move must be a turn, i.e. the directions of any two consecutive moves must be perpendicular. A *non-intersecting route* of the limp rook consists of a sequence of pairwise different squares that the limp rook can visit in that order by an admissible sequence of moves. Such a non-intersecting route is called *cyclic*, if the limp rook can, after reaching the last square of the route, move directly to the first square of the route and start over.

How many squares does the longest possible cyclic, non-intersecting route of a limp rook visit?

C7 RUS (Russian Federation)

Variant 1. A grasshopper jumps along the real axis. He starts at point 0 and makes 2009 jumps to the right with lengths $1, 2, \dots, 2009$ in an arbitrary order. Let M be a set of 2008 positive integers less than $1005 \cdot 2009$. Prove that the grasshopper can arrange his jumps in such a way that he never lands on a point from M.

Variant 2. Let n be a nonnegative integer. A grasshopper jumps along the real axis. He starts at point 0 and makes $n+1$ jumps to the right with pairwise different positive integral lengths $a_1, a_2, \dots, a_{n+1}$ in an arbitrary order. Let M be a set of n positive integers in the interval $(0, s)$, where $s = a_1 + a_2 + \cdots + a_{n+1}$. Prove that the grasshopper can arrange his jumps in such a way that he never lands on a point from M.

C8 AUT (Austria)

For any integer $n \geq 2$, we compute the integer $h(n)$ by applying the following procedure to its decimal representation. Let r be the rightmost digit of n.

(1) If $r = 0$, then the decimal representation of $h(n)$ results from the decimal representation of n by removing this rightmost digit 0.

(2) If $1 \leq r \leq 9$ we split the decimal representation of n into a maximal right part R that solely consists of digits not less than r and into a left part L that either is empty or ends with a digit strictly smaller than r. Then the decimal representation of $h(n)$ consists of the decimal representation of L, followed by two copies of the decimal representation of $R-1$. For instance, for the number $n = 17{,}151{,}345{,}543$, we will have $L = 17{,}151$, $R = 345{,}543$ and $h(n) = 17{,}151{,}345{,}542{,}345{,}542$.

Prove that, starting with an arbitrary integer $n \geq 2$, iterated application of h produces the integer 1 after finitely many steps.

Geometry

G1 BEL (Belgium)

Let ABC be a triangle with $AB = AC$. The angle bisectors of A and B meet the sides BC and AC in D and E, respectively. Let K be the incenter of triangle ADC. Suppose that $\angle BEK = 45°$. Find all possible values of $\angle BAC$.

G2 RUS (Russian Federation)

Let ABC be a triangle with circumcenter O. The points P and Q are interior points of the sides CA and AB, respectively. The circle k passes through the midpoints of the segments BP, CQ, and PQ. Prove that if the line PQ is tangent to circle k then $OP = OQ$.

G3 IRN (Islamic Republic of Iran)

Let ABC be a triangle. The incircle of ABC touches the sides AB and AC at the points Z and Y, respectively. Let G be the point where the lines BY and CZ meet, and let R and S be points such that the two quadrilaterals $BCYR$ and $BCSZ$ are parallelograms.

Prove that $GR = GS$.

G4 UNK (United Kingdom)

Given a cyclic quadrilateral $ABCD$, let the diagonals AC and BD meet at E and the lines AD and BC meet at F. The midpoints of AB and CD are G and H, respectively. Show that EF is tangent at E to the circle through the points E, G, and H.

G5 POL (Poland)

Let P be a polygon that is convex and symmetric to some point O. Prove that for some parallelogram R satisfying $P \subset R$ we have

$$\frac{|R|}{|P|} \leq \sqrt{2}$$

where $|R|$ and $|P|$ denote the area of the sets R and P, respectively.

G6 UKR (Ukraine)

Let the sides AD and BC of the quadrilateral $ABCD$ (such that AB is not parallel to CD) intersect at point P. Points O_1 and O_2 are the circumcenters and points H_1 and H_2 are the orthocenters of triangles ABP and DCP, respectively. Denote the midpoints of segments O_1H_1 and O_2H_2 by E_1 and E_2, respectively. Prove that the perpendicular from E_1 on CD, the perpendicular from E_2 on AB and the line H_1H_2 are concurrent.

G7 IRN (Islamic Republic of Iran)

Let ABC be a triangle with incenter I and let X, Y and Z be the incenters of the triangles BIC, CIA and AIB, respectively. Let the triangle XYZ be equilateral. Prove that ABC is equilateral too.

G8 BGR (Bulgaria)

Let $ABCD$ be a circumscribed quadrilateral. Let g be a line through A which meets the segment BC in M and the line CD in N. Denote by I_1, I_2, and I_3 the incenters of $\triangle ABM$, $\triangle MNC$, and $\triangle NDA$, respectively. Show that the orthocenter of $\triangle I_1I_2I_3$ lies on g.

Number Theory

N1 AUS (Australia)

A social club has n members. They have the membership numbers $1, 2, \ldots, n$, respectively. From time to time members send presents to other members, including items they have already received as presents from other members. In order to avoid the embarrassing situation that a member might receive a present that he or she has sent to other members, the club adds the following rule to its statutes at one of its annual general meetings:

"A member with membership number a is permitted to send a present to a member with membership number b if and only if $a(b-1)$ is a multiple of n."

Prove that, if each member follows this rule, none will receive a present from another member that he or she has already sent to other members.

Alternative formulation: Let G be a directed graph with n vertices $v_1, v_2, \ldots, v_n$, such that there is an edge going from v_a to v_b if and only if a and b are distinct and $a(b-1)$ is a multiple of n. Prove that this graph does not contain a directed cycle.

N2 PER (Peru)

A positive integer N is called *balanced*, if $N = 1$ or if N can be written as a product of an even number of not necessarily distinct primes. Given positive integers a and b, consider the polynomial P defined by $P(x) = (x+a)(x+b)$.

(a) Prove that there exist distinct positive integers a and b such that all the numbers $P(1), P(2), \ldots, P(50)$ are balanced.

(b) Prove that if $P(n)$ is balanced for all positive integers n, then $a = b$.

N3 EST (Estonia)

Let f be a non-constant function from the set of positive integers into the set of positive integers, such that $a - b$ divides $f(a) - f(b)$ for all distinct positive integers a, b. Prove that there exist infinitely many primes p such that p divides $f(c)$ for some positive integer c.

N4 PRK (Democratic People's Republic of Korea)

Find all positive integers n such that there exists a sequence of positive integers $a_1, a_2, \ldots, a_n$ satisfying

$$a_{k+1} = \frac{a_k^2 + 1}{a_{k-1} + 1} - 1$$

for every k with $2 \le k \le n-1$.

N5 HUN (Hungary)

Let $P(x)$ be a non-constant polynomial with integer coefficients. Prove that there is no function T from the set of integers into the set of integers such that the number of integers x with $T^n(x) = x$ is equal to $P(n)$ for every $n \ge 1$, where T^n denotes the n-fold application of T.

N6 TUR (Turkey)

Let k be a positive integer. Show that if there exists a sequence $a_0, a_1, \ldots$ of integers satisfying the condition

$$a_n = \frac{a_{n-1} + n^k}{n} \quad \text{for all } n \geq 1,$$

then $k - 2$ is divisible by 3.

N7 MNG (Mongolia)

Let a and b be distinct integers greater than 1. Prove that there exists a positive integer n such that $(a^n - 1)(b^n - 1)$ is not a perfect square.

Algebra

A1 CZE (Czech Republic)

Find the largest possible integer k, such that the following statement is true:

Let 2009 arbitrary non-degenerated triangles be given. In every triangle the three sides are colored, such that one is blue, one is red and one is white. Now, for every color separately, let us sort the lengths of the sides. We obtain

$$\begin{array}{rll} & b_1 \leq b_2 \leq \ldots \leq b_{2009} & \text{the lengths of the blue sides,} \\ & r_1 \leq r_2 \leq \ldots \leq r_{2009} & \text{the lengths of the red sides,} \\ \text{and} & w_1 \leq w_2 \leq \ldots \leq w_{2009} & \text{the lengths of the white sides.} \end{array}$$

Then there exist k indices j such that we can form a non-degenerated triangle with side lengths b_j, r_j, w_j.

Solution. We will prove that the largest possible number k of indices satisfying the given condition is one.

Firstly we prove that b_{2009}, r_{2009}, w_{2009} are always lengths of the sides of a triangle. Without loss of generality we may assume that $w_{2009} \geq r_{2009} \geq b_{2009}$. We show that the inequality $b_{2009} + r_{2009} > w_{2009}$ holds. Evidently, there exists a triangle with side lengths w, b, r for the white, blue and red side, respectively, such that $w_{2009} = w$. By the conditions of the problem we have $b + r > w$, $b_{2009} \geq b$ and $r_{2009} \geq r$. From these inequalities it follows

$$b_{2009} + r_{2009} \geq b + r > w = w_{2009}.$$

Secondly we will describe a sequence of triangles for which w_j, b_j, r_j with $j < 2009$ are not the lengths of the sides of a triangle. Let us define the sequence Δ_j, $j = 1, 2, \ldots, 2009$, of triangles, where Δ_j has

a blue side of length $2j$,
a red side of length j for all $j \leq 2008$ and 4018 for $j = 2009$,
and a white side of length $j+1$ for all $j \leq 2007$, 4018 for $j = 2008$ and 1 for $j = 2009$.

Since

$$\begin{array}{rll} (j+1) + j > 2j & \geq j+1 > j, & \text{if } j \leq 2007, \\ 2j + j > 4018 > 2j & > j, & \text{if } j = 2008, \\ 4018 + 1 > 2j & = 4018 > 1, & \text{if } j = 2009, \end{array}$$

such a sequence of triangles exists. Moreover, $w_j = j$, $r_j = j$ and $b_j = 2j$ for $1 \leq j \leq 2008$. Then

$$w_j + r_j = j + j = 2j = b_j,$$

i.e., b_j, r_j and w_j are not the lengths of the sides of a triangle for $1 \leq j \leq 2008$.

A2 EST (Estonia)

Let a, b, c be positive real numbers such that $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=a+b+c$. Prove that

$$\frac{1}{(2a+b+c)^2}+\frac{1}{(2b+c+a)^2}+\frac{1}{(2c+a+b)^2}\leq\frac{3}{16}.$$

Solution 1. For positive real numbers x, y, z, from the arithmetic-geometric-mean inequality,

$$2x+y+z=(x+y)+(x+z)\geq 2\sqrt{(x+y)(x+z)},$$

we obtain

$$\frac{1}{(2x+y+z)^2}\leq\frac{1}{4(x+y)(x+z)}.$$

Applying this to the left-hand side terms of the inequality to prove, we get

$$\begin{aligned}&\frac{1}{(2a+b+c)^2}+\frac{1}{(2b+c+a)^2}+\frac{1}{(2c+a+b)^2}\\ &\quad\leq\frac{1}{4(a+b)(a+c)}+\frac{1}{4(b+c)(b+a)}+\frac{1}{4(c+a)(c+b)}\\ &\quad=\frac{(b+c)+(c+a)+(a+b)}{4(a+b)(b+c)(c+a)}=\frac{a+b+c}{2(a+b)(b+c)(c+a)}.\end{aligned}\tag{1}$$

A second application of the inequality of the arithmetic-geometric mean yields

$$a^2b+a^2c+b^2a+b^2c+c^2a+c^2b\geq 6abc,$$

or, equivalently,

$$9(a+b)(b+c)(c+a)\geq 8(a+b+c)(ab+bc+ca).\tag{2}$$

The supposition $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=a+b+c$ can be written as

$$ab+bc+ca=abc(a+b+c).\tag{3}$$

Applying the arithmetic-geometric-mean inequality $x^2y^2+x^2z^2\geq 2x^2yz$ thrice, we get

$$a^2b^2+b^2c^2+c^2a^2\geq a^2bc+ab^2c+abc^2,$$

which is equivalent to

$$(ab+bc+ca)^2\geq 3abc(a+b+c).\tag{4}$$

Combining (1), (2), (3), and (4), we will finish the proof:

$$\begin{aligned}\frac{a+b+c}{2(a+b)(b+c)(c+a)} &= \frac{(a+b+c)(ab+bc+ca)}{2(a+b)(b+c)(c+a)} \cdot \frac{ab+bc+ca}{abc(a+b+c)} \cdot \frac{abc(a+b+c)}{(ab+bc+ca)^2} \\ &\leq \frac{9}{2\cdot 8} \cdot 1 \cdot \frac{1}{3} = \frac{3}{16}.\end{aligned}$$

Solution 2. Equivalently, we prove the homogenized inequality

$$\frac{(a+b+c)^2}{(2a+b+c)^2} + \frac{(a+b+c)^2}{(a+2b+c)^2} + \frac{(a+b+c)^2}{(a+b+2c)^2} \leq \frac{3}{16}(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$$

for all positive real numbers a, b, c. Without loss of generality we choose $a+b+c=1$. Thus, the problem is equivalent to prove for all $a,b,c>0$, fulfilling this condition, the inequality

$$\frac{1}{(1+a)^2} + \frac{1}{(1+b)^2} + \frac{1}{(1+c)^2} \leq \frac{3}{16}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right). \tag{5}$$

Applying Jensen's inequality to the function $f(x) = \dfrac{x}{(1+x)^2}$, which is concave for $0 \leq x \leq 2$ and increasing for $0 \leq x \leq 1$, we obtain

$$\alpha\frac{a}{(1+a)^2} + \beta\frac{b}{(1+b)^2} + \gamma\frac{c}{(1+c)^2} \leq (\alpha+\beta+\gamma)\frac{A}{(1+A)^2}, \quad \text{where} \quad A = \frac{\alpha a + \beta b + \gamma c}{\alpha+\beta+\gamma}.$$

Choosing $\alpha = \dfrac{1}{a}$, $\beta = \dfrac{1}{b}$, and $\gamma = \dfrac{1}{c}$, we can apply the harmonic-arithmetic-mean inequality

$$A = \frac{3}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}} \leq \frac{a+b+c}{3} = \frac{1}{3} < 1.$$

Finally we prove (5):

$$\begin{aligned}\frac{1}{(1+a)^2} + \frac{1}{(1+b)^2} + \frac{1}{(1+c)^2} &\leq \left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\frac{A}{(1+A)^2} \\ &\leq \left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\frac{\frac{1}{3}}{\left(1+\frac{1}{3}\right)^2} = \frac{3}{16}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right).\end{aligned}$$

A3 FRA (France)

Determine all functions f from the set of positive integers into the set of positive integers such that for all x and y there exists a non degenerated triangle with sides of lengths

$$x, \quad f(y) \quad \text{and} \quad f(y+f(x)-1).$$

Solution. The identity function $f(x)=x$ is the only solution of the problem.

If $f(x)=x$ for all positive integers x, the given three lengths are x, $y=f(y)$ and $z=f(y+f(x)-1)=x+y-1$. Because of $x\geq 1$, $y\geq 1$ we have $z\geq \max\{x,y\}>|x-y|$ and $z<x+y$. From this it follows that a triangle with these side lengths exists and does not degenerate. We prove in several steps that there is no other solution.

Step 1. We show $f(1)=1$.
If we had $f(1)=1+m>1$ we would conclude $f(y)=f(y+m)$ for all y considering the triangle with the side lengths 1, $f(y)$ and $f(y+m)$. Thus, f would be m-periodic and, consequently, bounded. Let B be a bound, $f(x)\leq B$. If we choose $x>2B$ we obtain the contradiction $x>2B\geq f(y)+f(y+f(x)-1)$.

Step 2. For all positive integers z, we have $f(f(z))=z$.
Setting $x=z$ and $y=1$ this follows immediately from Step 1.

Step 3. For all integers $z\geq 1$, we have $f(z)\leq z$.
Let us show, that the contrary leads to a contradiction. Assume $w+1=f(z)>z$ for some z. From Step 1 we know that $w\geq z\geq 2$. Let $M=\max\{f(1),f(2),\ldots,f(w)\}$ be the largest value of f for the first w integers. First we show, that no positive integer t exists with

$$f(t)>\frac{z-1}{w}\cdot t+M, \tag{1}$$

otherwise we decompose the smallest value t as $t=wr+s$ where r is an integer and $1\leq s\leq w$. Because of the definition of M, we have $t>w$. Setting $x=z$ and $y=t-w$ we get from the triangle inequality

$$z+f(t-w)>f((t-w)+f(z)-1)=f(t-w+w)=f(t).$$

Hence,

$$f(t-w)\geq f(t)-(z-1)>\frac{z-1}{w}(t-w)+M,$$

a contradiction to the minimality of t.

Therefore the inequality (1) fails for all $t\geq 1$, we have proven

$$f(t)\leq \frac{z-1}{w}\cdot t+M, \tag{2}$$

instead.

Now, using (2), we finish the proof of Step 3. Because of $z \leq w$ we have $\dfrac{z-1}{w} < 1$ and we can choose an integer t sufficiently large to fulfill the condition

$$\left(\frac{z-1}{w}\right)^2 t + \left(\frac{z-1}{w} + 1\right) M < t.$$

Applying (2) twice we get

$$f\left(f(t)\right) \leq \frac{z-1}{w} f(t) + M \leq \frac{z-1}{w}\left(\frac{z-1}{w} t + M\right) + M < t$$

in contradiction to Step 2, which proves Step 3.

Final step. Thus, following Step 2 and Step 3, we obtain

$$z = f(f(z)) \leq f(z) \leq z$$

and $f(z) = z$ for all positive integers z is proven.

A4 BLR (Belarus)

Let a, b, c be positive real numbers such that $ab+bc+ca \leq 3abc$. Prove that

$$\sqrt{\frac{a^2+b^2}{a+b}}+\sqrt{\frac{b^2+c^2}{b+c}}+\sqrt{\frac{c^2+a^2}{c+a}}+3 \leq \sqrt{2}\left(\sqrt{a+b}+\sqrt{b+c}+\sqrt{c+a}\right).$$

Solution. Starting with the terms of the right-hand side, the quadratic-arithmetic-mean inequality yields

$$\begin{aligned}\sqrt{2}\sqrt{a+b} &= 2\sqrt{\frac{ab}{a+b}}\sqrt{\frac{1}{2}\left(2+\frac{a^2+b^2}{ab}\right)} \\ &\geq 2\sqrt{\frac{ab}{a+b}}\cdot\frac{1}{2}\left(\sqrt{2}+\sqrt{\frac{a^2+b^2}{ab}}\right)=\sqrt{\frac{2ab}{a+b}}+\sqrt{\frac{a^2+b^2}{a+b}}\end{aligned}$$

and, analogously,

$$\sqrt{2}\sqrt{b+c} \geq \sqrt{\frac{2bc}{b+c}}+\sqrt{\frac{b^2+c^2}{b+c}}, \qquad \sqrt{2}\sqrt{c+a} \geq \sqrt{\frac{2ca}{c+a}}+\sqrt{\frac{c^2+a^2}{c+a}}.$$

Applying the inequality between the arithmetic mean and the squared harmonic mean will finish the proof:

$$\sqrt{\frac{2ab}{a+b}}+\sqrt{\frac{2bc}{b+c}}+\sqrt{\frac{2ca}{c+a}} \geq 3\cdot\sqrt{\frac{3}{\sqrt{\frac{a+b}{2ab}}^2+\sqrt{\frac{b+c}{2bc}}^2+\sqrt{\frac{c+a}{2ca}}^2}}=3\cdot\sqrt{\frac{3abc}{ab+bc+ca}} \geq 3.$$

A5 BLR (Belarus)

Let f be any function that maps the set of real numbers into the set of real numbers. Prove that there exist real numbers x and y such that

$$f\left(x - f(y)\right) > yf(x) + x.$$

Solution 1. Assume that

$$f(x - f(y)) \leq yf(x) + x \quad \text{for all real } x, y. \tag{1}$$

Let $a = f(0)$. Setting $y = 0$ in (1) gives $f(x - a) \leq x$ for all real x and, equivalently,

$$f(y) \leq y + a \quad \text{for all real } y. \tag{2}$$

Setting $x = f(y)$ in (1) yields in view of (2)

$$a = f(0) \leq yf(f(y)) + f(y) \leq yf(f(y)) + y + a.$$

This implies $0 \leq y(f(f(y)) + 1)$ and thus

$$f(f(y)) \geq -1 \quad \text{for all } y > 0. \tag{3}$$

From (2) and (3) we obtain $-1 \leq f(f(y)) \leq f(y) + a$ for all $y > 0$, so

$$f(y) \geq -a - 1 \quad \text{for all } y > 0. \tag{4}$$

Now we show that

$$f(x) \leq 0 \quad \text{for all real } x. \tag{5}$$

Assume the contrary, i.e. there is some x such that $f(x) > 0$. Take any y such that

$$y < x - a \quad \text{and} \quad y < \frac{-a - x - 1}{f(x)}.$$

Then in view of (2)

$$x - f(y) \geq x - (y + a) > 0$$

and with (1) and (4) we obtain

$$yf(x) + x \geq f(x - f(y)) \geq -a - 1,$$

whence

$$y \geq \frac{-a - x - 1}{f(x)}$$

contrary to our choice of y. Thereby, we have established (5).

Setting $x = 0$ in (5) leads to $a = f(0) \leq 0$ and (2) then yields

$$f(x) \leq x \quad \text{for all real } x. \tag{6}$$

Now choose y such that $y > 0$ and $y > -f(-1) - 1$ and set $x = f(y) - 1$. From (1), (5) and

(6) we obtain

$$f(-1) = f(x - f(y)) \le yf(x) + x = yf(f(y) - 1) + f(y) - 1 \le y(f(y) - 1) - 1 \le -y - 1,$$

i.e. $y \le -f(-1) - 1$, a contradiction to the choice of y.

Solution 2. Assume that

$$f(x - f(y)) \le yf(x) + x \quad \text{for all real } x, y. \tag{7}$$

Let $a = f(0)$. Setting $y = 0$ in (7) gives $f(x - a) \le x$ for all real x and, equivalently,

$$f(y) \le y + a \quad \text{for all real } y. \tag{8}$$

Now we show that

$$f(z) \ge 0 \quad \text{for all } z \ge 1. \tag{9}$$

Let $z \ge 1$ be fixed, set $b = f(z)$ and assume that $b < 0$. Setting $x = w + b$ and $y = z$ in (7) gives

$$f(w) - zf(w + b) \le w + b \quad \text{for all real } w. \tag{10}$$

Applying (10) to $w, w + b, \dots, w + (n - 1)b$, where $n = 1, 2, \dots$, leads to

$$\begin{aligned} f(w) - z^n f(w + nb) = &(f(w) - zf(w + b)) + z\,(f(w + b) - zf(w + 2b)) \\ &+ \cdots + z^{n-1}\,(f(w + (n - 1)b) - zf(w + nb)) \\ \le &(w + b) + z(w + 2b) + \cdots + z^{n-1}(w + nb). \end{aligned}$$

From (8) we obtain

$$f(w + nb) \le w + nb + a$$

and, thus, we have for all positive integers n

$$f(w) \le (1 + z + \cdots + z^{n-1} + z^n)w + (1 + 2z + \cdots + nz^{n-1} + nz^n)b + z^n a. \tag{11}$$

With $w = 0$ we get

$$a \le (1 + 2z + \cdots + nz^{n-1} + nz^n)b + az^n. \tag{12}$$

In view of the assumption $b < 0$ we find some n such that

$$a > (nb + a)z^n \tag{13}$$

because the right hand side tends to $-\infty$ as $n \to \infty$. Now (12) and (13) give the desired contradiction and (9) is established. In addition, we have for $z = 1$ the strict inequality

$$f(1) > 0. \tag{14}$$

Indeed, assume that $f(1) = 0$. Then setting $w = -1$ and $z = 1$ in (11) leads to

$$f(-1) \le -(n + 1) + a$$

which is false if n is sufficiently large.

To complete the proof we set $t = \min\{-a, -2/f(1)\}$. Setting $x = 1$ and $y = t$ in (7) gives

$$f(1 - f(t)) \le tf(1) + 1 \le -2 + 1 = -1. \tag{15}$$

On the other hand, by (8) and the choice of t we have $f(t) \leq t + a \leq 0$ and hence $1 - f(t) \geq 1$. The inequality (9) yields

$$f(1 - f(t)) \geq 0,$$

which contradicts (15).

A6 USA (United States of America)

Suppose that $s_1, s_2, s_3, \ldots$ is a strictly increasing sequence of positive integers such that the subsequences

$$s_{s_1},\ s_{s_2},\ s_{s_3},\ \ldots \qquad \text{and} \qquad s_{s_1+1},\ s_{s_2+1},\ s_{s_3+1},\ \ldots$$

are both arithmetic progressions. Prove that $s_1, s_2, s_3, \ldots$ is itself an arithmetic progression.

Solution 1. Let D be the common difference of the progression $s_{s_1}, s_{s_2}, \ldots$. Let for $n = 1, 2, \ldots$

$$d_n = s_{n+1} - s_n.$$

We have to prove that d_n is constant. First we show that the numbers d_n are bounded. Indeed, by supposition $d_n \geq 1$ for all n. Thus, we have for all n

$$d_n = s_{n+1} - s_n \leq d_{s_n} + d_{s_n+1} + \cdots + d_{s_{n+1}-1} = s_{s_{n+1}} - s_{s_n} = D.$$

The boundedness implies that there exist

$$m = \min\{d_n : n = 1, 2, \ldots\} \quad \text{and} \quad M = \max\{d_n : n = 1, 2, \ldots\}.$$

It suffices to show that $m = M$. Assume that $m < M$. Choose n such that $d_n = m$. Considering a telescoping sum of $m = d_n = s_{n+1} - s_n$ items not greater than M leads to

$$D = s_{s_{n+1}} - s_{s_n} = s_{s_n+m} - s_{s_n} = d_{s_n} + d_{s_n+1} + \cdots + d_{s_n+m-1} \leq mM \tag{1}$$

and equality holds if and only if all items of the sum are equal to M. Now choose n such that $d_n = M$. In the same way, considering a telescoping sum of M items not less than m we obtain

$$D = s_{s_{n+1}} - s_{s_n} = s_{s_n+M} - s_{s_n} = d_{s_n} + d_{s_n+1} + \cdots + d_{s_n+M-1} \geq Mm \tag{2}$$

and equality holds if and only if all items of the sum are equal to m. The inequalities (1) and (2) imply that $D = Mm$ and that

$$\begin{aligned} d_{s_n} = d_{s_n+1} = \cdots = d_{s_{n+1}-1} = M \quad &\text{if } d_n = m, \\ d_{s_n} = d_{s_n+1} = \cdots = d_{s_{n+1}-1} = m \quad &\text{if } d_n = M. \end{aligned}$$

Hence, $d_n = m$ implies $d_{s_n} = M$. Note that $s_n \geq s_1 + (n-1) \geq n$ for all n and moreover $s_n > n$ if $d_n = n$, because in the case $s_n = n$ we would have $m = d_n = d_{s_n} = M$ in contradiction to the assumption $m < M$. In the same way $d_n = M$ implies $d_{s_n} = m$ and $s_n > n$. Consequently, there is a strictly increasing sequence $n_1, n_2, \ldots$ such that

$$d_{s_{n_1}} = M, \quad d_{s_{n_2}} = m, \quad d_{s_{n_3}} = M, \quad d_{s_{n_4}} = m, \quad \ldots .$$

The sequence $d_{s_1}, d_{s_2}, \ldots$ is the sequence of pairwise differences of $s_{s_1+1}, s_{s_2+1}, \ldots$ and $s_{s_1}, s_{s_2}, \ldots$, hence also an arithmetic progression. Thus $m = M$.

Solution 2. Let the integers D and E be the common differences of the progressions $s_{s_1}, s_{s_2}, \dots$ and $s_{s_1+1}, s_{s_2+1}, \dots$, respectively. Let briefly $A = s_{s_1} - D$ and $B = s_{s_1+1} - E$. Then, for all positive integers n,

$$s_{s_n} = A + nD, \qquad s_{s_n+1} = B + nE.$$

Since the sequence $s_1, s_2, \dots$ is strictly increasing, we have for all positive integers n

$$s_{s_n} < s_{s_n+1} \le s_{s_{n+1}},$$

which implies

$$A + nD < B + nE \le A + (n+1)D,$$

and thereby

$$0 < B - A + n(E - D) \le D,$$

which implies $D - E = 0$ and thus

$$0 \le B - A \le D. \tag{3}$$

Let $m = \min\{s_{n+1} - s_n : n = 1, 2, \dots\}$. Then

$$B - A = (s_{s_1+1} - E) - (s_{s_1} - D) = s_{s_1+1} - s_{s_1} \ge m \tag{4}$$

and

$$D = A + (s_1 + 1)D - (A + s_1 D) = s_{s_{s_1+1}} - s_{s_{s_1}} = s_{B+D} - s_{A+D} \ge m(B - A). \tag{5}$$

From (3) we consider two cases.

Case 1. $B - A = D$.
Then, for each positive integer n, $s_{s_n+1} = B + nD = A + (n+1)D = s_{s_{n+1}}$, hence $s_{n+1} = s_n + 1$ and $s_1, s_2, \dots$ is an arithmetic progression with common difference 1.

Case 2. $B - A < D$. Choose some positive integer N such that $s_{N+1} - s_N = m$. Then

$$\begin{aligned} m(A - B + D - 1) &= m((A + (N+1)D) - (B + ND + 1)) \\ &\le s_{A+(N+1)D} - s_{B+ND+1} = s_{s_{s_{N+1}}} - s_{s_{s_N}+1} \\ &= (A + s_{N+1}D) - (B + (s_N + 1)D) = (s_{N+1} - s_N)D + A - B - D \\ &= mD + A - B - D, \end{aligned}$$

i.e.,

$$(B - A - m) + (D - m(B - A)) \le 0. \tag{6}$$

The inequalities (4)-(6) imply that

$$B - A = m \quad \text{and} \quad D = m(B - A).$$

Assume that there is some positive integer n such that $s_{n+1} > s_n + m$. Then

$$m(m+1) \le m(s_{n+1} - s_n) \le s_{s_{n+1}} - s_{s_n} = (A + (n+1)D) - (A + nD)) = D = m(B - A) = m^2,$$

a contradiction. Hence $s_1, s_2, \dots$ is an arithmetic progression with common difference m.

A7 JPN (Japan)

Find all functions f from the set of real numbers into the set of real numbers which satisfy for all real x, y the identity

$$f(xf(x+y)) = f(yf(x)) + x^2.$$

Solution 1. It is no hard to see that the two functions given by $f(x) = x$ and $f(x) = -x$ for all real x respectively solve the functional equation. In the sequel, we prove that there are no further solutions.

Let f be a function satisfying the given equation. It is clear that f cannot be a constant. Let us first show that $f(0) = 0$. Suppose that $f(0) \neq 0$. For any real t, substituting $(x, y) = (0, \frac{t}{f(0)})$ into the given functional equation, we obtain

$$f(0) = f(t), \tag{1}$$

contradicting the fact that f is not a constant function. Therefore, $f(0) = 0$. Next for any t, substituting $(x, y) = (t, 0)$ and $(x, y) = (t, -t)$ into the given equation, we get

$$f(tf(t)) = f(0) + t^2 = t^2,$$

and

$$f(tf(0)) = f(-tf(t)) + t^2,$$

respectively. Therefore, we conclude that

$$f(tf(t)) = t^2, \quad f(-tf(t)) = -t^2, \qquad \text{for every real } t. \tag{2}$$

Consequently, for every real v, there exists a real u, such that $f(u) = v$. We also see that if $f(t) = 0$, then $0 = f(tf(t)) = t^2$ so that $t = 0$, and thus 0 is the only real number satisfying $f(t) = 0$.

We next show that for any real number s,

$$f(-s) = -f(s). \tag{3}$$

This is clear if $f(s) = 0$. Suppose now $f(s) < 0$, then we can find a number t for which $f(s) = -t^2$. As $t \neq 0$ implies $f(t) \neq 0$, we can also find number a such that $af(t) = s$. Substituting $(x, y) = (t, a)$ into the given equation, we get

$$f(tf(t+a)) = f(af(t)) + t^2 = f(s) + t^2 = 0,$$

and therefore, $tf(t+a) = 0$, which implies $t + a = 0$, and hence $s = -tf(t)$. Consequently, $f(-s) = f(tf(t)) = t^2 = -(-t^2) = -f(s)$ holds in this case.

Finally, suppose $f(s) > 0$ holds. Then there exists a real number $t \neq 0$ for which $f(s) = t^2$. Choose a number a such that $tf(a) = s$. Substituting $(x, y) = (t, a-t)$ into the given equation, we get $f(s) = f(tf(a)) = f((a-t)f(t)) + t^2 = f((a-t)f(t)) + f(s)$. So we have $f((a-t)f(t)) = 0$, from which we conclude that $(a-t)f(t) = 0$. Since $f(t) \neq 0$, we get $a = t$ so that $s = tf(t)$ and thus we see $f(-s) = f(-tf(t)) = -t^2 = -f(s)$ holds in this case also. This observation finishes the proof of (3).

By substituting $(x, y) = (s, t)$, $(x, y) = (t, -s-t)$ and $(x, y) = (-s-t, s)$ into the given equation, we obtain

$$f(sf(s+t))) = f(tf(s)) + s^2,$$
$$f(tf(-s)) = f((-s-t)f(t)) + t^2,$$

and

$$f((-s-t)f(-t)) = f(sf(-s-t)) + (s+t)^2,$$

respectively. Using the fact that $f(-x) = -f(x)$ holds for all x to rewrite the second and the third equation, and rearranging the terms, we obtain

$$\begin{aligned} f(tf(s)) - f(sf(s+t)) &= -s^2, \\ f(tf(s)) - f((s+t)f(t)) &= -t^2, \\ f((s+t)f(t)) + f(sf(s+t)) &= (s+t)^2. \end{aligned}$$

Adding up these three equations now yields $2f(tf(s)) = 2ts$, and therefore, we conclude that $f(tf(s)) = ts$ holds for every pair of real numbers s, t. By fixing s so that $f(s) = 1$, we obtain $f(x) = sx$. In view of the given equation, we see that $s = \pm 1$. It is easy to check that both functions $f(x) = x$ and $f(x) = -x$ satisfy the given functional equation, so these are the desired solutions.

Solution 2. As in Solution 1 we obtain (1), (2) and (3).

Now we prove that f is injective. For this purpose, let us assume that $f(r) = f(s)$ for some $r \neq s$. Then, by (2)

$$r^2 = f(rf(r)) = f(rf(s)) = f((s-r)f(r)) + r^2,$$

where the last statement follows from the given functional equation with $x = r$ and $y = s - r$. Hence, $h = (s-r)f(r)$ satisfies $f(h) = 0$ which implies $h^2 = f(hf(h)) = f(0) = 0$, i.e., $h = 0$. Then, by $s \neq r$ we have $f(r) = 0$ which implies $r = 0$, and finally $f(s) = f(r) = f(0) = 0$. Analogously, it follows that $s = 0$ which gives the contradiction $r = s$.

To prove $|f(1)| = 1$ we apply (2) with $t = 1$ and also with $t = f(1)$ and obtain $f(f(1)) = 1$ and $(f(1))^2 = f(f(1) \cdot f(f(1))) = f(f(1)) = 1$.

Now we choose $\eta \in \{-1, 1\}$ with $f(1) = \eta$. Using that f is odd and the given equation with $x = 1, y = z$ (second equality) and with $x = -1, y = z + 2$ (fourth equality) we obtain

$$\begin{aligned} f(z) + 2\eta &= \eta(f(z\eta) + 2) = \eta(f(f(z+1)) + 1) = \eta(-f(-f(z+1)) + 1) \\ &= -\eta f((z+2)f(-1)) = -\eta f((z+2)(-\eta)) = \eta f((z+2)\eta) = f(z+2). \end{aligned} \tag{4}$$

Hence,

$$f(z + 2\eta) = \eta f(\eta z + 2) = \eta(f(\eta z) + 2\eta) = f(z) + 2.$$

Using this argument twice we obtain

$$f(z + 4\eta) = f(z + 2\eta) + 2 = f(z) + 4.$$

Substituting $z = 2f(x)$ we have

$$f(2f(x)) + 4 = f(2f(x) + 4\eta) = f(2f(x+2)),$$

where the last equality follows from (4). Applying the given functional equation we proceed to

$$f(2f(x+2)) = f(xf(2)) + 4 = f(2\eta x) + 4$$

where the last equality follows again from (4) with $z = 0$, i.e., $f(2) = 2\eta$. Finally, $f(2f(x)) = f(2\eta x)$ and by injectivity of f we get $2f(x) = 2\eta x$ and hence the two solutions.

Combinatorics

C1 NZL (New Zealand)

Consider 2009 cards, each having one gold side and one black side, lying in parallel on a long table. Initially all cards show their gold sides. Two players, standing by the same long side of the table, play a game with alternating moves. Each move consists of choosing a block of 50 consecutive cards, the leftmost of which is showing gold, and turning them all over, so those which showed gold now show black and vice versa. The last player who can make a legal move wins.

(a) Does the game necessarily end?

(b) Does there exist a winning strategy for the starting player?

Solution. (a) We interpret a card showing black as the digit 0 and a card showing gold as the digit 1. Thus each position of the 2009 cards, read from left to right, corresponds bijectively to a nonnegative integer written in binary notation of 2009 digits, where leading zeros are allowed. Each move decreases this integer, so the game must end.

(b) We show that there is no winning strategy for the starting player. We label the cards from right to left by $1, \ldots, 2009$ and consider the set S of cards with labels $50i$, $i = 1, 2, \ldots, 40$. Let g_n be the number of cards from S showing gold after n moves. Obviously, $g_0 = 40$. Moreover, $|g_n - g_{n+1}| = 1$ as long as the play goes on. Thus, after an odd number of moves, the non-starting player finds a card from S showing gold and hence can make a move. Consequently, this player always wins.

C2 ROU (Romania)

For any integer $n \geq 2$, let $N(n)$ be the maximal number of triples (a_i, b_i, c_i), $i = 1, \dots, N(n)$, consisting of nonnegative integers a_i, b_i and c_i such that the following two conditions are satisfied:

(1) $a_i + b_i + c_i = n$ for all $i = 1, \dots, N(n)$,

(2) If $i \neq j$, then $a_i \neq a_j, b_i \neq b_j$ and $c_i \neq c_j$.

Determine $N(n)$ for all $n \geq 2$.

Comment. The original problem was formulated for m-tuples instead for triples. The numbers $N(m, n)$ are then defined similarly to $N(n)$ in the case $m = 3$. The numbers $N(3, n)$ and $N(n, n)$ should be determined. The case $m = 3$ is the same as in the present problem. The upper bound for $N(n, n)$ can be proved by a simple generalization. The construction of a set of triples attaining the bound can be easily done by induction from n to $n + 2$.

Solution. Let $n \geq 2$ be an integer and let $\{T_1, \dots, T_N\}$ be any set of triples of nonnegative integers satisfying the conditions (1) and (2). Since the a-coordinates are pairwise distinct we have

$$\sum_{i=1}^{N} a_i \geq \sum_{i=1}^{N} (i-1) = \frac{N(N-1)}{2}.$$

Analogously,

$$\sum_{i=1}^{N} b_i \geq \frac{N(N-1)}{2} \quad \text{and} \quad \sum_{i=1}^{N} c_i \geq \frac{N(N-1)}{2}.$$

Summing these three inequalities and applying (1) yields

$$3\frac{N(N-1)}{2} \leq \sum_{i=1}^{N} a_i + \sum_{i=1}^{N} b_i + \sum_{i=1}^{N} c_i = \sum_{i=1}^{N} (a_i + b_i + c_i) = nN,$$

hence $3\frac{N-1}{2} \leq n$ and, consequently,

$$N \leq \left\lfloor \frac{2n}{3} \right\rfloor + 1.$$

By constructing examples, we show that this upper bound can be attained, so $N(n) = \lfloor \frac{2n}{3} \rfloor + 1$.

We distinguish the cases $n = 3k - 1$, $n = 3k$ and $n = 3k + 1$ for $k \geq 1$ and present the extremal examples in form of a table.

$n = 3k - 1$ $\lfloor \frac{2n}{3} \rfloor + 1 = 2k$		
a_i	b_i	c_i
0	$k+1$	$2k-2$
1	$k+2$	$2k-4$
$\vdots$	$\vdots$	$\vdots$
$k-1$	$2k$	0
k	0	$2k-1$
$k+1$	1	$2k-3$
$\vdots$	$\vdots$	$\vdots$
$2k-1$	$k-1$	1

$n = 3k$ $\lfloor \frac{2n}{3} \rfloor + 1 = 2k + 1$		
a_i	b_i	c_i
0	k	$2k$
1	$k+1$	$2k-2$
$\vdots$	$\vdots$	$\vdots$
k	$2k$	0
$k+1$	0	$2k-1$
$k+2$	1	$2k-3$
$\vdots$	$\vdots$	$\vdots$
$2k$	$k-1$	1

$n = 3k + 1$ $\lfloor \frac{2n}{3} \rfloor + 1 = 2k + 1$		
a_i	b_i	c_i
0	k	$2k+1$
1	$k+1$	$2k-1$
$\vdots$	$\vdots$	$\vdots$
k	$2k$	1
$k+1$	0	$2k$
$k+2$	1	$2k-2$
$\vdots$	$\vdots$	$\vdots$
$2k$	$k-1$	2

It can be easily seen that the conditions (1) and (2) are satisfied and that we indeed have $\lfloor \frac{2n}{3} \rfloor + 1$ triples in each case.

Comment. A cute combinatorial model is given by an equilateral triangle, partitioned into n^2 congruent equilateral triangles by $n - 1$ equidistant parallels to each of its three sides. Two chess-like bishops placed at any two vertices of the small triangles are said to *menace* one another if they lie on a same parallel. The problem is to determine the largest number of bishops that can be placed so that none menaces another. A bishop may be assigned three coordinates a, b, c, namely the numbers of sides of small triangles they are off each of the sides of the big triangle. It is readily seen that the sum of these coordinates is always n, therefore fulfilling the requirements.

C3 RUS (Russian Federation)

Let n be a positive integer. Given a sequence $\varepsilon_1, \ldots, \varepsilon_{n-1}$ with $\varepsilon_i = 0$ or $\varepsilon_i = 1$ for each $i = 1, \ldots, n-1$, the sequences $a_0, \ldots, a_n$ and $b_0, \ldots, b_n$ are constructed by the following rules:

$$a_0 = b_0 = 1, \quad a_1 = b_1 = 7,$$

$$a_{i+1} = \begin{cases} 2a_{i-1} + 3a_i, & \text{if } \varepsilon_i = 0, \\ 3a_{i-1} + a_i, & \text{if } \varepsilon_i = 1, \end{cases} \quad \text{for each } i = 1, \ldots, n-1,$$

$$b_{i+1} = \begin{cases} 2b_{i-1} + 3b_i, & \text{if } \varepsilon_{n-i} = 0, \\ 3b_{i-1} + b_i, & \text{if } \varepsilon_{n-i} = 1, \end{cases} \quad \text{for each } i = 1, \ldots, n-1.$$

Prove that $a_n = b_n$.

Solution. For a binary word $w = \sigma_1 \ldots \sigma_n$ of length n and a letter $\sigma \in \{0,1\}$ let $w\sigma = \sigma_1 \ldots \sigma_n \sigma$ and $\sigma w = \sigma\sigma_1 \ldots \sigma_n$. Moreover let $\overline{w} = \sigma_n \ldots \sigma_1$ and let $\emptyset$ be the empty word (of length 0 and with $\overline{\emptyset} = \emptyset$). Let (u,v) be a pair of two real numbers. For binary words w we define recursively the numbers $(u,v)^w$ as follows:

$$(u,v)^{\emptyset} = v, \quad (u,v)^0 = 2u + 3v, \quad (u,v)^1 = 3u + v,$$

$$(u,v)^{w\sigma\varepsilon} = \begin{cases} 2(u,v)^w + 3(u,v)^{w\sigma}, & \text{if } \varepsilon = 0, \\ 3(u,v)^w + (u,v)^{w\sigma}, & \text{if } \varepsilon = 1. \end{cases}$$

It easily follows by induction on the length of w that for all real numbers $u_1, v_1, u_2, v_2, \lambda_1$ and λ_2

$$(\lambda_1 u_1 + \lambda_2 u_2, \lambda_1 v_1 + \lambda_2 v_2)^w = \lambda_1 (u_1, v_1)^w + \lambda_2 (u_2, v_2)^w \tag{1}$$

and that for $\varepsilon \in \{0,1\}$

$$(u,v)^{\varepsilon w} = (v, (u,v)^{\varepsilon})^w. \tag{2}$$

Obviously, for $n \geq 1$ and $w = \varepsilon_1 \ldots \varepsilon_{n-1}$, we have $a_n = (1,7)^w$ and $b_n = (1,7)^{\overline{w}}$. Thus it is sufficient to prove that

$$(1,7)^w = (1,7)^{\overline{w}} \tag{3}$$

for each binary word w. We proceed by induction on the length of w. The assertion is obvious if w has length 0 or 1. Now let $w\sigma\varepsilon$ be a binary word of length $n \geq 2$ and suppose that the assertion is true for all binary words of length at most $n-1$.

Note that $(2,1)^{\sigma} = 7 = (1,7)^{\emptyset}$ for $\sigma \in \{0,1\}$, $(1,7)^0 = 23$, and $(1,7)^1 = 10$.

First let $\varepsilon = 0$. Then in view of the induction hypothesis and the equalities (1) and (2), we obtain

$$(1,7)^{w\sigma 0} = 2(1,7)^w + 3(1,7)^{w\sigma} = 2(1,7)^{\overline{w}} + 3(1,7)^{\sigma\overline{w}} = 2(2,1)^{\sigma\overline{w}} + 3(1,7)^{\sigma\overline{w}}$$
$$= (7,23)^{\sigma\overline{w}} = (1,7)^{0\sigma\overline{w}}.$$

Now let $\varepsilon = 1$. Analogously, we obtain

$$(1,7)^{w\sigma 1} = 3(1,7)^{w} + (1,7)^{w\sigma} = 3(1,7)^{\overline{w}} + (1,7)^{\sigma\overline{w}} = 3(2,1)^{\sigma\overline{w}} + (1,7)^{\sigma\overline{w}}$$
$$= (7,10)^{\sigma\overline{w}} = (1,7)^{1\sigma\overline{w}}.$$

Thus the induction step is complete, (3) and hence also $a_n = b_n$ are proved.

Comment. The original solution uses the relation

$$(1,7)^{\alpha\beta w} = ((1,7)^{w}, (1,7)^{\beta w})^{\alpha}, \quad \alpha, \beta \in \{0,1\},$$

which can be proved by induction on the length of w. Then (3) also follows by induction on the length of w:

$$(1,7)^{\alpha\beta w} = ((1,7)^{w}, (1,7)^{\beta w})^{\alpha} = ((1,7)^{\overline{w}}, (1,7)^{\overline{w}\beta})^{\alpha} = (1,7)^{\overline{w}\beta\alpha}.$$

Here w may be the empty word.

C4 NLD (Netherlands)

For an integer $m \geq 1$, we consider partitions of a $2^m \times 2^m$ chessboard into rectangles consisting of cells of the chessboard, in which each of the 2^m cells along one diagonal forms a separate rectangle of side length 1. Determine the smallest possible sum of rectangle perimeters in such a partition.

Solution 1. For a $k \times k$ chessboard, we introduce in a standard way coordinates of the vertices of the cells and assume that the cell C_{ij} in row i and column j has vertices $(i-1, j-1), (i-1, j), (i, j-1), (i, j)$, where $i, j \in \{1, \ldots, k\}$. Without loss of generality assume that the cells C_{ii}, $i = 1, \ldots, k$, form a separate rectangle. Then we may consider the boards $B_k = \bigcup_{1 \leq i < j \leq k} C_{ij}$ below that diagonal and the congruent board $B'_k = \bigcup_{1 \leq j < i \leq k} C_{ij}$ above that diagonal separately because no rectangle can simultaneously cover cells from B_k and B'_k. We will show that for $k = 2^m$ the smallest total perimeter of a rectangular partition of B_k is $m2^{m+1}$. Then the overall answer to the problem is $2 \cdot m2^{m+1} + 4 \cdot 2^m = (m+1)2^{m+2}$.

First we inductively construct for $m \geq 1$ a partition of B_{2^m} with total perimeter $m2^{m+1}$. If $m = 0$, the board B_{2^m} is empty and the total perimeter is 0. For $m \geq 0$, the board $B_{2^{m+1}}$ consists of a $2^m \times 2^m$ square in the lower right corner with vertices $(2^m, 2^m), (2^m, 2^{m+1}), (2^{m+1}, 2^m)$, $(2^{m+1}, 2^{m+1})$ to which two boards congruent to B_{2^m} are glued along the left and the upper margin. The square together with the inductive partitions of these two boards yield a partition with total perimeter $4 \cdot 2^m + 2 \cdot m2^{m+1} = (m+1)2^{m+2}$ and the induction step is complete.

Let

$$D_k = 2k \log_2 k.$$

Note that $D_k = m2^{m+1}$ if $k = 2^m$. Now we show by induction on k that the total perimeter of a rectangular partition of B_k is at least D_k. The case $k = 1$ is trivial (see $m = 0$ from above). Let the assertion be true for all positive integers less than k. We investigate a fixed rectangular partition of B_k that attains the minimal total perimeter. Let R be the rectangle that covers the cell C_{1k} in the lower right corner. Let (i, j) be the upper left corner of R. First we show that $i = j$. Assume that $i < j$. Then the line from (i, j) to $(i+1, j)$ or from (i, j) to $(i, j-1)$ must belong to the boundary of some rectangle in the partition. Without loss of generality assume that this is the case for the line from (i, j) to $(i+1, j)$.

Case 1. No line from (i, l) to $(i+1, l)$ where $j < l < k$ belongs to the boundary of some rectangle of the partition.
Then there is some rectangle R' of the partition that has with R the common side from (i, j) to (i, k). If we join these two rectangles to one rectangle we get a partition with smaller total perimeter, a contradiction.

Case 2. There is some l such that $j < l < k$ and the line from (i, l) to $(i+1, l)$ belongs to the boundary of some rectangle of the partition.
Then we replace the upper side of R by the line $(i+1, j)$ to $(i+1, k)$ and for the rectangles whose lower side belongs to the line from (i, j) to (i, k) we shift the lower side upwards so that the new lower side belongs to the line from $(i+1, j)$ to $(i+1, k)$. In such a way we obtain a rectangular partition of B_k with smaller total perimeter, a contradiction.

Now the fact that the upper left corner of R has the coordinates (i, i) is established. Consequently, the partition consists of R, of rectangles of a partition of a board congruent to B_i and of rectangles of a partition of a board congruent to B_{k-i}. By the induction hypothesis, its total perimeter is at least

$$2(k-i)+2i+D_i+D_{k-i} \geq 2k+2i\log_2 i+2(k-i)\log_2(k-i). \tag{1}$$

Since the function $f(x)=2x\log_2 x$ is convex for $x>0$, JENSEN's inequality immediately shows that the minimum of the right hand sight of (1) is attained for $i=k/2$. Hence the total perimeter of the optimal partition of B_k is at least $2k+2k/2\log_2 k/2+2(k/2)\log_2(k/2)=D_k$.

Solution 2. We start as in Solution 1 and present another proof that $m2^{m+1}$ is a lower bound for the total perimeter of a partition of B_{2^m} into n rectangles. Let briefly $M=2^m$. For $1\leq i\leq M$, let r_i denote the number of rectangles in the partition that cover some cell from row i and let c_j be the number of rectangles that cover some cell from column j. Note that the total perimeter p of all rectangles in the partition is

$$p=2\left(\sum_{i=1}^{M} r_i+\sum_{i=1}^{M} c_i\right).$$

No rectangle can simultaneously cover cells from row i and from column i since otherwise it would also cover the cell C_{ii}. We classify subsets S of rectangles of the partition as follows. We say that S is of type i, $1\leq i\leq M$, if S contains all r_i rectangles that cover some cell from row i, but none of the c_i rectangles that cover some cell from column i. Altogether there are $2^{n-r_i-c_i}$ subsets of type i. Now we show that no subset S can be simultaneously of type i and of type j if $i\neq j$. Assume the contrary and let without loss of generality $i<j$. The cell C_{ij} must be covered by some rectangle R. The subset S is of type i, hence R is contained in S. S is of type j, thus R does not belong to S, a contradiction. Since there are 2^n subsets of rectangles of the partition, we infer

$$2^n \geq \sum_{i=1}^{M} 2^{n-r_i-c_i}=2^n\sum_{i=1}^{M} 2^{-(r_i+c_i)}. \tag{2}$$

By applying JENSEN'S inequality to the convex function $f(x)=2^{-x}$ we derive

$$\frac{1}{M}\sum_{i=1}^{M} 2^{-(r_i+c_i)} \geq 2^{-\frac{1}{M}\sum_{i=1}^{M}(r_i+c_i)}=2^{-\frac{p}{2M}}. \tag{3}$$

From (2) and (3) we obtain

$$1\geq M2^{-\frac{p}{2M}}$$

and equivalently

$$p\geq m2^{m+1}.$$

C5 NLD (Netherlands)

Five identical empty buckets of 2-liter capacity stand at the vertices of a regular pentagon. Cinderella and her wicked Stepmother go through a sequence of rounds: At the beginning of every round, the Stepmother takes one liter of water from the nearby river and distributes it arbitrarily over the five buckets. Then Cinderella chooses a pair of neighboring buckets, empties them into the river, and puts them back. Then the next round begins. The Stepmother's goal is to make one of these buckets overflow. Cinderella's goal is to prevent this. Can the wicked Stepmother enforce a bucket overflow?

Solution 1. No, the Stepmother cannot enforce a bucket overflow and Cinderella can keep playing forever. Throughout we denote the five buckets by B_0, B_1, B_2, B_3, and B_4, where B_k is adjacent to bucket B_{k-1} and B_{k+1} $(k = 0, 1, 2, 3, 4)$ and all indices are taken modulo 5. Cinderella enforces that the following three conditions are satisfied at the beginning of every round:

(1) Two adjacent buckets (say B_1 and B_2) are empty.

(2) The two buckets standing next to these adjacent buckets (here B_0 and B_3) have total contents at most 1.

(3) The remaining bucket (here B_4) has contents at most 1.

These conditions clearly hold at the beginning of the first round, when all buckets are empty.

Assume that Cinderella manages to maintain them until the beginning of the r-th round ($r \geq 1$). Denote by x_k $(k = 0, 1, 2, 3, 4)$ the contents of bucket B_k at the beginning of this round and by y_k the corresponding contents after the Stepmother has distributed her liter of water in this round.

By the conditions, we can assume $x_1 = x_2 = 0$, $x_0 + x_3 \leq 1$ and $x_4 \leq 1$. Then, since the Stepmother adds one liter, we conclude $y_0 + y_1 + y_2 + y_3 \leq 2$. This inequality implies $y_0 + y_2 \leq 1$ or $y_1 + y_3 \leq 1$. For reasons of symmetry, we only consider the second case.

Then Cinderella empties buckets B_0 and B_4.

At the beginning of the next round B_0 and B_4 are empty (condition (1) is fulfilled), due to $y_1 + y_3 \leq 1$ condition (2) is fulfilled and finally since $x_2 = 0$ we also must have $y_2 \leq 1$ (condition (3) is fulfilled).

Therefore, Cinderella can indeed manage to maintain the three conditions (1)–(3) also at the beginning of the $(r + 1)$-th round. By induction, she thus manages to maintain them at the beginning of every round. In particular she manages to keep the contents of every single bucket at most 1 liter. Therefore, the buckets of 2-liter capacity will never overflow.

Solution 2. We prove that Cinderella can maintain the following two conditions and hence she can prevent the buckets from overflow:

(1′) Every two non-adjacent buckets contain a total of at most 1.

(2′) The total contents of all five buckets is at most $\frac{3}{2}$.

We use the same notations as in the first solution. The two conditions again clearly hold at the beginning. Assume that Cinderella maintained these two conditions until the beginning of the r-th round. A pair of non-neighboring buckets (B_i, B_{i+2}), $i = 0, 1, 2, 3, 4$ is called *critical* if $y_i + y_{i+2} > 1$. By condition (2′), after the Stepmother has distributed her water we have $y_0 + y_1 + y_2 + y_3 + y_4 \le \frac{5}{2}$. Therefore,

$$(y_0 + y_2) + (y_1 + y_3) + (y_2 + y_4) + (y_3 + y_0) + (y_4 + y_1) = 2(y_0 + y_1 + y_2 + y_3 + y_4) \le 5,$$

and hence there is a pair of non-neighboring buckets which is not critical, say (B_0, B_2). Now, if both of the pairs (B_3, B_0) and (B_2, B_4) are critical, we must have $y_1 < \frac{1}{2}$ and Cinderella can empty the buckets B_3 and B_4. This clearly leaves no critical pair of buckets and the total contents of all the buckets is then $y_1 + (y_0 + y_2) \le \frac{3}{2}$. Therefore, conditions (1′) and (2′) are fulfilled.

Now suppose that without loss of generality the pair (B_3, B_0) is not critical. If in this case $y_0 \le \frac{1}{2}$, then one of the inequalities $y_0 + y_1 + y_2 \le \frac{3}{2}$ and $y_0 + y_3 + y_4 \le \frac{3}{2}$ must hold. But then Cinderella can empty B_3 and B_4 or B_1 and B_2, respectively and clearly fulfill the conditions.

Finally consider the case $y_0 > \frac{1}{2}$. By $y_0 + y_1 + y_2 + y_3 + y_4 \le \frac{5}{2}$, at least one of the pairs (B_1, B_3) and (B_2, B_4) is not critical. Without loss of generality let this be the pair (B_1, B_3). Since the pair (B_3, B_0) is not critical and $y_0 > \frac{1}{2}$, we must have $y_3 \le \frac{1}{2}$. But then, as before, Cinderella can maintain the two conditions at the beginning of the next round by either emptying B_1 and B_2 or B_4 and B_0.

Comments on GREEDY approaches. A natural approach for Cinderella would be a GREEDY strategy as for example: *Always remove as much water as possible from the system.* It is straightforward to prove that GREEDY can avoid buckets of capacity $\frac{5}{2}$ from overflowing: If before the Stepmothers move one has $x_0 + x_1 + x_2 + x_3 + x_4 \le \frac{3}{2}$ then after her move the inequality $Y = y_0 + y_1 + y_2 + y_3 + y_4 \le \frac{5}{2}$ holds. If now Cinderella removes the two adjacent buckets with maximum total contents she removes at least $\frac{2Y}{5}$ and thus the remaining buckets contain at most $\frac{3}{5} \cdot Y \le \frac{3}{2}$.

But GREEDY is in general not strong enough to settle this problem as can be seen in the following example:

- In an initial phase, the Stepmother brings all the buckets (after her move) to contents of at least $\frac{1}{2} - 2\epsilon$, where ϵ is an arbitrary small positive number. This can be done by always splitting the 1 liter she has to distribute so that all buckets have the same contents. After her r-th move the total contents of each of the buckets is then c_r with $c_1 = 1$ and $c_{r+1} = 1 + \frac{3}{5} \cdot c_r$ and hence $c_r = \frac{5}{2} - \frac{3}{2} \cdot \left(\frac{3}{5}\right)^{r-1}$. So the contents of each single bucket indeed approaches $\frac{1}{2}$ (from below). In particular, any two adjacent buckets have total contents strictly less than 1 which enables the Stepmother to always refill the buckets that Cinderella just emptied and then distribute the remaining water evenly over all buckets.
- After that phase GREEDY faces a situation like this $(\frac{1}{2} - 2\epsilon, \frac{1}{2} - 2\epsilon, \frac{1}{2} - 2\epsilon, \frac{1}{2} - 2\epsilon, \frac{1}{2} - 2\epsilon)$ and leaves a situation of the form $(x_0, x_1, x_2, x_3, x_4) = (\frac{1}{2} - 2\epsilon, \frac{1}{2} - 2\epsilon, \frac{1}{2} - 2\epsilon, 0, 0)$.
- Then the Stepmother can add the amounts $(0, \frac{1}{4} + \epsilon, \epsilon, \frac{3}{4} - 2\epsilon, 0)$ to achieve a situation like this: $(y_0, y_1, y_2, y_3, y_4) = (\frac{1}{2} - 2\epsilon, \frac{3}{4} - \epsilon, \frac{1}{2} - \epsilon, \frac{3}{4} - 2\epsilon, 0)$.
- Now B_1 and B_2 are the adjacent buckets with the maximum total contents and thus GREEDY empties them to yield $(x_0, x_1, x_2, x_3, x_4) = (\frac{1}{2} - 2\epsilon, 0, 0, \frac{3}{4} - 2\epsilon, 0)$.
- Then the Stepmother adds $(\frac{5}{8}, 0, 0, \frac{3}{8}, 0)$, which yields $(\frac{9}{8} - 2\epsilon, 0, 0, \frac{9}{8} - 2\epsilon, 0)$.
- Now GREEDY can only empty one of the two nonempty buckets and in the next step the Stepmother adds her liter to the other bucket and brings it to $\frac{17}{8} - 2\epsilon$, i.e. an overflow.

A harder variant. Five identical empty buckets of capacity b stand at the vertices of a regular pentagon. Cinderella and her wicked Stepmother go through a sequence of rounds: At the beginning of every round, the Stepmother takes one liter of water from the nearby river and distributes it arbitrarily over the five buckets. Then Cinderella chooses a pair of neighboring buckets, empties them into the river, and puts them back. Then the next round begins. The Stepmother's goal is to make one of these buckets overflow. Cinderella's goal is to prevent this. Determine all bucket capacities b for which the Stepmother can enforce a bucket to overflow.

Solution to the harder variant. The answer is $b < 2$.

The previous proof shows that for all $b \geq 2$ the Stepmother cannot enforce overflowing. Now if $b < 2$, let R be a positive integer such that $b < 2 - 2^{1-R}$. In the first R rounds the Stepmother now ensures that at least one of the (nonadjacent) buckets B_1 and B_3 have contents of at least $1 - 2^{1-r}$ at the beginning of round r ($r = 1, 2, \ldots, R$). This is trivial for $r = 1$ and if it holds at the beginning of round r, she can fill the bucket which contains at least $1 - 2^{1-r}$ liters with another 2^{-r} liters and put the rest of her water – $1 - 2^{-r}$ liters – in the other bucket. As Cinderella now can remove the water of at most one of the two buckets, the other bucket carries its contents into the next round.

At the beginning of the R-th round there are $1 - 2^{1-R}$ liters in B_1 or B_3. The Stepmother puts the entire liter into that bucket and produces an overflow since $b < 2 - 2^{1-R}$.

C6 BGR (Bulgaria)

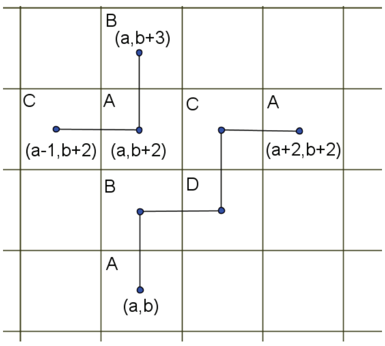
On a 999×999 board a *limp rook* can move in the following way: From any square it can move to any of its adjacent squares, i.e. a square having a common side with it, and every move must be a turn, i.e. the directions of any two consecutive moves must be perpendicular. A *non-intersecting route* of the limp rook consists of a sequence of pairwise different squares that the limp rook can visit in that order by an admissible sequence of moves. Such a non-intersecting route is called *cyclic*, if the limp rook can, after reaching the last square of the route, move directly to the first square of the route and start over.

How many squares does the longest possible cyclic, non-intersecting route of a limp rook visit?

Solution. The answer is $998^2 - 4 = 4 \cdot (499^2 - 1)$ squares.

First we show that this number is an upper bound for the number of cells a limp rook can visit. To do this we color the cells with four colors A, B, C and D in the following way: for $(i,j) \equiv (0,0) \mod 2$ use A, for $(i,j) \equiv (0,1) \mod 2$ use B, for $(i,j) \equiv (1,0) \mod 2$ use C and for $(i,j) \equiv (1,1) \mod 2$ use D. From an A-cell the rook has to move to a B-cell or a C-cell. In the first case, the order of the colors of the cells visited is given by $A, B, D, C, A, B, D, C, A, \ldots$, in the second case it is $A, C, D, B, A, C, D, B, A, \ldots$. Since the route is closed it must contain the same number of cells of each color. There are only 499^2 A-cells. In the following we will show that the rook cannot visit all the A-cells on its route and hence the maximum possible number of cells in a route is $4 \cdot (499^2 - 1)$.

Assume that the route passes through every single A-cell. Color the A-cells in black and white in a chessboard manner, i.e. color any two A-cells at distance 2 in different color. Since the number of A-cells is odd the rook cannot always alternate between visiting black and white A-cells along its route. Hence there are two A-cells of the same color which are four rook-steps apart that are visited directly one after the other. Let these two A-cells have row and column numbers (a, b) and $(a+2, b+2)$ respectively.



There is up to reflection only one way the rook can take from (a, b) to $(a+2, b+2)$. Let this way be $(a, b) \to (a, b+1) \to (a+1, b+1) \to (a+1, b+2) \to (a+2, b+2)$. Also let without loss of generality the color of the cell $(a, b+1)$ be B (otherwise change the roles of columns and rows).

Now consider the A-cell $(a, b+2)$. The only way the rook can pass through it is via $(a-1, b+2) \to (a, b+2) \to (a, b+3)$ in this order, since according to our assumption after every A-cell the rook passes through a B-cell. Hence, to connect these two parts of the path, there must be a path connecting the cell $(a, b+3)$ and (a, b) and also a path connecting $(a+2, b+2)$ and $(a-1, b+2)$.

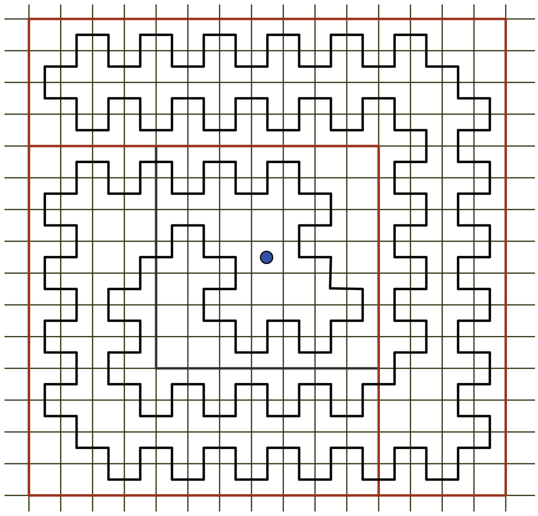
But these four cells are opposite vertices of a convex quadrilateral and the paths are outside of that quadrilateral and hence they must intersect. This is due to the following fact:

The path from (a, b) to $(a, b+3)$ together with the line segment joining these two cells form a closed loop that has one of the cells $(a-1, b+2)$ and $(a+2, b+2)$ in its inside and the other one on the outside. Thus the path between these two points must cross the previous path.

But an intersection is only possible if a cell is visited twice. This is a contradiction.

Hence the number of cells visited is at most $4 \cdot (499^2 - 1)$.

The following picture indicates a recursive construction for all $n \times n$-chessboards with $n \equiv 3 \mod 4$ which clearly yields a path that misses exactly one A-cell (marked with a dot, the center cell of the 15×15-chessboard) and hence, in the case of $n = 999$ crosses exactly $4 \cdot (499^2 - 1)$ cells.



C7 RUS (Russian Federation)

Variant 1. A grasshopper jumps along the real axis. He starts at point 0 and makes 2009 jumps to the right with lengths $1, 2, \ldots, 2009$ in an arbitrary order. Let M be a set of 2008 positive integers less than $1005 \cdot 2009$. Prove that the grasshopper can arrange his jumps in such a way that he never lands on a point from M.

Variant 2. Let n be a nonnegative integer. A grasshopper jumps along the real axis. He starts at point 0 and makes $n+1$ jumps to the right with pairwise different positive integral lengths $a_1, a_2, \ldots, a_{n+1}$ in an arbitrary order. Let M be a set of n positive integers in the interval $(0, s)$, where $s = a_1 + a_2 + \cdots + a_{n+1}$. Prove that the grasshopper can arrange his jumps in such a way that he never lands on a point from M.

Solution of Variant 1. We construct the set of landing points of the grasshopper.

Case 1. M does not contain numbers divisible by 2009.
We fix the numbers $2009k$ as landing points, $k = 1, 2, \ldots, 1005$. Consider the open intervals $I_k = (2009(k-1), 2009k)$, $k = 1, 2, \ldots, 1005$. We show that we can choose exactly one point outside of M as a landing point in 1004 of these intervals such that all lengths from 1 to 2009 are realized. Since there remains one interval without a chosen point, the length 2009 indeed will appear. Each interval has length 2009, hence a new landing point in an interval yields with a length d also the length $2009 - d$. Thus it is enough to implement only the lengths from $D = \{1, 2, \ldots, 1004\}$. We will do this in a greedy way. Let n_k, $k = 1, 2, \ldots, 1005$, be the number of elements of M that belong to the interval I_k. We order these numbers in a decreasing way, so let $p_1, p_2, \ldots, p_{1005}$ be a permutation of $\{1, 2, \ldots, 1005\}$ such that $n_{p_1} \geq n_{p_2} \geq \cdots \geq n_{p_{1005}}$. In I_{p_1} we do not choose a landing point. Assume that landing points have already been chosen in the intervals $I_{p_2}, \ldots, I_{p_m}$ and the lengths $d_2, \ldots, d_m$ from D are realized, $m = 1, \ldots, 1004$. We show that there is some $d \in D \setminus \{d_2, \ldots, d_m\}$ that can be implemented with a new landing point in $I_{p_{m+1}}$. Assume the contrary. Then the $1004 - (m-1)$ other lengths are obstructed by the $n_{p_{m+1}}$ points of M in $I_{p_{m+1}}$. Each length d can be realized by two landing points, namely $2009(p_{m+1} - 1) + d$ and $2009p_{m+1} - d$, hence

$$n_{p_{m+1}} \geq 2(1005 - m). \tag{1}$$

Moreover, since $|M| = 2008 = n_1 + \cdots + n_{1005}$,

$$2008 \geq n_{p_1} + n_{p_2} + \cdots + n_{p_{m+1}} \geq (m+1)n_{p_{m+1}}. \tag{2}$$

Consequently, by (1) and (2),

$$2008 \geq 2(m+1)(1005 - m).$$

The right hand side of the last inequality obviously attains its minimum for $m = 1004$ and this minimum value is greater than 2008, a contradiction.

Case 2. M does contain a number μ divisible by 2009.
By the pigeonhole principle there exists some $r \in \{1, \ldots, 2008\}$ such that M does not contain numbers with remainder r modulo 2009. We fix the numbers $2009(k-1) + r$ as landing points, $k = 1, 2, \ldots, 1005$. Moreover, $1005 \cdot 2009$ is a landing point. Consider the open intervals

$I_k = (2009(k-1)+r, 2009k+r)$, $k = 1,2,\ldots,1004$. Analogously to Case 1, it is enough to show that we can choose in 1003 of these intervals exactly one landing point outside of $M \setminus \{\mu\}$ such that each of the lengths of $D = \{1,2,\ldots,1004\} \setminus \{r\}$ are implemented. Note that r and $2009-r$ are realized by the first and last jump and that choosing μ would realize these two differences again. Let n_k, $k = 1,2,\ldots,1004$, be the number of elements of $M \setminus \{\mu\}$ that belong to the interval I_k and $p_1, p_2, \ldots, p_{1004}$ be a permutation of $\{1,2,\ldots,1004\}$ such that $n_{p_1} \geq n_{p_2} \geq \cdots \geq n_{p_{1004}}$. With the same reasoning as in Case 1 we can verify that a greedy choice of the landing points in $I_{p_2}, I_{p_3}, \ldots, I_{p_{1004}}$ is possible. We only have to replace (1) by

$$n_{p_{m+1}} \geq 2(1004-m)$$

(D has one element less) and (2) by

$$2007 \geq n_{p_1} + n_{p_2} + \cdots + n_{p_{m+1}} \geq (m+1)n_{p_{m+1}}.$$

Comment. The cardinality 2008 of M in the problem is the maximum possible value. For $M = \{1,2,\ldots,2009\}$, the grasshopper necessarily lands on a point from M.

Solution of Variant 2. First of all we remark that the statement in the problem implies a strengthening of itself: Instead of $|M| = n$ it is sufficient to suppose that $|M \cap (0, s-\overline{a}]| \leq n$, where $\overline{a} = \min\{a_1, a_2, \ldots, a_{n+1}\}$. This fact will be used in the proof.

We prove the statement by induction on n. The case $n = 0$ is obvious. Let $n > 0$ and let the assertion be true for all nonnegative integers less than n. Moreover let $a_1, a_2, \ldots, a_{n+1}, s$ and M be given as in the problem. Without loss of generality we may assume that $a_{n+1} < a_n < \cdots < a_2 < a_1$. Set

$$T_k = \sum_{i=1}^{k} a_i \quad \text{for } k = 0,1,\ldots,n+1.$$

Note that $0 = T_0 < T_1 < \cdots < T_{n+1} = s$. We will make use of the induction hypothesis as follows:

Claim 1. It suffices to show that for some $m \in \{1,2,\ldots,n+1\}$ the grasshopper is able to do at least m jumps without landing on a point of M and, in addition, after these m jumps he has jumped over at least m points of M.

Proof. Note that $m = n+1$ is impossible by $|M| = n$. Now set $n' = n-m$. Then $0 \leq n' < n$. The remaining $n'+1$ jumps can be carried out without landing on one of the remaining at most n' forbidden points by the induction hypothesis together with a shift of the origin. This proves the claim.

An integer $k \in \{1,2,\ldots,n+1\}$ is called *smooth*, if the grasshopper is able to do k jumps with the lengths $a_1, a_2, \ldots, a_k$ in such a way that he never lands on a point of M except for the very last jump, when he may land on a point of M.

Obviously, 1 is smooth. Thus there is a largest number k^*, such that all the numbers $1,2,\ldots,k^*$ are smooth. If $k^* = n+1$, the proof is complete. In the following let $k^* \leq n$.

Claim 2. We have

$$T_{k^*} \in M \quad \text{and} \quad |M \cap (0, T_{k^*})| \geq k^*. \tag{3}$$

Proof. In the case $T_{k^*} \notin M$ any sequence of jumps that verifies the smoothness of k^* can be extended by appending a_{k^*+1}, which is a contradiction to the maximality of k^*. Therefore we have $T_{k^*} \in M$. If $|M \cap (0, T_{k^*})| < k^*$, there exists an $l \in \{1,2,\ldots,k^*\}$ with $T_{k^*+1} - a_l \notin M$. By the induction hypothesis with k^*-1 instead of n, the grasshopper is able to reach $T_{k^*+1} - a_l$

with k^* jumps of lengths from $\{a_1, a_2, \ldots, a_{k^*+1}\} \setminus \{a_l\}$ without landing on any point of M. Therefore k^*+1 is also smooth, which is a contradiction to the maximality of k^*. Thus Claim 2 is proved.

Now, by Claim 2, there exists a smallest integer $\overline{k} \in \{1, 2, \ldots, k^*\}$ with

$$T_{\overline{k}} \in M \quad \text{and} \quad |M \cap (0, T_{\overline{k}})| \geq \overline{k}.$$

Claim 3. It is sufficient to consider the case

$$|M \cap (0, T_{\overline{k}-1}]| \leq \overline{k} - 1. \tag{4}$$

Proof. If $\overline{k} = 1$, then (4) is clearly satisfied. In the following let $\overline{k} > 1$. If $T_{\overline{k}-1} \in M$, then (4) follows immediately by the minimality of $\overline{k}$. If $T_{\overline{k}-1} \notin M$, by the smoothness of $\overline{k}-1$, we obtain a situation as in Claim 1 with $m = \overline{k}-1$ provided that $|M \cap (0, T_{\overline{k}-1}]| \geq \overline{k}-1$. Hence, we may even restrict ourselves to $|M \cap (0, T_{\overline{k}-1}]| \leq \overline{k}-2$ in this case and Claim 3 is proved.

Choose an integer $v \geq 0$ with $|M \cap (0, T_{\overline{k}})| = \overline{k} + v$. Let $r_1 > r_2 > \cdots > r_l$ be exactly those indices r from $\{\overline{k}+1, \overline{k}+2, \ldots, n+1\}$ for which $T_{\overline{k}} + a_r \notin M$. Then

$$n = |M| = |M \cap (0, T_{\overline{k}})| + 1 + |M \cap (T_{\overline{k}}, s)| \geq \overline{k} + v + 1 + (n + 1 - \overline{k} - l)$$

and consequently $l \geq v + 2$. Note that

$$T_{\overline{k}} + a_{r_1} - a_1 < T_{\overline{k}} + a_{r_1} - a_2 < \cdots < T_{\overline{k}} + a_{r_1} - a_{\overline{k}} < T_{\overline{k}} + a_{r_2} - a_{\overline{k}} < \cdots < T_{\overline{k}} + a_{r_{v+2}} - a_{\overline{k}} < T_{\overline{k}}$$

and that this are $\overline{k} + v + 1$ numbers from $(0, T_{\overline{k}})$. Therefore we find some $r \in \{\overline{k}+1, \overline{k}+2, \ldots, n+1\}$ and some $s \in \{1, 2, \ldots, \overline{k}\}$ with $T_{\overline{k}} + a_r \notin M$ and $T_{\overline{k}} + a_r - a_s \notin M$. Consider the set of jump lengths $B = \{a_1, a_2, \ldots, a_{\overline{k}}, a_r\} \setminus \{a_s\}$. We have

$$\sum_{x \in B} x = T_{\overline{k}} + a_r - a_s$$

and

$$T_{\overline{k}} + a_r - a_s - \min(B) = T_{\overline{k}} - a_s \leq T_{\overline{k}-1}.$$

By (4) and the strengthening, mentioned at the very beginning with $\overline{k}-1$ instead of n, the grasshopper is able to reach $T_{\overline{k}} + a_r - a_s$ by $\overline{k}$ jumps with lengths from B without landing on any point of M. From there he is able to jump to $T_{\overline{k}} + a_r$ and therefore we reach a situation as in Claim 1 with $m = \overline{k} + 1$, which completes the proof.

C8 AUT (Austria)

For any integer $n \geq 2$, we compute the integer $h(n)$ by applying the following procedure to its decimal representation. Let r be the rightmost digit of n.

(1) If $r = 0$, then the decimal representation of $h(n)$ results from the decimal representation of n by removing this rightmost digit 0.

(2) If $1 \leq r \leq 9$ we split the decimal representation of n into a maximal right part R that solely consists of digits not less than r and into a left part L that either is empty or ends with a digit strictly smaller than r. Then the decimal representation of $h(n)$ consists of the decimal representation of L, followed by two copies of the decimal representation of $R - 1$. For instance, for the number $n = 17{,}151{,}345{,}543$, we will have $L = 17{,}151$, $R = 345{,}543$ and $h(n) = 17{,}151{,}345{,}542{,}345{,}542$.

Prove that, starting with an arbitrary integer $n \geq 2$, iterated application of h produces the integer 1 after finitely many steps.

Solution 1. We identify integers $n \geq 2$ with the digit-strings, briefly strings, of their decimal representation and extend the definition of h to all non-empty strings with digits from 0 to 9. We recursively define ten functions $f_0, \ldots, f_9$ that map some strings into integers for $k = 9, 8, \ldots, 1, 0$. The function f_9 is only defined on strings x (including the empty string ε) that entirely consist of nines. If x consists of m nines, then $f_9(x) = m + 1$, $m = 0, 1, \ldots$. For $k \leq 8$, the domain of $f_k(x)$ is the set of all strings consisting only of digits that are $\geq k$. We write x in the form $x_0 k x_1 k x_2 k \ldots x_{m-1} k x_m$ where the strings x_s only consist of digits $\geq k + 1$. Note that some of these strings might equal the empty string ε and that $m = 0$ is possible, i.e. the digit k does not appear in x. Then we define

$$f_k(x) = \sum_{s=0}^{m} 4^{f_{k+1}(x_s)}.$$

We will use the following obvious fact:

Fact 1. If x does not contain digits smaller than k, then $f_i(x) = 4^{f_{i+1}(x)}$ for all $i = 0, \ldots, k-1$. In particular, $f_i(\varepsilon) = 4^{9-i}$ for all $i = 0, 1, \ldots, 9$.

Moreover, by induction on $k = 9, 8, \ldots, 0$ it follows easily:

Fact 2. If the nonempty string x does not contain digits smaller than k, then $f_i(x) > f_i(\varepsilon)$ for all $i = 0, \ldots, k$.

We will show the essential fact:

Fact 3. $f_0(n) > f_0(h(n))$.

Then the empty string will necessarily be reached after a finite number of applications of h. But starting from a string without leading zeros, ε can only be reached via the strings $1 \to 00 \to 0 \to \varepsilon$. Hence also the number 1 will appear after a finite number of applications of h.

Proof of Fact 3. If the last digit r of n is 0, then we write $n = x_0 0 \ldots 0 x_{m-1} 0 \varepsilon$ where the x_i do not contain the digit 0. Then $h(n) = x_0 0 \ldots 0 x_{m-1}$ and $f_0(n) - f_0(h(n)) = f_0(\varepsilon) > 0$.

So let the last digit r of n be at least 1. Let $L = yk$ and $R = zr$ be the corresponding left and right parts where y is some string, $k \leq r - 1$ and the string z consists only of digits not less than r. Then $n = ykzr$ and $h(n) = ykz(r-1)z(r-1)$. Let $d(y)$ be the smallest digit of y. We consider two cases which do not exclude each other.

Case 1. $d(y) \geq k$.
Then

$$f_k(n) - f_k(h(n)) = f_k(zr) - f_k(z(r-1)z(r-1)).$$

In view of Fact 1 this difference is positive if and only if

$$f_{r-1}(zr) - f_{r-1}(z(r-1)z(r-1)) > 0.$$

We have, using Fact 2,

$$f_{r-1}(zr) = 4^{f_r(zr)} = 4^{f_r(z)+4^{f_{r+1}(\varepsilon)}} \geq 4 \cdot 4^{f_r(z)} > 4^{f_r(z)} + 4^{f_r(z)} + 4^{f_r(\varepsilon)} = f_{r-1}(z(r-1)z(r-1)).$$

Here we use the additional definition $f_{10}(\varepsilon) = 0$ if $r = 9$. Consequently, $f_k(n) - f_k(h(n)) > 0$ and according to Fact 1, $f_0(n) - f_0(h(n)) > 0$.

Case 2. $d(y) \leq k$.
We prove by induction on $d(y) = k, k-1, \ldots, 0$ that $f_i(n) - f_i(h(n)) > 0$ for all $i = 0, \ldots, d(y)$. By Fact 1, it suffices to do so for $i = d(y)$. The initialization $d(y) = k$ was already treated in Case 1. Let $t = d(y) < k$. Write y in the form utv where v does not contain digits $\leq t$. Then, in view of the induction hypothesis,

$$f_t(n) - f_t(h(n)) = f_t(vkzr) - f_t(vkz(r-1)z(r-1)) = 4^{f_{t+1}(vkzr)} - 4^{f_{t+1}(vkz(r-1)z(r-1))} > 0.$$

Thus the inequality $f_{d(y)}(n) - f_{d(y)}(h(n)) > 0$ is established and from Fact 1 it follows that $f_0(n) - f_0(h(n)) > 0$.

Solution 2. We identify integers $n \geq 2$ with the digit-strings, briefly strings, of their decimal representation and extend the definition of h to all non-empty strings with digits from 0 to 9. Moreover, let us define that the empty string, ε, is being mapped to the empty string. In the following all functions map the set of strings into the set of strings. For two functions f and g let $g \circ f$ be defined by $(g \circ f)(x) = g(f(x))$ for all strings x and let, for non-negative integers n, f^n denote the n-fold application of f. For any string x let $s(x)$ be the smallest digit of x, and for the empty string let $s(\varepsilon) = \infty$. We define nine functions $g_1, \ldots, g_9$ as follows: Let $k \in \{1, \ldots, 9\}$ and let x be a string. If $x = \varepsilon$ then $g_k(x) = \varepsilon$. Otherwise, write x in the form $x = yzr$ where y is either the empty string or ends with a digit smaller than k, $s(z) \geq k$ and r is the rightmost digit of x. Then $g_k(x) = zr$.

Lemma 1. We have $g_k \circ h = g_k \circ h \circ g_k$ for all $k = 1, \ldots, 9$.

Proof of Lemma 1. Let $x = yzr$ be as in the definition of g_k. If $y = \varepsilon$, then $g_k(x) = x$, whence

$$g_k(h(x)) = g_k(h(g_k(x))). \tag{1}$$

So let $y \neq \varepsilon$.

Case 1. z contains a digit smaller than r.
Let $z = uav$ where $a < r$ and $s(v) \geq r$. Then

$$h(x) = \begin{cases} yuav & \text{if } r = 0, \\ yuav(r-1)v(r-1) & \text{if } r > 0 \end{cases}$$

and

$$h(g_k(x)) = h(zr) = h(uavr) = \begin{cases} uav & \text{if } r = 0, \\ uav(r-1)v(r-1) & \text{if } r > 0. \end{cases}$$

Since y ends with a digit smaller than k, (1) is obviously true.

Case 2. z does not contain a digit smaller than r.
Let $y = uv$ where u is either the empty string or ends with a digit smaller than r and $s(v) \geq r$. We have

$$h(x) = \begin{cases} uvz & \text{if } r = 0, \\ uvz(r-1)vz(r-1) & \text{if } r > 0 \end{cases}$$

and

$$h(g_k(x)) = h(zr) = \begin{cases} z & \text{if } r = 0, \\ z(r-1)z(r-1) & \text{if } r > 0. \end{cases}$$

Recall that y and hence v ends with a digit smaller than k, but all digits of v are at least r. Now if $r > k$, then $v = \varepsilon$, whence the terminal digit of u is smaller than k, which entails

$$g_k(h(x)) = z(r-1)z(r-1) = g_k(h(g_k(x))).$$

If $r \leq k$, then

$$g_k(h(x)) = z(r-1) = g_k(h(g_k(x))) ,$$

so that in both cases (1) is true. Thus Lemma 1 is proved.

Lemma 2. Let $k \in \{1, \ldots, 9\}$, let x be a non-empty string and let n be a positive integer. If $h^n(x) = \varepsilon$ then $(g_k \circ h)^n(x) = \varepsilon$.

Proof of Lemma 2. We proceed by induction on n. If $n = 1$ we have

$$\varepsilon = h(x) = g_k(h(x)) = (g_k \circ h)(x).$$

Now consider the step from $n-1$ to n where $n \geq 2$. Let $h^n(x) = \varepsilon$ and let $y = h(x)$. Then $h^{n-1}(y) = \varepsilon$ and by the induction hypothesis $(g_k \circ h)^{n-1}(y) = \varepsilon$. In view of Lemma 1,

$$\begin{aligned} \varepsilon = (g_k \circ h)^{n-2}((g_k \circ h)(y)) &= (g_k \circ h)^{n-2}(g_k(h(y)) \\ &= (g_k \circ h)^{n-2}(g_k(h(g_k(y))) = (g_k \circ h)^{n-2}(g_k(h(g_k(h(x)))) = (g_k \circ h)^n(x). \end{aligned}$$

Thus the induction step is complete and Lemma 2 is proved.

We say that the non-empty string x *terminates* if $h^n(x) = \varepsilon$ for some non-negative integer n.

Lemma 3. Let $x = yzr$ where $s(y) \geq k$, $s(z) \geq k$, y ends with the digit k and z is possibly empty. If y and zr terminate then also x terminates.

Proof of Lemma 3. Suppose that y and zr terminate. We proceed by induction on k. Let $k = 0$. Obviously, $h(yw) = yh(w)$ for any non-empty string w. Let $h^n(zr) = \epsilon$. It follows easily by induction on m that $h^m(yzr) = yh^m(zr)$ for $m = 1, \ldots, n$. Consequently, $h^n(yzr) = y$. Since y terminates, also $x = yzr$ terminates.

Now let the assertion be true for all nonnegative integers less than k and let us prove it for k where $k \geq 1$. It turns out that it is sufficient to prove that $yg_k(h(zr))$ terminates. Indeed:

Case 1. $r = 0$.
Then $h(yzr) = yz = yg_k(h(zr))$.

Case 2. $0 < r \leq k$.
We have $h(zr) = z(r-1)z(r-1)$ and $g_k(h(zr)) = z(r-1)$. Then $h(yzr) = yz(r-1)yz(r-$

$1) = yg_k(h(zr))yg_k(h(zr))$ and we may apply the induction hypothesis to see that if $yg_kh(zr))$ terminates, then $h(yzr)$ terminates.

Case 3. $r > k$.
Then $h(yzr) = yh(zr) = yg_k(h(zr))$.

Note that $yg_k(h(zr))$ has the form $yz'r'$ where $s(z') \geq k$. By the same arguments it is sufficient to prove that $yg_k(h(z'r')) = y(g_k \circ h)^2(zr)$ terminates and, by induction, that $y(g_k \circ h)^m(zr)$ terminates for some positive integer m. In view of Lemma 2 there is some m such that $(g_k \circ h)^m(zr) = \epsilon$, so $x = yzr$ terminates if y terminates. Thus Lemma 3 is proved.

Now assume that there is some string x that does not terminate. We choose x minimal. If $x \geq 10$, we can write x in the form $x = yzr$ of Lemma 3 and by this lemma x terminates since y and zr are smaller than x. If $x \leq 9$, then $h(x) = (x-1)(x-1)$ and $h(x)$ terminates again by Lemma 3 and the minimal choice of x.

Solution 3. We commence by introducing some terminology. Instead of integers, we will consider the set S of all strings consisting of the digits $0, 1, \ldots, 9$, including the empty string ϵ. If $(a_1, a_2, \ldots, a_n)$ is a nonempty string, we let $\rho(a) = a_n$ denote the *terminal digit of* a and $\lambda(a)$ be the string with the last digit removed. We also define $\lambda(\epsilon) = \epsilon$ and denote the set of non-negative integers by $\mathbb{N}_0$.

Now let $k \in \{0, 1, 2, \ldots, 9\}$ denote any digit. We define a function $f_k : S \longrightarrow S$ on the set of strings: First, if the terminal digit of n belongs to $\{0, 1, \ldots, k\}$, then $f_k(n)$ is obtained from n by deleting this terminal digit, i.e $f_k(n) = \lambda(n)$. Secondly, if the terminal digit of n belongs to $\{k+1, \ldots, 9\}$, then $f_k(n)$ is obtained from n by the process described in the problem. We also define $f_k(\epsilon) = \epsilon$. Note that up to the definition for integers $n \leq 1$, the function f_0 coincides with the function h in the problem, through interpreting integers as digit strings. The argument will be roughly as follows. We begin by introducing a straightforward generalization of our claim about f_0. Then it will be easy to see that f_9 has all these stronger properties, which means that is suffices to show for $k \in \{0, 1, \ldots, 8\}$ that f_k possesses these properties provided that f_{k+1} does.

We continue to use k to denote any digit. The operation f_k is said to be *separating*, if the followings holds: Whenever a is an initial segment of b, there is some $N \in \mathbb{N}_0$ such that $f_k^N(b) = a$. The following two notions only apply to the case where f_k is indeed separating, otherwise they remain undefined. For every $a \in S$ we denote the least $N \in \mathbb{N}_0$ for which $f_k^N(a) = \epsilon$ occurs by $g_k(a)$ (because ϵ is an initial segment of a, such an N exists if f_k is separating). If for every two strings a and b such that a is a terminal segment of b one has $g_k(a) \leq g_k(b)$, we say that f_k is *coherent*. In case that f_k is separating and coherent we call the digit k *seductive*.

As $f_9(a) = \lambda(a)$ for all a, it is obvious that 9 is seductive. Hence in order to show that 0 is seductive, which clearly implies the statement of the problem, it suffices to take any $k \in \{0, 1, \ldots, 8\}$ such that $k+1$ is seductive and to prove that k has to be seductive as well. Note that in doing so, we have the function g_{k+1} at our disposal. We have to establish two things and we begin with

Step 1. f_k is separating.

Before embarking on the proof of this, we record a useful observation which is easily proved by induction on M.

Claim 1. For any strings A, B and any positive integer M such that $f_k^{M-1}(B) \neq \epsilon$, we have

$$f_k^M(AkB) = Akf_k^M(B).$$

Now we call a pair (a, b) of strings *wicked* provided that a is an initial segment of b, but there is no $N \in \mathbb{N}_0$ such that $f_k^N(b) = a$. We need to show that there are none, so assume that there were such pairs. Choose a wicked pair (a, b) for which $g_{k+1}(b)$ attains its minimal possible value. Obviously $b \neq \epsilon$ for any wicked pair (a, b). Let z denote the terminal digit of b. Observe that $a \neq b$, which means that a is also an initial segment of $\lambda(b)$. To facilitate the construction of the eventual contradiction, we prove

Claim 2. There cannot be an $N \in \mathbb{N}_0$ such that

$$f_k^N(b) = \lambda(b).$$

Proof of Claim 2. For suppose that such an N existed. Because $g_{k+1}(\lambda(b)) < g_{k+1}(b)$ in view of the coherency of f_{k+1}, the pair $(a, \lambda(b))$ is not wicked. But then there is some N' for which $f_k^{N'}(\lambda(b)) = a$ which entails $f_k^{N+N'}(b) = a$, contradiction. Hence Claim 2 is proved.

It follows that $z \leq k$ is impossible, for otherwise $N = 1$ violated Claim 2.

Also $z > k+1$ is impossible: Set $B = f_k(b)$. Then also $f_{k+1}(b) = B$, but $g_{k+1}(B) < g_{k+1}(b)$ and a is an initial segment of B. Thus the pair (a, B) is not wicked. Hence there is some $N \in \mathbb{N}_0$ with $a = f_k^N(B)$, which, however, entails $a = f_k^{N+1}(b)$.

We are left with the case $z = k+1$. Let L denote the left part and $R = R^*(k+1)$ the right part of b. Then we have symbolically

$$f_k(b) = LR^*kR^*k\ ,\ f_k^2(b) = LR^*kR^* \quad \text{and} \quad f_{k+1}(b) = LR^*.$$

Using that R^* is a terminal segment of LR^* and the coherency of f_{k+1}, we infer

$$g_{k+1}(R^*) \leq g_{k+1}(LR^*) < g_{k+1}(b).$$

Hence the pair (ϵ, R^*) is not wicked, so there is some minimal $M \in \mathbb{N}_0$ with $f_k^M(R^*) = \epsilon$ and by Claim 1 it follows that $f_k^{2+M}(b) = LR^*k$. Finally, we infer that $\lambda(b) = LR^* = f_k(LR^*k) = f_k^{3+M}(b)$, which yields a contradiction to Claim 2.

This final contradiction establishes that f_k is indeed separating.

Step 2. f_k is coherent.

To prepare the proof of this, we introduce some further pieces of terminology. A nonempty string $(a_1, a_2, \ldots, a_n)$ is called a *hypostasis*, if $a_n < a_i$ for all $i = 1, \ldots, n-1$. Reading an arbitrary string a backwards, we easily find a, possibly empty, sequence $(A_1, A_2, \ldots, A_m)$ of hypostases such that $\rho(A_1) \leq \rho(A_2) \leq \cdots \leq \rho(A_m)$ and, symbolically, $a = A_1A_2 \ldots A_m$. The latter sequence is referred to as the *decomposition* of a. So, for instance, $(20, 0, 9)$ is the decomposition of 2009 and the string 50 is a hypostasis. Next we explain when we say about two strings a and b that a is *injectible* into b. The definition is by induction on the length of b. Let $(B_1, B_2, \ldots, B_n)$ be the decomposition of b into hypostases. Then a is injectible into b if for the decomposition $(A_1, A_2, \ldots, A_m)$ of a there is a strictly increasing function $H : \{1, 2, \ldots, m\} \longrightarrow \{1, 2, \ldots, n\}$ satisfying

$\rho(A_i) = \rho(B_{H(i)})$ for all $i = 1, \ldots, m$;

$\lambda(A_i)$ is injectible into $\lambda(B_{H(i)})$ for all $i = 1, \ldots, m$.

If one can choose H with $H(m) = n$, then we say that a is *strongly injectible* into b. Obviously, if a is a terminal segment of b, then a is strongly injectible into b.

Claim 3. If a and b are two nonempty strings such that a is strongly injectible into b, then $\lambda(a)$ is injectible into $\lambda(b)$.

Proof of Claim 3. Let $(B_1, B_2, \ldots, B_n)$ be the decomposition of b and let $(A_1, A_2, \ldots, A_m)$ be the decomposition of a. Take a function H exemplifying that a is strongly injectible into b. Let $(C_1, C_2, \ldots, C_r)$ be the decomposition of $\lambda(A_m)$ and let $(D_1, D_2, \ldots, D_s)$ be the decomposition of $\lambda(B_n)$. Choose a strictly increasing $H' : \{1, 2, \ldots, r\} \longrightarrow \{1, 2, \ldots s\}$ witnessing that $\lambda(A_m)$ is injectible into $\lambda(B_n)$. Clearly, $(A_1, A_2, \ldots, A_{m-1}, C_1, C_2, \ldots, C_r)$ is the decomposition of $\lambda(a)$ and $(B_1, B_2, \ldots, B_{n-1}, D_1, D_2, \ldots, D_s)$ is the decomposition of $\lambda(b)$. Then the function $H'' : \{1, 2, \ldots, m+r-1\} \longrightarrow \{1, 2, \ldots, n+s-1\}$ given by $H''(i) = H(i)$ for $i = 1, 2, \ldots, m-1$ and $H''(m-1+i) = n-1+H'(i)$ for $i = 1, 2, \ldots, r$ exemplifies that $\lambda(a)$ is injectible into $\lambda(b)$, which finishes the proof of the claim.

A pair (a, b) of strings is called *aggressive* if a is injectible into b and nevertheless $g_k(a) > g_k(b)$. Observe that if f_k was incoherent, which we shall assume from now on, then such pairs existed. Now among all aggressive pairs we choose one, say (a, b), for which $g_k(b)$ attains its least possible value. Obviously $f_k(a)$ cannot be injectible into $f_k(b)$, for otherwise the pair $(f_k(a), f_k(b))$ was aggressive and contradicted our choice of (a, b). Let $(A_1, A_2, \ldots, A_m)$ and $(B_1, B_2, \ldots, B_n)$ be the decompositions of a and b and take a function $H : \{1, 2, \ldots, m\} \longrightarrow \{1, 2, \ldots, n\}$ exemplifying that a is indeed injectible into b. If we had $H(m) < n$, then a was also injectible into the number b' whose decomposition is $(B_1, B_2, \ldots, B_{n-1})$ and by separativity of f_k we obtained $g_k(b') < g_k(b)$, whence the pair (a, b') was also aggressive, contrary to the minimality condition imposed on b. Therefore a is strongly injectible into b. In particular, a and b have a common terminal digit, say z. If we had $z \leq k$, then $f_k(a) = \lambda(a)$ and $f_k(b) = \lambda(b)$, so that by Claim 3, $f_k(a)$ was injectible into $f_k(b)$, which is a contradiction. Hence, $z \geq k+1$.

Now let r be the minimal element of $\{1, 2, \ldots, m\}$ for which $\rho(A_r) = z$. Then the maximal right part of a consisting of digits $\geq z$ is equal to R_a, the string whose decomposition is $(A_r, A_{r+1}, \ldots, A_m)$. Then $R_a - 1$ is a hypostasis and $(A_1, \ldots, A_{r-1}, R_a - 1, R_a - 1)$ is the decomposition of $f_k(a)$. Defining s and R_b in a similar fashion with respect to b, we see that $(B_1, \ldots, B_{s-1}, R_b - 1, R_b - 1)$ is the decomposition of $f_k(b)$. The definition of injectibility then easily entails that R_a is strongly injectible into R_b. It follows from Claim 3 that $\lambda(R_a) = \lambda(R_a - 1)$ is injectible into $\lambda(R_b) = \lambda(R_b - 1)$, whence the function $H' : \{1, 2, \ldots, r+1\} \longrightarrow \{1, 2, \ldots, s+1\}$, given by $H'(i) = H(i)$ for $i = 1, 2, \ldots, r-1$, $H'(r) = s$ and $H'(r+1) = s+1$ exemplifies that $f_k(a)$ is injectible into $f_k(b)$, which yields a contradiction as before.

This shows that aggressive pairs cannot exist, whence f_k is indeed coherent, which finishes the proof of the seductivity of k, whereby the problem is finally solved.

Geometry

G1 BEL (Belgium)

Let ABC be a triangle with $AB = AC$. The angle bisectors of A and B meet the sides BC and AC in D and E, respectively. Let K be the incenter of triangle ADC. Suppose that $\angle BEK = 45^\circ$. Find all possible values of $\angle BAC$.

Solution 1. Answer: $\angle BAC = 60^\circ$ or $\angle BAC = 90^\circ$ are possible values and the only possible values.

Let I be the incenter of triangle ABC, then K lies on the line CI. Let F be the point, where the incircle of triangle ABC touches the side AC; then the segments IF and ID have the same length and are perpendicular to AC and BC, respectively.

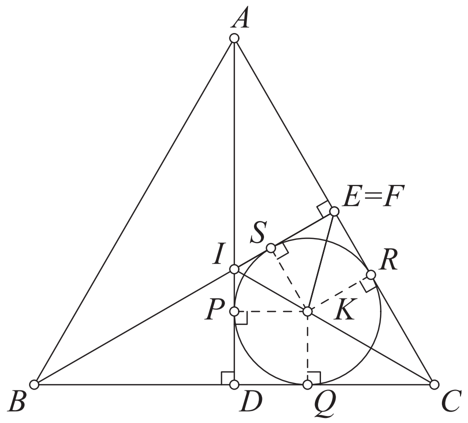


Figure 1

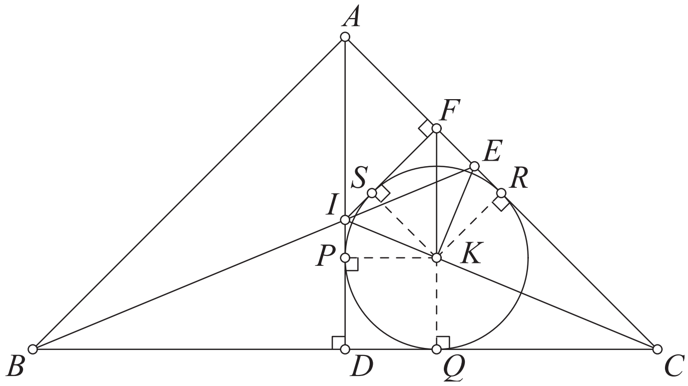


Figure 2

Let P, Q and R be the points where the incircle of triangle ADC touches the sides AD, DC and CA, respectively. Since K and I lie on the angle bisector of $\angle ACD$, the segments ID and IF are symmetric with respect to the line IC. Hence there is a point S on IF where the incircle of triangle ADC touches the segment IF. Then segments KP, KQ, KR and KS all have the same length and are perpendicular to AD, DC, CA and IF, respectively. So – regardless of the value of $\angle BEK$ – the quadrilateral $KRFS$ is a square and $\angle SFK = \angle KFC = 45^\circ$.

Consider the case $\angle BAC = 60^\circ$ (see Figure 1). Then triangle ABC is equilateral. Furthermore we have $F = E$, hence $\angle BEK = \angle IFK = \angle SEK = 45^\circ$. So 60° is a possible value for $\angle BAC$.

Now consider the case $\angle BAC = 90^\circ$ (see Figure 2). Then $\angle CBA = \angle ACB = 45^\circ$. Furthermore, $\angle KIE = \frac{1}{2}\angle CBA + \frac{1}{2}\angle ACB = 45^\circ$, $\angle AEB = 180^\circ - 90^\circ - 22.5^\circ = 67.5^\circ$ and $\angle EIA = \angle BID = 180^\circ - 90^\circ - 22.5^\circ = 67.5^\circ$. Hence triangle IEA is isosceles and a reflection of the bisector of $\angle IAE$ takes I to E and K to itself. So triangle IKE is symmetric with respect to this axis, i.e. $\angle KIE = \angle IEK = \angle BEK = 45^\circ$. So 90° is a possible value for $\angle BAC$, too.

If, on the other hand, $\angle BEK = 45^\circ$ then $\angle BEK = \angle IEK = \angle IFK = 45^\circ$. Then

- either $F = E$, which makes the angle bisector BI be an altitude, i.e., which makes triangle ABC isosceles with base AC and hence equilateral and so $\angle BAC = 60^\circ$,
- or E lies between F and C, which makes the points K, E, F and I concyclic, so $45^\circ = \angle KFC = \angle KFE = \angle KIE = \angle CBI + \angle ICB = 2 \cdot \angle ICB = 90^\circ - \frac{1}{2}\angle BAC$, and so $\angle BAC = 90^\circ$,

- or F lies between E and C, then again, K, E, F and I are concyclic, so $45^\circ = \angle KFC = 180^\circ - \angle KFE = \angle KIE$, which yields the same result $\angle BAC = 90^\circ$. (However, for $\angle BAC = 90^\circ$ E lies, in fact, between F and C, see Figure 2. So this case does not occur.)

This proves 90° and 60° to be the only possible values for $\angle BAC$.

Solution 2. Denote angles at A, B and C as usual by α, β and γ. Since triangle ABC is isosceles, we have $\beta = \gamma = 90^\circ - \frac{\alpha}{2} < 90^\circ$, so $\angle ECK = 45^\circ - \frac{\alpha}{4} = \angle KCD$. Since K is the incenter of triangle ADC, we have $\angle CDK = \angle KDA = 45^\circ$; furthermore $\angle DIC = 45^\circ + \frac{\alpha}{4}$. Now, if $\angle BEK = 45^\circ$, easy calculations within triangles BCE and KCE yield

$$\begin{aligned}
&\angle KEC = 180^\circ - \tfrac{\beta}{2} - 45^\circ - \beta = 135^\circ - \tfrac{3}{2}\beta = \tfrac{3}{2}(90^\circ - \beta) = \tfrac{3}{4}\alpha,\\
&\angle IKE = \tfrac{3}{4}\alpha + 45^\circ - \tfrac{\alpha}{4} = 45^\circ + \tfrac{\alpha}{2}.
\end{aligned}$$

So in triangles ICE, IKE, IDK and IDC we have (see Figure 3)

$$\frac{IC}{IE} = \frac{\sin\angle IEC}{\sin\angle ECI} = \frac{\sin(45^\circ + \frac{3}{4}\alpha)}{\sin(45^\circ - \frac{\alpha}{4})}, \qquad \frac{IE}{IK} = \frac{\sin\angle EKI}{\sin\angle IEK} = \frac{\sin(45^\circ + \frac{\alpha}{2})}{\sin 45^\circ},$$

$$\frac{IK}{ID} = \frac{\sin\angle KDI}{\sin\angle IKD} = \frac{\sin 45^\circ}{\sin(90^\circ - \frac{\alpha}{4})}, \qquad \frac{ID}{IC} = \frac{\sin\angle ICD}{\sin\angle CDI} = \frac{\sin(45^\circ - \frac{\alpha}{4})}{\sin 90^\circ}.$$

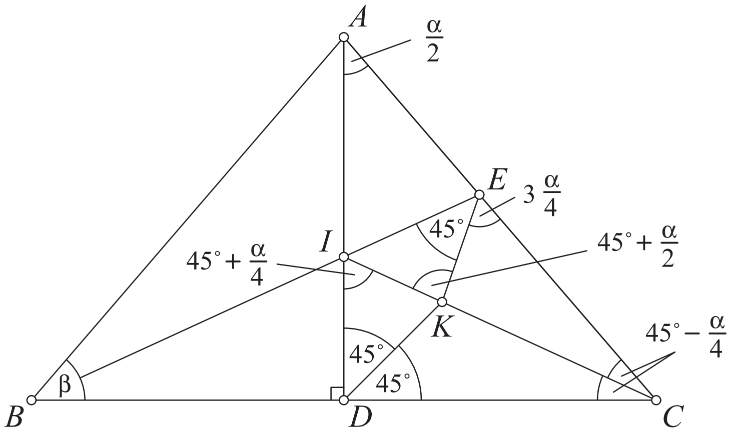


Figure 3

Multiplication of these four equations yields

$$1 = \frac{\sin(45^\circ + \frac{3}{4}\alpha)\sin(45^\circ + \frac{\alpha}{2})}{\sin(90^\circ - \frac{\alpha}{4})}.$$

But, since

$$\begin{aligned}
\sin\left(90^\circ - \tfrac{\alpha}{4}\right) &= \cos\tfrac{\alpha}{4} = \cos\left((45^\circ + \tfrac{3}{4}\alpha) - (45^\circ + \tfrac{\alpha}{2})\right)\\
&= \cos\left(45^\circ + \tfrac{3}{4}\alpha\right)\cos\left(45^\circ + \tfrac{\alpha}{2}\right) + \sin\left(45^\circ + \tfrac{3}{4}\alpha\right)\sin\left(45^\circ + \tfrac{\alpha}{2}\right),
\end{aligned}$$

this is equivalent to

$$\sin(45^\circ + \tfrac{3}{4}\alpha)\sin(45^\circ + \tfrac{\alpha}{2}) = \cos\left(45^\circ + \tfrac{3}{4}\alpha\right)\cos\left(45^\circ + \tfrac{\alpha}{2}\right) + \sin\left(45^\circ + \tfrac{3}{4}\alpha\right)\sin\left(45^\circ + \tfrac{\alpha}{2}\right)$$

and finally

$$\cos\left(45^\circ + \tfrac{3}{4}\alpha\right)\cos\left(45^\circ + \tfrac{\alpha}{2}\right) = 0.$$

But this means $\cos\left(45^\circ + \frac{3}{4}\alpha\right) = 0$, hence $45^\circ + \frac{3}{4}\alpha = 90^\circ$, i.e. $\alpha = 60^\circ$ or $\cos\left(45^\circ + \frac{\alpha}{2}\right) = 0$, hence $45^\circ + \frac{\alpha}{2} = 90^\circ$, i.e. $\alpha = 90^\circ$. So these values are the only two possible values for α.

On the other hand, both $\alpha = 90^\circ$ and $\alpha = 60^\circ$ yield $\angle BEK = 45^\circ$, this was shown in Solution 1.

G2 RUS (Russian Federation)

Let ABC be a triangle with circumcenter O. The points P and Q are interior points of the sides CA and AB, respectively. The circle k passes through the midpoints of the segments BP, CQ, and PQ. Prove that if the line PQ is tangent to circle k then $OP = OQ$.

Solution 1. Let K, L, M, B', C' be the midpoints of BP, CQ, PQ, CA, and AB, respectively (see Figure 1). Since $CA \parallel LM$, we have $\angle LMP = \angle QPA$. Since k touches the segment PQ at M, we find $\angle LMP = \angle LKM$. Thus $\angle QPA = \angle LKM$. Similarly it follows from $AB \parallel MK$ that $\angle PQA = \angle KLM$. Therefore, triangles APQ and MKL are similar, hence

$$\frac{AP}{AQ} = \frac{MK}{ML} = \frac{\frac{QB}{2}}{\frac{PC}{2}} = \frac{QB}{PC}. \tag{1}$$

Now (1) is equivalent to $AP \cdot PC = AQ \cdot QB$ which means that the power of points P and Q with respect to the circumcircle of $\triangle ABC$ are equal, hence $OP = OQ$.

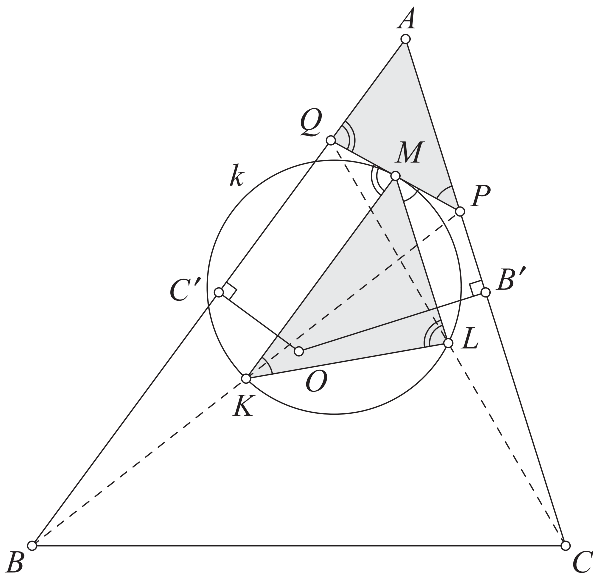


Figure 1

Comment. The last argument can also be established by the following calculation:

$$\begin{aligned} OP^2 - OQ^2 &= OB'^2 + B'P^2 - OC'^2 - C'Q^2 \\ &= (OA^2 - AB'^2) + B'P^2 - (OA^2 - AC'^2) - C'Q^2 \\ &= (AC'^2 - C'Q^2) - (AB'^2 - B'P^2) \\ &= (AC' - C'Q)(AC' + C'Q) - (AB' - B'P)(AB' + B'P) \\ &= AQ \cdot QB - AP \cdot PC. \end{aligned}$$

With (1), we conclude $OP^2 - OQ^2 = 0$, as desired.

Solution 2. Again, denote by K, L, M the midpoints of segments BP, CQ, and PQ, respectively. Let O, S, T be the circumcenters of triangles ABC, KLM, and APQ, respectively (see Figure 2). Note that MK and LM are the midlines in triangles BPQ and CPQ, respectively, so $\overrightarrow{MK} = \frac{1}{2}\overrightarrow{QB}$ and $\overrightarrow{ML} = \frac{1}{2}\overrightarrow{PC}$. Denote by $\mathrm{pr}_l(\overrightarrow{v})$ the projection of vector $\overrightarrow{v}$ onto line l. Then $\mathrm{pr}_{AB}(\overrightarrow{OT}) = \mathrm{pr}_{AB}(\overrightarrow{OA} - \overrightarrow{TA}) = \frac{1}{2}\overrightarrow{BA} - \frac{1}{2}\overrightarrow{QA} = \frac{1}{2}\overrightarrow{BQ} = \overrightarrow{KM}$ and $\mathrm{pr}_{AB}(\overrightarrow{SM}) = \mathrm{pr}_{MK}(\overrightarrow{SM}) = \frac{1}{2}\overrightarrow{KM} = \frac{1}{2}\mathrm{pr}_{AB}(\overrightarrow{OT})$. Analogously we get $\mathrm{pr}_{CA}(\overrightarrow{SM}) = \frac{1}{2}\mathrm{pr}_{CA}(\overrightarrow{OT})$. Since AB and CA are not parallel, this implies that $\overrightarrow{SM} = \frac{1}{2}\overrightarrow{OT}$.

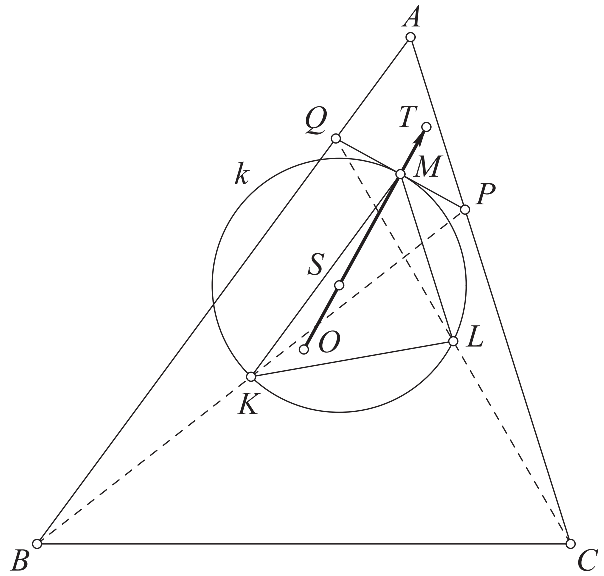


Figure 2

Now, since the circle k touches PQ at M, we get $SM \perp PQ$, hence $OT \perp PQ$. Since T is equidistant from P and Q, the line OT is a perpendicular bisector of segment PQ, and hence O is equidistant from P and Q which finishes the proof.

G3 IRN (Islamic Republic of Iran)

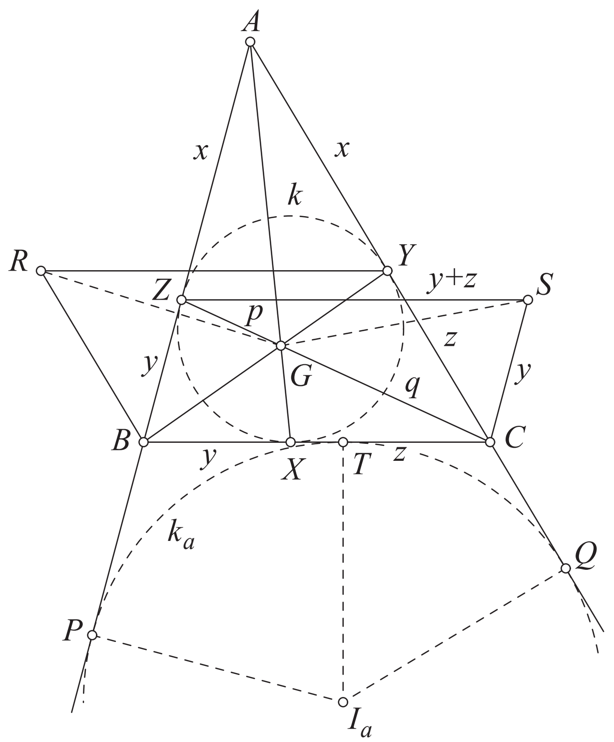
Let ABC be a triangle. The incircle of ABC touches the sides AB and AC at the points Z and Y, respectively. Let G be the point where the lines BY and CZ meet, and let R and S be points such that the two quadrilaterals $BCYR$ and $BCSZ$ are parallelograms.

Prove that $GR = GS$.

Solution 1. Denote by k the incircle and by k_a the excircle opposite to A of triangle ABC. Let k and k_a touch the side BC at the points X and T, respectively, let k_a touch the lines AB and AC at the points P and Q, respectively. We use several times the fact that opposing sides of a parallelogram are of equal length, that points of contact of the excircle and incircle to a side of a triangle lie symmetric with respect to the midpoint of this side and that segments on two tangents to a circle defined by the points of contact and their point of intersection have the same length. So we conclude

$$ZP = ZB + BP = XB + BT = BX + CX = ZS \quad \text{and}$$

$$CQ = CT = BX = BZ = CS.$$



So for each of the points Z, C, their distances to S equal the length of a tangent segment from this point to k_a. It is well-known, that all points with this property lie on the line ZC, which is the radical axis of S and k_a. Similar arguments yield that BY is the radical axis of R and k_a. So the point of intersection of ZC and BY, which is G by definition, is the radical center of R, S and k_a, from which the claim $GR = GS$ follows immediately.

Solution 2. Denote $x = AZ = AY$, $y = BZ = BX$, $z = CX = CY$, $p = ZG$, $q = GC$. Several lengthy calculations (Menelaos' theorem in triangle AZC, law of Cosines in triangles ABC and AZC and Stewart's theorem in triangle ZCS) give four equations for p, q, $\cos\alpha$

and GS in terms of x, y, and z that can be resolved for GS. The result is symmetric in y and z, so $GR = GS$. More in detail this means:

The line BY intersects the sides of triangle AZC, so MENELAOS' theorem yields $\frac{p}{q} \cdot \frac{z}{x} \cdot \frac{x+y}{y} = 1$, hence

$$\frac{p}{q} = \frac{xy}{yz + zx}. \tag{1}$$

Since we only want to show that the term for GS is symmetric in y and z, we abbreviate terms that are symmetric in y and z by capital letters, starting with $N = xy+yz+zx$. So (1) implies

$$\frac{p}{p+q} = \frac{xy}{xy + yz + zx} = \frac{xy}{N} \quad \text{and} \quad \frac{q}{p+q} = \frac{yz + zx}{xy + yz + zx} = \frac{yz + zx}{N}. \tag{2}$$

Now the law of Cosines in triangle ABC yields

$$\cos\alpha = \frac{(x+y)^2 + (x+z)^2 - (y+z)^2}{2(x+y)(x+z)} = \frac{2x^2 + 2xy + 2xz - 2yz}{2(x+y)(x+z)} = 1 - \frac{2yz}{(x+y)(x+z)}.$$

We use this result to apply the law of Cosines in triangle AZC:

$$\begin{aligned}(p+q)^2 &= x^2 + (x+z)^2 - 2x(x+z)\cos\alpha \\ &= x^2 + (x+z)^2 - 2x(x+z) \cdot \left(1 - \frac{2yz}{(x+y)(x+z)}\right) \\ &= z^2 + \frac{4xyz}{x+y}.\end{aligned} \tag{3}$$

Now in triangle ZCS the segment GS is a cevian, so with STEWART's theorem we have $py^2 + q(y+z)^2 = (p+q)(GS^2 + pq)$, hence

$$GS^2 = \frac{p}{p+q} \cdot y^2 + \frac{q}{p+q} \cdot (y+z)^2 - \frac{p}{p+q} \cdot \frac{q}{p+q} \cdot (p+q)^2.$$

Replacing the p's and q's herein by (2) and (3) yields

$$\begin{aligned}GS^2 &= \frac{xy}{N}y^2 + \frac{yz+zx}{N}(y+z)^2 - \frac{xy}{N} \cdot \frac{yz+zx}{N} \cdot \left(z^2 + \frac{4xyz}{x+y}\right) \\ &= \frac{xy^3}{N} + \underbrace{\frac{yz(y+z)^2}{N}}_{M_1} + \frac{zx(y+z)^2}{N} - \frac{xyz^3(x+y)}{N^2} - \underbrace{\frac{4x^2y^2z^2}{N^2}}_{M_2} \\ &= \frac{xy^3 + zx(y+z)^2}{N} - \frac{xyz^3(x+y)}{N^2} + M_1 - M_2 \\ &= \underbrace{\frac{x(y^3 + y^2z + yz^2 + z^3)}{N}}_{M_3} + \frac{xyz^2N}{N^2} - \frac{xyz^3(x+y)}{N^2} + M_1 - M_2 \\ &= \frac{x^2y^2z^2 + xy^2z^3 + x^2yz^3 - x^2yz^3 - xy^2z^3}{N^2} + M_1 - M_2 + M_3 \\ &= \frac{x^2y^2z^2}{N^2} + M_1 - M_2 + M_3,\end{aligned}$$

a term that is symmetric in y and z, indeed.

Comment. G is known as GERGONNE's point of $\triangle ABC$.

G4 UNK (United Kingdom)

Given a cyclic quadrilateral $ABCD$, let the diagonals AC and BD meet at E and the lines AD and BC meet at F. The midpoints of AB and CD are G and H, respectively. Show that EF is tangent at E to the circle through the points E, G, and H.

Solution 1. It suffices to show that $\angle HEF = \angle HGE$ (see Figure 1), since in circle EGH the angle over the chord EH at G equals the angle between the tangent at E and EH.

First, $\angle BAD = 180^\circ - \angle DCB = \angle FCD$. Since triangles FAB and FCD have also a common interior angle at F, they are similar.

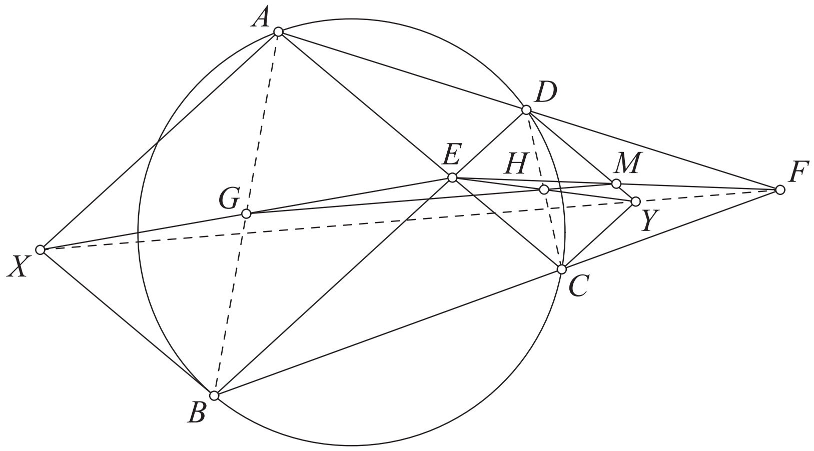


Figure 1

Denote by $\mathcal{T}$ the transformation consisting of a reflection at the bisector of $\angle DFC$ followed by a dilation with center F and factor of $\frac{FA}{FC}$. Then $\mathcal{T}$ maps F to F, C to A, D to B, and H to G. To see this, note that $\triangle FCA \sim \triangle FDB$, so $\frac{FA}{FC} = \frac{FB}{FD}$. Moreover, as $\angle ADB = \angle ACB$, the image of the line DE under $\mathcal{T}$ is parallel to AC (and passes through B) and similarly the image of CE is parallel to DB and passes through A. Hence E is mapped to the point X which is the fourth vertex of the parallelogram $BEAX$. Thus, in particular $\angle HEF = \angle FXG$.

As G is the midpoint of the diagonal AB of the parallelogram $BEAX$, it is also the midpoint of EX. In particular, E, G, X are collinear, and $EX = 2 \cdot EG$.

Denote by Y the fourth vertex of the parallelogram $DECY$. By an analogous reasoning as before, it follows that $\mathcal{T}$ maps Y to E, thus E, H, Y are collinear with $EY = 2 \cdot EH$. Therefore, by the intercept theorem, $HG \parallel XY$.

From the construction of $\mathcal{T}$ it is clear that the lines FX and FE are symmetric with respect to the bisector of $\angle DFC$, as are FY and FE. Thus, F, X, Y are collinear, which together with $HG \parallel XY$ implies $\angle FXE = \angle HGE$. This completes the proof.

Solution 2. We use the following

Lemma (Gauß). Let $ABCD$ be a quadrilateral. Let AB and CD intersect at P, and BC and DA intersect at Q. Then the midpoints K, L, M of AC, BD, and PQ, respectively, are collinear.

Proof: Let us consider the points Z that fulfill the equation

$$(ABZ) + (CDZ) = (BCZ) + (DAZ), \tag{1}$$

where (RST) denotes the oriented area of the triangle RST (see Figure 2).

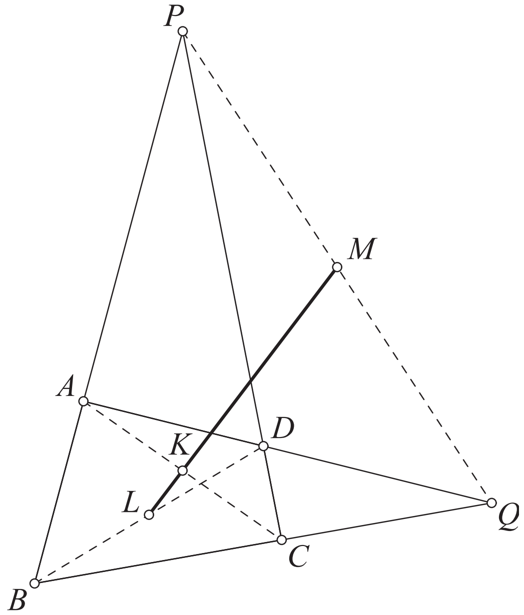


Figure 2

As (1) is linear in Z, it can either characterize a line, or be contradictory, or be trivially fulfilled for all Z in the plane. If (1) was fulfilled for all Z, then it would hold for $Z = A$, $Z = B$, which gives $(CDA) = (BCA)$, $(CDB) = (DAB)$, respectively, i.e. the diagonals of $ABCD$ would bisect each other, thus $ABCD$ would be a parallelogram. This contradicts the hypothesis that AD and BC intersect. Since E, F, G fulfill (1), it is the equation of a line which completes the proof of the lemma.

Now consider the parallelograms $EAXB$ and $ECYD$ (see Figure 1). Then G, H are the midpoints of EX, EY, respectively. Let M be the midpoint of EF. By applying the Lemma to the (re-entrant) quadrilateral $ADBC$, it is evident that G, H, and M are collinear. A dilation by a factor of 2 with center E shows that X, Y, F are collinear. Since $AX \parallel DE$ and $BX \parallel CE$, we have pairwise equal interior angles in the quadrilaterals $FDEC$ and $FBXA$. Since we have also $\angle EBA = \angle DCA = \angle CDY$, the quadrilaterals are similar. Thus, $\angle FXA = \angle CEF$.

Clearly the parallelograms $ECYD$ and $EBXA$ are similar, too, thus $\angle EXA = \angle CEY$. Consequently, $\angle FXE = \angle FXA - \angle EXA = \angle CEF - \angle CEY = \angle YEF$. By the converse of the tangent-chord angle theorem EF is tangent to the circle XEY. A dilation by a factor of $\frac{1}{2}$ completes the proof.

Solution 3. As in Solution 2, G, H, M are proven to be collinear. It suffices to show that $ME^2 = MG \cdot MH$. If $\boldsymbol{p} = \overrightarrow{OP}$ denotes the vector from circumcenter O to point P, the claim becomes

$$\left(\frac{\boldsymbol{e}-\boldsymbol{f}}{2}\right)^2 = \left(\frac{\boldsymbol{e}+\boldsymbol{f}}{2} - \frac{\boldsymbol{a}+\boldsymbol{b}}{2}\right)\left(\frac{\boldsymbol{e}+\boldsymbol{f}}{2} - \frac{\boldsymbol{c}+\boldsymbol{d}}{2}\right),$$

or equivalently

$$4\,\boldsymbol{ef} - (\boldsymbol{e}+\boldsymbol{f})(\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c}+\boldsymbol{d}) + (\boldsymbol{a}+\boldsymbol{b})(\boldsymbol{c}+\boldsymbol{d}) = 0. \tag{2}$$

With R as the circumradius of $ABCD$, we obtain for the powers $\mathcal{P}(E)$ and $\mathcal{P}(F)$ of E and F, respectively, with respect to the circumcircle

$$\mathcal{P}(E) = (\boldsymbol{e}-\boldsymbol{a})(\boldsymbol{e}-\boldsymbol{c}) = (\boldsymbol{e}-\boldsymbol{b})(\boldsymbol{e}-\boldsymbol{d}) = \boldsymbol{e}^2 - R^2,$$
$$\mathcal{P}(F) = (\boldsymbol{f}-\boldsymbol{a})(\boldsymbol{f}-\boldsymbol{d}) = (\boldsymbol{f}-\boldsymbol{b})(\boldsymbol{f}-\boldsymbol{c}) = \boldsymbol{f}^2 - R^2,$$

hence

$$(\boldsymbol{e}-\boldsymbol{a})(\boldsymbol{e}-\boldsymbol{c}) = \boldsymbol{e}^2 - R^2, \tag{3}$$
$$(\boldsymbol{e}-\boldsymbol{b})(\boldsymbol{e}-\boldsymbol{d}) = \boldsymbol{e}^2 - R^2, \tag{4}$$
$$(\boldsymbol{f}-\boldsymbol{a})(\boldsymbol{f}-\boldsymbol{d}) = \boldsymbol{f}^2 - R^2, \tag{5}$$
$$(\boldsymbol{f}-\boldsymbol{b})(\boldsymbol{f}-\boldsymbol{c}) = \boldsymbol{f}^2 - R^2. \tag{6}$$

Since F lies on the polar to E with respect to the circumcircle, we have

$$4\,\boldsymbol{ef} = 4R^2. \tag{7}$$

Adding up (3) to (7) yields (2), as desired.

G5 POL (Poland)

Let P be a polygon that is convex and symmetric to some point O. Prove that for some parallelogram R satisfying $P \subset R$ we have

$$\frac{|R|}{|P|} \leq \sqrt{2}$$

where $|R|$ and $|P|$ denote the area of the sets R and P, respectively.

Solution 1. We will construct two parallelograms R_1 and R_3, each of them containing P, and prove that at least one of the inequalities $|R_1| \leq \sqrt{2}\,|P|$ and $|R_3| \leq \sqrt{2}\,|P|$ holds (see Figure 1).

First we will construct a parallelogram $R_1 \supseteq P$ with the property that the midpoints of the sides of R_1 are points of the boundary of P.

Choose two points A and B of P such that the triangle OAB has maximal area. Let a be the line through A parallel to OB and b the line through B parallel to OA. Let A', B', a' and b' be the points or lines, that are symmetric to A, B, a and b, respectively, with respect to O. Now let R_1 be the parallelogram defined by a, b, a' and b'.

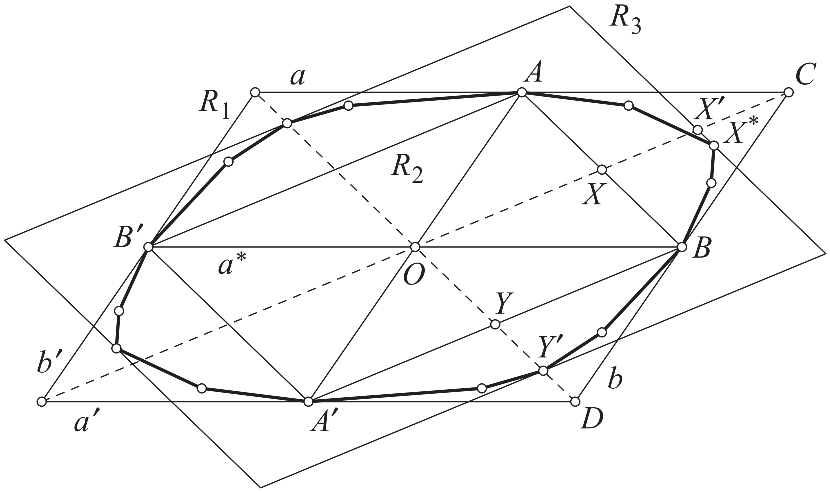


Figure 1

Obviously, A and B are located on the boundary of the polygon P, and A, B, A' and B' are midpoints of the sides of R_1. We note that $P \subseteq R_1$. Otherwise, there would be a point $Z \in P$ but $Z \notin R_1$, i.e., one of the lines a, b, a' or b' were between O and Z. If it is a, we have $|OZB| > |OAB|$, which is contradictory to the choice of A and B. If it is one of the lines b, a' or b' almost identical arguments lead to a similar contradiction.

Let R_2 be the parallelogram $ABA'B'$. Since A and B are points of P, segment $AB \subset P$ and so $R_2 \subset R_1$. Since A, B, A' and B' are midpoints of the sides of R_1, an easy argument yields

$$|R_1| = 2 \cdot |R_2|. \tag{1}$$

Let R_3 be the smallest parallelogram enclosing P defined by lines parallel to AB and BA'. Obviously $R_2 \subset R_3$ and every side of R_3 contains at least one point of the boundary of P. Denote by C the intersection point of a and b, by X the intersection point of AB and OC, and by X' the intersection point of XC and the boundary of R_3. In a similar way denote by D

the intersection point of b and a', by Y the intersection point of $A'B$ and OD, and by Y' the intersection point of YD and the boundary of R_3.

Note that $OC = 2 \cdot OX$ and $OD = 2 \cdot OY$, so there exist real numbers x and y with $1 \le x, y \le 2$ and $OX' = x \cdot OX$ and $OY' = y \cdot OY$. Corresponding sides of R_3 and R_2 are parallel which yields

$$|R_3| = xy \cdot |R_2|. \tag{2}$$

The side of R_3 containing X' contains at least one point X^* of P; due to the convexity of P we have $AX^*B \subset P$. Since this side of the parallelogram R_3 is parallel to AB we have $|AX^*B| = |AX'B|$, so $|OAX'B|$ does not exceed the area of P confined to the sector defined by the rays OB and OA. In a similar way we conclude that $|OB'Y'A'|$ does not exceed the area of P confined to the sector defined by the rays OB and OA'. Putting things together we have $|OAX'B| = x \cdot |OAB|$, $|OBDA'| = y \cdot |OBA'|$. Since $|OAB| = |OBA'|$, we conclude that $|P| \ge 2 \cdot |AX'BY'A'| = 2 \cdot (x \cdot |OAB| + y \cdot |OBA'|) = 4 \cdot \frac{x+y}{2} \cdot |OAB| = \frac{x+y}{2} \cdot R_2$; this is in short

$$\frac{x+y}{2} \cdot |R_2| \le |P|. \tag{3}$$

Since all numbers concerned are positive, we can combine (1)–(3). Using the arithmetic-geometric-mean inequality we obtain

$$|R_1| \cdot |R_3| = 2 \cdot |R_2| \cdot xy \cdot |R_2| \le 2 \cdot |R_2|^2 \left(\frac{x+y}{2}\right)^2 \le 2 \cdot |P|^2.$$

This implies immediately the desired result $|R_1| \le \sqrt{2} \cdot |P|$ or $|R_3| \le \sqrt{2} \cdot |P|$.

Solution 2. We construct the parallelograms R_1, R_2 and R_3 in the same way as in Solution 1 and will show that $\frac{|R_1|}{|P|} \le \sqrt{2}$ or $\frac{|R_3|}{|P|} \le \sqrt{2}$.

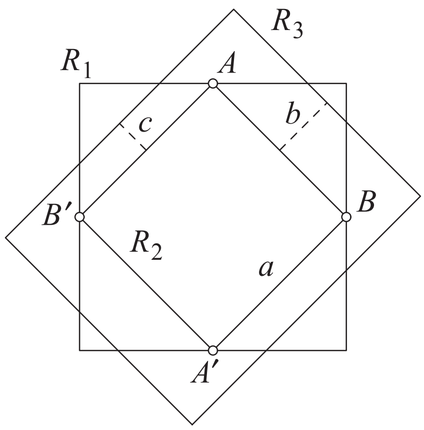


Figure 2

Recall that affine one-to-one maps of the plane preserve the ratio of areas of subsets of the plane. On the other hand, every parallelogram can be transformed with an affine map onto a square. It follows that without loss of generality we may assume that R_1 is a square (see Figure 2).

Then R_2, whose vertices are the midpoints of the sides of R_1, is a square too, and R_3, whose sides are parallel to the diagonals of R_1, is a rectangle.

Let $a > 0$, $b \ge 0$ and $c \ge 0$ be the distances introduced in Figure 2. Then $|R_1| = 2a^2$ and

$|R_3| = (a+2b)(a+2c)$.

Points A, A', B and B' are in the convex polygon P. Hence the square $ABA'B'$ is a subset of P. Moreover, each of the sides of the rectangle R_3 contains a point of P, otherwise R_3 would not be minimal. It follows that

$$|P| \geq a^2 + 2 \cdot \frac{ab}{2} + 2 \cdot \frac{ac}{2} = a(a+b+c).$$

Now assume that both $\frac{|R_1|}{|P|} > \sqrt{2}$ and $\frac{|R_3|}{|P|} > \sqrt{2}$, then

$$2a^2 = |R_1| > \sqrt{2} \cdot |P| \geq \sqrt{2} \cdot a(a+b+c)$$

and

$$(a+2b)(a+2c) = |R_3| > \sqrt{2} \cdot |P| \geq \sqrt{2} \cdot a(a+b+c).$$

All numbers concerned are positive, so after multiplying these inequalities we get

$$2a^2(a+2b)(a+2c) > 2a^2(a+b+c)^2.$$

But the arithmetic-geometric-mean inequality implies the contradictory result

$$2a^2(a+2b)(a+2c) \leq 2a^2 \left(\frac{(a+2b)+(a+2c)}{2} \right)^2 = 2a^2(a+b+c)^2.$$

Hence $\frac{|R_1|}{|P|} \leq \sqrt{2}$ or $\frac{|R_3|}{|P|} \leq \sqrt{2}$, as desired.

G6 UKR (Ukraine)

Let the sides AD and BC of the quadrilateral $ABCD$ (such that AB is not parallel to CD) intersect at point P. Points O_1 and O_2 are the circumcenters and points H_1 and H_2 are the orthocenters of triangles ABP and DCP, respectively. Denote the midpoints of segments O_1H_1 and O_2H_2 by E_1 and E_2, respectively. Prove that the perpendicular from E_1 on CD, the perpendicular from E_2 on AB and the line H_1H_2 are concurrent.

Solution 1. We keep triangle ABP fixed and move the line CD parallel to itself uniformly, i.e. linearly dependent on a single parameter λ (see Figure 1). Then the points C and D also move uniformly. Hence, the points O_2, H_2 and E_2 move uniformly, too. Therefore also the perpendicular from E_2 on AB moves uniformly. Obviously, the points O_1, H_1, E_1 and the perpendicular from E_1 on CD do not move at all. Hence, the intersection point S of these two perpendiculars moves uniformly. Since H_1 does not move, while H_2 and S move uniformly along parallel lines (both are perpendicular to CD), it is sufficient to prove their collinearity for two different positions of CD.

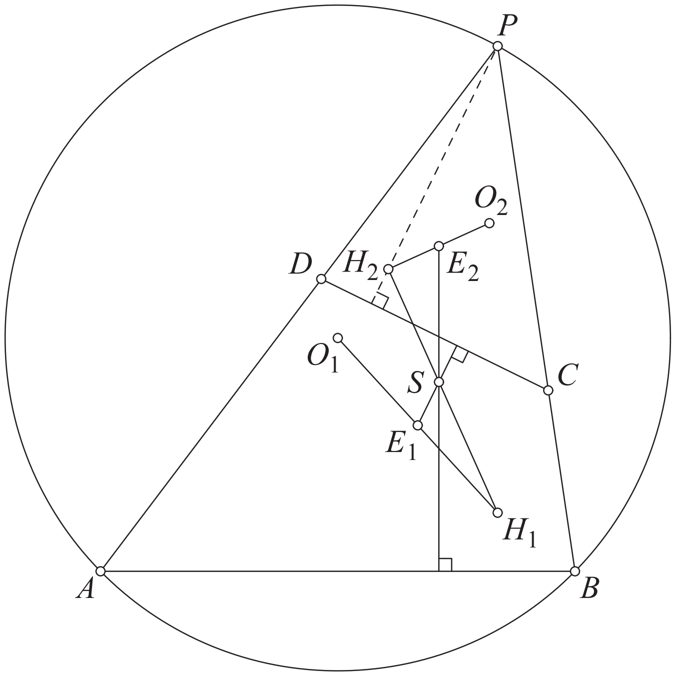


Figure 1

Let CD pass through either point A or point B. Note that by hypothesis these two cases are different. We will consider the case $A \in CD$, i.e. $A = D$. So we have to show that the perpendiculars from E_1 on AC and from E_2 on AB intersect on the altitude AH of triangle ABC (see Figure 2).

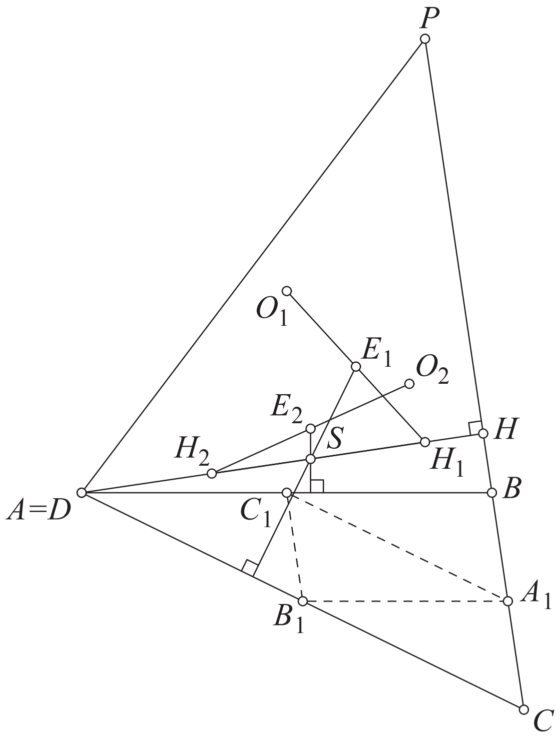


Figure 2

To this end, we consider the midpoints A_1, B_1, C_1 of BC, CA, AB, respectively. As E_1 is the center of FEUERBACH's circle (nine-point circle) of $\triangle ABP$, we have $E_1C_1 = E_1H$. Similarly, $E_2B_1 = E_2H$. Note further that a point X lies on the perpendicular from E_1 on A_1C_1 if and only if

$$XC_1^2 - XA_1^2 = E_1C_1^2 - E_1A_1^2.$$

Similarly, the perpendicular from E_2 on A_1B_1 is characterized by

$$XA_1^2 - XB_1^2 = E_2A_1^2 - E_2B_1^2.$$

The line H_1H_2, which is perpendicular to B_1C_1 and contains A, is given by

$$XB_1^2 - XC_1^2 = AB_1^2 - AC_1^2.$$

The three lines are concurrent if and only if

$$\begin{aligned} 0 &= XC_1^2 - XA_1^2 + XA_1^2 - XB_1^2 + XB_1^2 - XC_1^2 \\ &= E_1C_1^2 - E_1A_1^2 + E_2A_1^2 - E_2B_1^2 + AB_1^2 - AC_1^2 \\ &= -E_1A_1^2 + E_2A_1^2 + E_1H^2 - E_2H^2 + AB_1^2 - AC_1^2, \end{aligned}$$

i.e. it suffices to show that

$$E_1A_1^2 - E_2A_1^2 - E_1H^2 + E_2H^2 = \frac{AC^2 - AB^2}{4}.$$

We have

$$\frac{AC^2 - AB^2}{4} = \frac{HC^2 - HB^2}{4} = \frac{(HC + HB)(HC - HB)}{4} = \frac{HA_1 \cdot BC}{2}.$$

Let F_1, F_2 be the projections of E_1, E_2 on BC. Obviously, these are the midpoints of HP_1,

HP_2, where P_1, P_2 are the midpoints of PB and PC respectively. Then

$$\begin{aligned}
&E_1A_1^2 - E_2A_1^2 - E_1H^2 + E_2H^2 \\
&= F_1A_1^2 - F_1H^2 - F_2A_1^2 + F_2H^2 \\
&= (F_1A_1 - F_1H)(F_1A_1 + F_1H) - (F_2A_1 - F_2H)(F_2A_1 + F_2H) \\
&= A_1H \cdot (A_1P_1 - A_1P_2) \\
&= \frac{A_1H \cdot BC}{2} \\
&= \frac{AC^2 - AB^2}{4},
\end{aligned}$$

which proves the claim.

Solution 2. Let the perpendicular from E_1 on CD meet PH_1 at X, and the perpendicular from E_2 on AB meet PH_2 at Y (see Figure 3). Let φ be the intersection angle of AB and CD. Denote by M, N the midpoints of PH_1, PH_2 respectively.

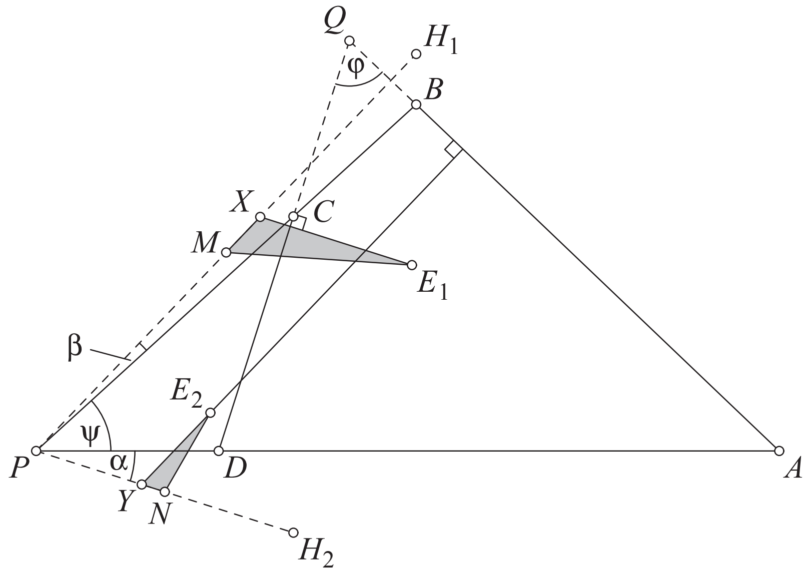


Figure 3

We will prove now that triangles E_1XM and E_2YN have equal angles at E_1, E_2, and supplementary angles at X, Y.

In the following, angles are understood as oriented, and equalities of angles modulo 180°.

Let $\alpha = \angle H_2PD$, $\psi = \angle DPC$, $\beta = \angle CPH_1$. Then $\alpha + \psi + \beta = \varphi$, $\angle E_1XH_1 = \angle H_2YE_2 = \varphi$, thus $\angle MXE_1 + \angle NYE_2 = 180°$.

By considering the Feuerbach circle of $\triangle ABP$ whose center is E_1 and which goes through M, we have $\angle E_1MH_1 = \psi + 2\beta$. Analogous considerations with the Feuerbach circle of $\triangle DCP$ yield $\angle H_2NE_2 = \psi + 2\alpha$. Hence indeed $\angle XE_1M = \varphi - (\psi + 2\beta) = (\psi + 2\alpha) - \varphi = \angle YE_2N$.

It follows now that

$$\frac{XM}{ME_1} = \frac{YN}{NE_2}.$$

Furthermore, ME_1 is half the circumradius of $\triangle ABP$, while PH_1 is the distance of P to the orthocenter of that triangle, which is twice the circumradius times the cosine of ψ. Together with analogous reasoning for $\triangle DCP$ we have

$$\frac{ME_1}{PH_1} = \frac{1}{4\cos\psi} = \frac{NE_2}{PH_2} \ .$$

By multiplication,

$$\frac{XM}{PH_1} = \frac{YN}{PH_2},$$

and therefore

$$\frac{PX}{XH_1} = \frac{H_2Y}{YP}.$$

Let E_1X, E_2Y meet H_1H_2 in R, S respectively.

Applying the intercept theorem to the parallels E_1X, PH_2 and center H_1 gives

$$\frac{H_2R}{RH_1} = \frac{PX}{XH_1},$$

while with parallels E_2Y, PH_1 and center H_2 we obtain

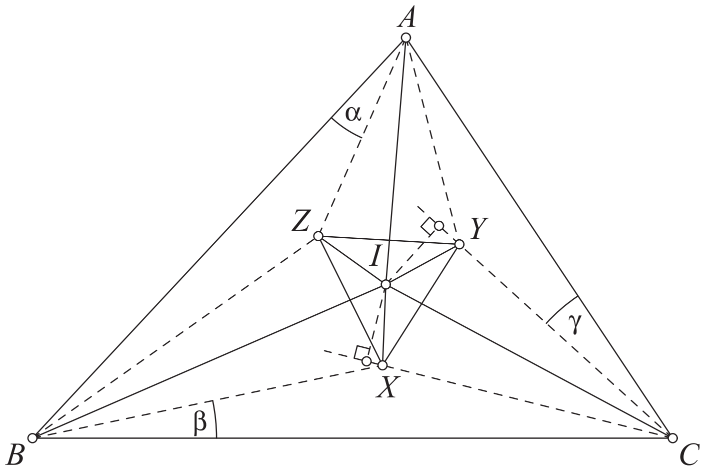
$$\frac{H_2S}{SH_1} = \frac{H_2Y}{YP}.$$

Combination of the last three equalities yields that R and S coincide.

G7 IRN (Islamic Republic of Iran)

Let ABC be a triangle with incenter I and let X, Y and Z be the incenters of the triangles BIC, CIA and AIB, respectively. Let the triangle XYZ be equilateral. Prove that ABC is equilateral too.

Solution. AZ, AI and AY divide $\angle BAC$ into four equal angles; denote them by α. In the same way we have four equal angles β at B and four equal angles γ at C. Obviously $\alpha+\beta+\gamma=\frac{180^\circ}{4}=45^\circ$; and $0^\circ<\alpha,\beta,\gamma<45^\circ$.



Easy calculations in various triangles yield $\angle BIC = 180^\circ - 2\beta - 2\gamma = 180^\circ - (90^\circ - 2\alpha) = 90^\circ + 2\alpha$, hence (for X is the incenter of triangle BCI, so IX bisects $\angle BIC$) we have $\angle XIC = \angle BIX = \frac{1}{2}\angle BIC = 45^\circ + \alpha$ and with similar aguments $\angle CIY = \angle YIA = 45^\circ + \beta$ and $\angle AIZ = \angle ZIB = 45^\circ + \gamma$. Furthermore, we have $\angle XIY = \angle XIC + \angle CIY = (45^\circ + \alpha) + (45^\circ + \beta) = 135^\circ - \gamma$, $\angle YIZ = 135^\circ - \alpha$, and $\angle ZIX = 135^\circ - \beta$.

Now we calculate the lengths of IX, IY and IZ in terms of α, β and γ. The perpendicular from I on CX has length $IX \cdot \sin\angle CXI = IX \cdot \sin(90^\circ + \beta) = IX \cdot \cos\beta$. But CI bisects $\angle YCX$, so the perpendicular from I on CY has the same length, and we conclude

$$IX \cdot \cos\beta = IY \cdot \cos\alpha.$$

To make calculations easier we choose a length unit that makes $IX = \cos\alpha$. Then $IY = \cos\beta$ and with similar arguments $IZ = \cos\gamma$.

Since XYZ is equilateral we have $ZX = ZY$. The law of Cosines in triangles XYI, YZI yields

$$\begin{aligned}
& ZX^2 = ZY^2 \\
\Longrightarrow\quad & IZ^2 + IX^2 - 2\cdot IZ\cdot IX\cdot\cos\angle ZIX = IZ^2 + IY^2 - 2\cdot IZ\cdot IY\cdot\cos\angle YIZ \\
\Longrightarrow\quad & IX^2 - IY^2 = 2\cdot IZ\cdot(IX\cdot\cos\angle ZIX - IY\cdot\cos\angle YIZ) \\
\Longrightarrow\quad & \underbrace{\cos^2\alpha - \cos^2\beta}_{\text{L.H.S.}} = \underbrace{2\cdot\cos\gamma\cdot(\cos\alpha\cdot\cos(135^\circ-\beta) - \cos\beta\cdot\cos(135^\circ-\alpha))}_{\text{R.H.S.}}.
\end{aligned}$$

A transformation of the left-hand side (L.H.S.) yields

$$\begin{aligned}
\text{L.H.S.} &= \cos^2\alpha\cdot\left(\sin^2\beta+\cos^2\beta\right) - \cos^2\beta\cdot\left(\sin^2\alpha+\cos^2\alpha\right) \\
&= \cos^2\alpha\cdot\sin^2\beta - \cos^2\beta\cdot\sin^2\alpha
\end{aligned}$$

$$
\begin{aligned}
&= (\cos\alpha \cdot \sin\beta + \cos\beta \cdot \sin\alpha) \cdot (\cos\alpha \cdot \sin\beta - \cos\beta \cdot \sin\alpha) \\
&= \sin(\beta+\alpha) \cdot \sin(\beta-\alpha) = \sin(45^\circ - \gamma) \cdot \sin(\beta - \alpha)
\end{aligned}
$$

whereas a transformation of the right-hand side (R.H.S.) leads to

$$
\begin{aligned}
\text{R.H.S.} &= 2 \cdot \cos\gamma \cdot (\cos\alpha \cdot (-\cos(45^\circ+\beta)) - \cos\beta \cdot (-\cos(45^\circ+\alpha))) \\
&= 2 \cdot \frac{\sqrt{2}}{2} \cdot \cos\gamma \cdot (\cos\alpha \cdot (\sin\beta - \cos\beta) + \cos\beta \cdot (\cos\alpha - \sin\alpha)) \\
&= \sqrt{2} \cdot \cos\gamma \cdot (\cos\alpha \cdot \sin\beta - \cos\beta \cdot \sin\alpha) \\
&= \sqrt{2} \cdot \cos\gamma \cdot \sin(\beta - \alpha).
\end{aligned}
$$

Equating L.H.S. and R.H.S. we obtain

$$
\begin{aligned}
& \sin(45^\circ - \gamma) \cdot \sin(\beta - \alpha) = \sqrt{2} \cdot \cos\gamma \cdot \sin(\beta - \alpha) \\
\Longrightarrow\quad & \sin(\beta - \alpha) \cdot \left(\sqrt{2} \cdot \cos\gamma - \sin(45^\circ - \gamma)\right) = 0 \\
\Longrightarrow\quad & \alpha = \beta \quad \text{or} \quad \sqrt{2} \cdot \cos\gamma = \sin(45^\circ - \gamma).
\end{aligned}
$$

But $\gamma < 45^\circ$; so $\sqrt{2} \cdot \cos\gamma > \cos\gamma > \cos 45^\circ = \sin 45^\circ > \sin(45^\circ - \gamma)$. This leaves $\alpha = \beta$. With similar reasoning we have $\alpha = \gamma$, which means triangle ABC must be equilateral.

G8 BGR (Bulgaria)

Let $ABCD$ be a circumscribed quadrilateral. Let g be a line through A which meets the segment BC in M and the line CD in N. Denote by I_1, I_2, and I_3 the incenters of $\triangle ABM$, $\triangle MNC$, and $\triangle NDA$, respectively. Show that the orthocenter of $\triangle I_1I_2I_3$ lies on g.

Solution 1. Let k_1, k_2 and k_3 be the incircles of triangles ABM, MNC, and NDA, respectively (see Figure 1). We shall show that the tangent h from C to k_1 which is different from CB is also tangent to k_3.

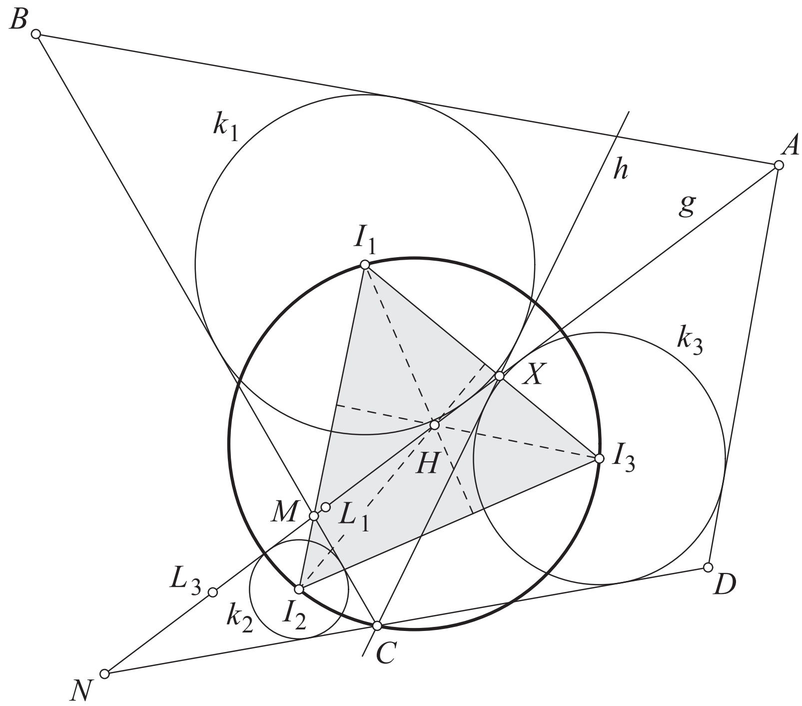


Figure 1

To this end, let X denote the point of intersection of g and h. Then $ABCX$ and $ABCD$ are circumscribed quadrilaterals, whence

$$CD - CX = (AB + CD) - (AB + CX) = (BC + AD) - (BC + AX) = AD - AX,$$

i.e.

$$AX + CD = CX + AD$$

which in turn reveals that the quadrilateral $AXCD$ is also circumscribed. Thus h touches indeed the circle k_3.

Moreover, we find that $\angle I_3CI_1 = \angle I_3CX + \angle XCI_1 = \frac{1}{2}(\angle DCX + \angle XCB) = \frac{1}{2}\angle DCB = \frac{1}{2}(180^\circ - \angle MCN) = 180^\circ - \angle MI_2N = \angle I_3I_2I_1$, from which we conclude that C, I_1, I_2, I_3 are concyclic.

Let now L_1 and L_3 be the reflection points of C with respect to the lines I_2I_3 and I_1I_2 respectively. Since I_1I_2 is the angle bisector of $\angle NMC$, it follows that L_3 lies on g. By analogous reasoning, L_1 lies on g.

Let H be the orthocenter of $\triangle I_1I_2I_3$. We have $\angle I_2L_3I_1 = \angle I_1CI_2 = \angle I_1I_3I_2 = 180^\circ - \angle I_1HI_2$, which entails that the quadrilateral $I_2HI_1L_3$ is cyclic. Analogously, $I_3HL_1I_2$ is cyclic.

Then, working with oriented angles modulo 180°, we have

$$\angle L_3HI_2 = \angle L_3I_1I_2 = \angle I_2I_1C = \angle I_2I_3C = \angle L_1I_3I_2 = \angle L_1HI_2,$$

whence L_1, L_3, and H are collinear. By $L_1 \neq L_3$, the claim follows.

Comment. The last part of the argument essentially reproves the following fact: The Simson line of a point P lying on the circumcircle of a triangle ABC with respect to that triangle bisects the line segment connecting P with the orthocenter of ABC.

Solution 2. We start by proving that C, I_1, I_2, and I_3 are concyclic.

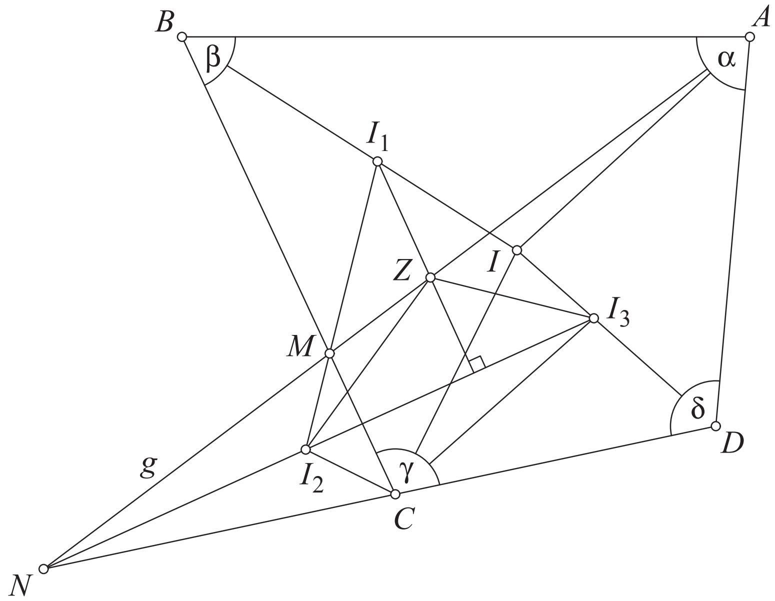


Figure 2

To this end, notice first that I_2, M, I_1 are collinear, as are N, I_2, I_3 (see Figure 2). Denote by α, β, γ, δ the internal angles of $ABCD$. By considerations in triangle CMN, it follows that $\angle I_3I_2I_1 = \frac{\gamma}{2}$. We will show that $\angle I_3CI_1 = \frac{\gamma}{2}$, too. Denote by I the incenter of $ABCD$. Clearly, $I_1 \in BI$, $I_3 \in DI$, $\angle I_1AI_3 = \frac{\alpha}{2}$.

Using the abbreviation $[X, YZ]$ for the distance from point X to the line YZ, we have because of $\angle BAI_1 = \angle IAI_3$ and $\angle I_1AI = \angle I_3AD$ that

$$\frac{[I_1, AB]}{[I_1, AI]} = \frac{[I_3, AI]}{[I_3, AD]}.$$

Furthermore, consideration of the angle sums in AIB, BIC, CID and DIA implies $\angle AIB + \angle CID = \angle BIC + \angle DIA = 180°$, from which we see

$$\frac{[I_1, AI]}{[I_3, CI]} = \frac{I_1I}{I_3I} = \frac{[I_1, CI]}{[I_3, AI]}.$$

Because of $[I_1, AB] = [I_1, BC]$, $[I_3, AD] = [I_3, CD]$, multiplication yields

$$\frac{[I_1, BC]}{[I_3, CI]} = \frac{[I_1, CI]}{[I_3, CD]}.$$

By $\angle DCI = \angle ICB = \gamma/2$ it follows that $\angle I_1CB = \angle I_3CI$ which concludes the proof of the above statement.

Let the perpendicular from I_1 on I_2I_3 intersect g at Z. Then $\angle MI_1Z = 90^\circ - \angle I_3I_2I_1 = 90^\circ - \gamma/2 = \angle MCI_2$. Since we have also $\angle ZMI_1 = \angle I_2MC$, triangles MZI_1 and MI_2C are similar. From this one easily proves that also MI_2Z and MCI_1 are similar. Because C, I_1, I_2, and I_3 are concyclic, $\angle MZI_2 = \angle MI_1C = \angle NI_3C$, thus NI_2Z and NCI_3 are similar, hence NCI_2 and NI_3Z are similar. We conclude $\angle ZI_3I_2 = \angle I_2CN = 90^\circ - \gamma/2$, hence $I_1I_2 \perp ZI_3$. This completes the proof.

Number Theory

N1 AUS (Australia)

A social club has n members. They have the membership numbers $1, 2, \ldots, n$, respectively. From time to time members send presents to other members, including items they have already received as presents from other members. In order to avoid the embarrassing situation that a member might receive a present that he or she has sent to other members, the club adds the following rule to its statutes at one of its annual general meetings:

"A member with membership number a is permitted to send a present to a member with membership number b if and only if $a(b-1)$ is a multiple of n."

Prove that, if each member follows this rule, none will receive a present from another member that he or she has already sent to other members.

Alternative formulation: Let G be a directed graph with n vertices $v_1, v_2, \ldots, v_n$, such that there is an edge going from v_a to v_b if and only if a and b are distinct and $a(b-1)$ is a multiple of n. Prove that this graph does not contain a directed cycle.

Solution 1. Suppose there is an edge from v_i to v_j. Then $i(j-1) = ij - i = kn$ for some integer k, which implies $i = ij - kn$. If $\gcd(i,n) = d$ and $\gcd(j,n) = e$, then e divides $ij - kn = i$ and thus e also divides d. Hence, if there is an edge from v_i to v_j, then $\gcd(j,n) \mid \gcd(i,n)$.

If there is a cycle in G, say $v_{i_1} \to v_{i_2} \to \cdots \to v_{i_r} \to v_{i_1}$, then we have

$$\gcd(i_1,n) \mid \gcd(i_r,n) \mid \gcd(i_{r-1},n) \mid \ldots \mid \gcd(i_2,n) \mid \gcd(i_1,n),$$

which implies that all these greatest common divisors must be equal, say be equal to t.

Now we pick any of the i_k, without loss of generality let it be i_1. Then $i_r(i_1-1)$ is a multiple of n and hence also (by dividing by t), $i_1 - 1$ is a multiple of $\frac{n}{t}$. Since i_1 and $i_1 - 1$ are relatively prime, also t and $\frac{n}{t}$ are relatively prime. So, by the Chinese remainder theorem, the value of i_1 is uniquely determined modulo $n = t \cdot \frac{n}{t}$ by the value of t. But, as i_1 was chosen arbitrarily among the i_k, this implies that all the i_k have to be equal, a contradiction.

Solution 2. If a, b, c are integers such that $ab - a$ and $bc - b$ are multiples of n, then also $ac - a = a(bc - b) + (ab - a) - (ab - a)c$ is a multiple of n. This implies that if there is an edge from v_a to v_b and an edge from v_b to v_c, then there also must be an edge from v_a to v_c. Therefore, if there are any cycles at all, the smallest cycle must have length 2. But suppose the vertices v_a and v_b form such a cycle, i.e., $ab - a$ and $ab - b$ are both multiples of n. Then $a - b$ is also a multiple of n, which can only happen if $a = b$, which is impossible.

Solution 3. Suppose there was a cycle $v_{i_1} \to v_{i_2} \to \cdots \to v_{i_r} \to v_{i_1}$. Then $i_1(i_2 - 1)$ is a multiple of n, i.e., $i_1 \equiv i_1 i_2 \bmod n$. Continuing in this manner, we get $i_1 \equiv i_1 i_2 \equiv i_1 i_2 i_3 \equiv i_1 i_2 i_3 \ldots i_r \bmod n$. But the same holds for all i_k, i.e., $i_k \equiv i_1 i_2 i_3 \ldots i_r \bmod n$. Hence $i_1 \equiv i_2 \equiv \cdots \equiv i_r \bmod n$, which means $i_1 = i_2 = \cdots = i_r$, a contradiction.

Solution 4. Let $n = k$ be the smallest value of n for which the corresponding graph has a cycle. We show that k is a prime power.
If k is not a prime power, it can be written as a product $k = de$ of relatively prime integers greater than 1. Reducing all the numbers modulo d yields a single vertex or a cycle in the corresponding graph on d vertices, because if $a(b-1) \equiv 0 \bmod k$ then this equation also holds modulo d. But since the graph on d vertices has no cycles, by the minimality of k, we must have that all the indices of the cycle are congruent modulo d. The same holds modulo e and hence also modulo $k = de$. But then all the indices are equal, which is a contradiction.
Thus k must be a prime power $k = p^m$. There are no edges ending at v_k, so v_k is not contained in any cycle. All edges not starting at v_k end at a vertex belonging to a non-multiple of p, and all edges starting at a non-multiple of p must end at v_1. But there is no edge starting at v_1. Hence there is no cycle.

Solution 5. Suppose there was a cycle $v_{i_1} \to v_{i_2} \to \cdots \to v_{i_r} \to v_{i_1}$. Let $q = p^m$ be a prime power dividing n. We claim that either $i_1 \equiv i_2 \equiv \cdots \equiv i_r \equiv 0 \bmod q$ or $i_1 \equiv i_2 \equiv \cdots \equiv i_r \equiv 1 \bmod q$.

Suppose that there is an i_s not divisible by q. Then, as $i_s(i_{s+1}-1)$ is a multiple of q, $i_{s+1} \equiv 1 \bmod p$. Similarly, we conclude $i_{s+2} \equiv 1 \bmod p$ and so on. So none of the labels is divisible by p, but since $i_s(i_{s+1}-1)$ is a multiple of $q = p^m$ for all s, all i_{s+1} are congruent to 1 modulo q. This proves the claim.

Now, as all the labels are congruent modulo all the prime powers dividing n, they must all be equal by the Chinese remainder theorem. This is a contradiction.

N2 PER (Peru)

A positive integer N is called *balanced*, if $N = 1$ or if N can be written as a product of an even number of not necessarily distinct primes. Given positive integers a and b, consider the polynomial P defined by $P(x) = (x + a)(x + b)$.

(a) Prove that there exist distinct positive integers a and b such that all the numbers $P(1), P(2), \dots, P(50)$ are balanced.

(b) Prove that if $P(n)$ is balanced for all positive integers n, then $a = b$.

Solution. Define a function f on the set of positive integers by $f(n) = 0$ if n is balanced and $f(n) = 1$ otherwise. Clearly, $f(nm) \equiv f(n) + f(m) \mod 2$ for all positive integers n, m.

(a) Now for each positive integer n consider the binary sequence $(f(n+1), f(n+2), \dots, f(n+50))$. As there are only 2^{50} different such sequences there are two different positive integers a and b such that

$$(f(a+1), f(a+2), \dots, f(a+50)) = (f(b+1), f(b+2), \dots, f(b+50)).$$

But this implies that for the polynomial $P(x) = (x+a)(x+b)$ all the numbers $P(1), P(2), \dots, P(50)$ are balanced, since for all $1 \le k \le 50$ we have $f(P(k)) \equiv f(a+k) + f(b+k) \equiv 2f(a+k) \equiv 0 \mod 2$.

(b) Now suppose $P(n)$ is balanced for all positive integers n and $a < b$. Set $n = k(b-a) - a$ for sufficiently large k, such that n is positive. Then $P(n) = k(k+1)(b-a)^2$, and this number can only be balanced, if $f(k) = f(k+1)$ holds. Thus, the sequence $f(k)$ must become constant for sufficiently large k. But this is not possible, as for every prime p we have $f(p) = 1$ and for every square t^2 we have $f(t^2) = 0$.

Hence $a = b$.

Comment. Given a positive integer k, a computer search for the pairs of positive integers (a, b), for which $P(1), P(2), \dots, P(k)$ are all balanced yields the following results with minimal sum $a + b$ and $a < b$:

k	3	4	5	10	20
(a,b)	$(2,4)$	$(6,11)$	$(8,14)$	$(20,34)$	$(1751,3121)$

Therefore, trying to find a and b in part (a) of the problem cannot be done by elementary calculations.

N3 EST (Estonia)

Let f be a non-constant function from the set of positive integers into the set of positive integers, such that $a-b$ divides $f(a)-f(b)$ for all distinct positive integers a,b. Prove that there exist infinitely many primes p such that p divides $f(c)$ for some positive integer c.

Solution 1. Denote by $v_p(a)$ the exponent of the prime p in the prime decomposition of a.

Assume that there are only finitely many primes $p_1, p_2, \ldots, p_m$ that divide some function value produced of f.

There are infinitely many positive integers a such that $v_{p_i}(a) > v_{p_i}(f(1))$ for all $i = 1, 2, \ldots, m$, e.g. $a = (p_1 p_2 \ldots p_m)^\alpha$ with α sufficiently large. Pick any such a. The condition of the problem then yields $a \mid (f(a+1) - f(1))$. Assume $f(a+1) \neq f(1)$. Then we must have $v_{p_i}(f(a+1)) \neq v_{p_i}(f(1))$ for at least one i. This yields $v_{p_i}(f(a+1) - f(1)) = \min\{v_{p_i}(f(a+1)), v_{p_i}(f(1))\} \leq v_{p_1}(f(1)) < v_{p_i}(a)$. But this contradicts the fact that $a \mid (f(a+1) - f(1))$.

Hence we must have $f(a+1) = f(1)$ for all such a.

Now, for any positive integer b and all such a, we have $(a+1-b) | (f(a+1) - f(b))$, i.e., $(a+1-b) | (f(1) - f(b))$. Since this is true for infinitely many positive integers a we must have $f(b) = f(1)$. Hence f is a constant function, a contradiction. Therefore, our initial assumption was false and there are indeed infinitely many primes p dividing $f(c)$ for some positive integer c.

Solution 2. Assume that there are only finitely many primes $p_1, p_2, \ldots, p_m$ that divide some function value of f. Since f is not identically 1, we must have $m \geq 1$.

Then there exist non-negative integers $\alpha_1, \ldots, \alpha_m$ such that

$$f(1) = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_m^{\alpha_m}.$$

We can pick a positive integer r such that $f(r) \neq f(1)$. Let

$$M = 1 + p_1^{\alpha_1+1} p_2^{\alpha_2+1} \ldots p_m^{\alpha_m+1} \cdot (f(r) + r).$$

Then for all $i \in \{1, \ldots, m\}$ we have that $p_i^{\alpha_i+1}$ divides $M-1$ and hence by the condition of the problem also $f(M) - f(1)$. This implies that $f(M)$ is divisible by $p_i^{\alpha_i}$ but not by $p_i^{\alpha_i+1}$ for all i and therefore $f(M) = f(1)$.

Hence

$$\begin{aligned} M - r &> p_1^{\alpha_1+1} p_2^{\alpha_2+1} \ldots p_m^{\alpha_m+1} \cdot (f(r) + r) - r \\ &\geq p_1^{\alpha_1+1} p_2^{\alpha_2+1} \ldots p_m^{\alpha_m+1} + (f(r) + r) - r \\ &> p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_m^{\alpha_m} + f(r) \\ &\geq |f(M) - f(r)|. \end{aligned}$$

But since $M - r$ divides $f(M) - f(r)$ this can only be true if $f(r) = f(M) = f(1)$, which contradicts the choice of r.

Comment. In the case that f is a polynomial with integer coefficients the result is well-known, see e.g. W. Schwarz, Einführung in die Methoden der Primzahltheorie, 1969.

N4 PRK (Democratic People's Republic of Korea)

Find all positive integers n such that there exists a sequence of positive integers $a_1, a_2, \ldots, a_n$ satisfying

$$a_{k+1} = \frac{a_k^2 + 1}{a_{k-1} + 1} - 1$$

for every k with $2 \leq k \leq n-1$.

Solution 1. Such a sequence exists for $n = 1, 2, 3, 4$ and no other n. Since the existence of such a sequence for some n implies the existence of such a sequence for all smaller n, it suffices to prove that $n = 5$ is not possible and $n = 4$ is possible.

Assume first that for $n = 5$ there exists a sequence of positive integers $a_1, a_2, \ldots, a_5$ satisfying the conditions

$$\begin{aligned} a_2^2 + 1 &= (a_1 + 1)(a_3 + 1), \\ a_3^2 + 1 &= (a_2 + 1)(a_4 + 1), \\ a_4^2 + 1 &= (a_3 + 1)(a_5 + 1). \end{aligned}$$

Assume a_1 is odd, then a_2 has to be odd as well and as then $a_2^2 + 1 \equiv 2 \mod 4$, a_3 has to be even. But this is a contradiction, since then the even number $a_2 + 1$ cannot divide the odd number $a_3^2 + 1$.

Hence a_1 is even.

If a_2 is odd, $a_3^2 + 1$ is even (as a multiple of $a_2 + 1$) and hence a_3 is odd, too. Similarly we must have a_4 odd as well. But then $a_3^2 + 1$ is a product of two even numbers $(a_2 + 1)(a_4 + 1)$ and thus is divisible by 4, which is a contradiction as for odd a_3 we have $a_3^2 + 1 \equiv 2 \mod 4$.

Hence a_2 is even. Furthermore $a_3 + 1$ divides the odd number $a_2^2 + 1$ and so a_3 is even. Similarly, a_4 and a_5 are even as well.

Now set $x = a_2$ and $y = a_3$. From the given condition we get $(x+1) | (y^2+1)$ and $(y+1) | (x^2+1)$. We will prove that there is no pair of positive even numbers (x, y) satisfying these two conditions, thus yielding a contradiction to the assumption.

Assume there exists a pair (x_0, y_0) of positive even numbers satisfying the two conditions $(x_0 + 1) | (y_0^2 + 1)$ and $(y_0 + 1) | (x_0^2 + 1)$.

Then one has $(x_0+1) | (y_0^2 + 1 + x_0^2 - 1)$, i.e., $(x_0+1) | (x_0^2 + y_0^2)$, and similarly $(y_0+1) | (x_0^2 + y_0^2)$. Any common divisor d of $x_0 + 1$ and $y_0 + 1$ must hence also divide the number $(x_0^2+1) + (y_0^2+1) - (x_0^2 + y_0^2) = 2$. But as $x_0 + 1$ and $y_0 + 1$ are both odd, we must have $d = 1$. Thus $x_0 + 1$ and $y_0 + 1$ are relatively prime and therefore there exists a positive integer k such that

$$k(x+1)(y+1) = x^2 + y^2$$

has the solution (x_0, y_0). We will show that the latter equation has no solution (x, y) in positive even numbers.

Assume there is a solution. Pick the solution (x_1, y_1) with the smallest sum $x_1 + y_1$ and assume $x_1 \geq y_1$. Then x_1 is a solution to the quadratic equation

$$x^2 - k(y_1 + 1)x + y_1^2 - k(y_1 + 1) = 0.$$

Let x_2 be the second solution, which by VIETA's theorem fulfills $x_1 + x_2 = k(y_1 + 1)$ and $x_1x_2 = y_1^2 - k(y_1 + 1)$. If $x_2 = 0$, the second equation implies $y_1^2 = k(y_1 + 1)$, which is impossible, as $y_1 + 1 > 1$ cannot divide the relatively prime number y_1^2. Therefore $x_2 \neq 0$.

Also we get $(x_1 + 1)(x_2 + 1) = x_1x_2 + x_1 + x_2 + 1 = y_1^2 + 1$ which is odd, and hence x_2 must be even and positive. Also we have $x_2 + 1 = \frac{y_1^2+1}{x_1+1} \leq \frac{y_1^2+1}{y_1+1} \leq y_1 \leq x_1$. But this means that the pair (x', y') with $x' = y_1$ and $y' = x_2$ is another solution of $k(x + 1)(y + 1) = x^2 + y^2$ in even positive numbers with $x' + y' < x_1 + y_1$, a contradiction.

Therefore we must have $n \leq 4$.

When $n = 4$, a possible example of a sequence is $a_1 = 4$, $a_2 = 33$, $a_3 = 217$ and $a_4 = 1384$.

Solution 2. It is easy to check that for $n = 4$ the sequence $a_1 = 4$, $a_2 = 33$, $a_3 = 217$ and $a_4 = 1384$ is possible.

Now assume there is a sequence with $n \geq 5$. Then we have in particular

$$\begin{aligned} a_2^2 + 1 &= (a_1 + 1)(a_3 + 1), \\ a_3^2 + 1 &= (a_2 + 1)(a_4 + 1), \\ a_4^2 + 1 &= (a_3 + 1)(a_5 + 1). \end{aligned}$$

Also assume without loss of generality that among all such quintuples $(a_1, a_2, a_3, a_4, a_5)$ we have chosen one with minimal a_1.

One shows quickly the following fact:

> If three positive integers x, y, z fulfill $y^2 + 1 = (x + 1)(z + 1)$ and if y is even, then x and z are even as well and either $x < y < z$ or $z < y < x$ holds. (1)

Indeed, the first part is obvious and from $x < y$ we conclude

$$z + 1 = \frac{y^2 + 1}{x + 1} \geq \frac{y^2 + 1}{y} > y,$$

and similarly in the other case.

Now, if a_3 was odd, then $(a_2 + 1)(a_4 + 1) = a_3^2 + 1 \equiv 2 \mod 4$ would imply that one of a_2 or a_4 is even, this contradicts (1). Thus a_3 and hence also a_1, a_2, a_4 and a_5 are even. According to (1), one has $a_1 < a_2 < a_3 < a_4 < a_5$ or $a_1 > a_2 > a_3 > a_4 > a_5$ but due to the minimality of a_1 the first series of inequalities must hold.

Consider the identity

$$(a_3+1)(a_1+a_3) = a_3^2-1+(a_1+1)(a_3+1) = a_2^2+a_3^2 = a_2^2-1+(a_2+1)(a_4+1) = (a_2+1)(a_2+a_4).$$

Any common divisor of the two odd numbers $a_2 + 1$ and $a_3 + 1$ must also divide $(a_2 + 1)(a_4 + 1) - (a_3 + 1)(a_3 - 1) = 2$, so these numbers are relatively prime. Hence the last identity shows that $a_1 + a_3$ must be a multiple of $a_2 + 1$, i.e. there is an integer k such that

$$a_1 + a_3 = k(a_2 + 1). \tag{2}$$

Now set $a_0 = k(a_1 + 1) - a_2$. This is an integer and we have

$$\begin{aligned} (a_0 + 1)(a_2 + 1) &= k(a_1 + 1)(a_2 + 1) - (a_2 - 1)(a_2 + 1) \\ &= (a_1 + 1)(a_1 + a_3) - (a_1 + 1)(a_3 + 1) + 2 \\ &= (a_1 + 1)(a_1 - 1) + 2 = a_1^2 + 1. \end{aligned}$$

Thus $a_0 \geq 0$. If $a_0 > 0$, then by (1) we would have $a_0 < a_1 < a_2$ and then the quintuple $(a_0, a_1, a_2, a_3, a_4)$ would contradict the minimality of a_1.

Hence $a_0 = 0$, implying $a_2 = a_1^2$. But also $a_2 = k(a_1 + 1)$, which finally contradicts the fact that $a_1 + 1 > 1$ is relatively prime to a_1^2 and thus cannot be a divisor of this number.

Hence $n \geq 5$ is not possible.

Comment 1. Finding the example for $n = 4$ is not trivial and requires a tedious calculation, but it can be reduced to checking a few cases. The equations $(a_1 + 1)(a_3 + 1) = a_2^2 + 1$ and $(a_2 + 1)(a_4 + 1) = a_3^2 + 1$ imply, as seen in the proof, that a_1 is even and a_2, a_3, a_4 are odd. The case $a_1 = 2$ yields $a_2^2 \equiv -1 \mod 3$ which is impossible. Hence $a_1 = 4$ is the smallest possibility. In this case $a_2^2 \equiv -1 \mod 5$ and a_2 is odd, which implies $a_2 \equiv 3$ or $a_2 \equiv 7 \mod 10$. Hence we have to start checking $a_2 = 7, 13, 17, 23, 27, 33$ and in the last case we succeed.

Comment 2. The choice of $a_0 = k(a_1 + 1) - a_2$ in the second solution appears more natural if one considers that by the previous calculations one has $a_1 = k(a_2+1)-a_3$ and $a_2 = k(a_3+1)-a_4$.

Alternatively, one can solve the equation (2) for a_3 and use $a_2^2 + 1 = (a_1 + 1)(a_3 + 1)$ to get $a_2^2 - k(a_1 + 1)a_2 + a_1^2 - k(a_1 + 1) = 0$. Now a_0 is the second solution to this quadratic equation in a_2 (Vieta jumping).

N5 HUN (Hungary)

Let $P(x)$ be a non-constant polynomial with integer coefficients. Prove that there is no function T from the set of integers into the set of integers such that the number of integers x with $T^n(x) = x$ is equal to $P(n)$ for every $n \geq 1$, where T^n denotes the n-fold application of T.

Solution 1. Assume there is a polynomial P of degree at least 1 with the desired property for a given function T. Let $A(n)$ denote the set of all $x \in \mathbb{Z}$ such that $T^n(x) = x$ and let $B(n)$ denote the set of all $x \in \mathbb{Z}$ for which $T^n(x) = x$ and $T^k(x) \neq x$ for all $1 \leq k < n$. Both sets are finite under the assumption made. For each $x \in A(n)$ there is a smallest $k \geq 1$ such that $T^k(x) = x$, i.e., $x \in B(k)$. Let $d = \gcd(k, n)$. There are positive integers r, s such that $rk - sn = d$ and hence $x = T^{rk}(x) = T^{sn+d}(x) = T^d(T^{sn}(x)) = T^d(x)$. The minimality of k implies $d = k$, i.e., $k|n$. On the other hand one clearly has $B(k) \subset A(n)$ if $k|n$ and thus we have $A(n) = \bigcup_{d|n} B(d)$ as a disjoint union and hence

$$|A(n)| = \sum_{d|n} |B(d)|.$$

Furthermore, for every $x \in B(n)$ the elements $x, T^1(x), T^2(x), \ldots, T^{n-1}(x)$ are n distinct elements of $B(n)$. The fact that they are in $A(n)$ is obvious. If for some $k < n$ and some $0 \leq i < n$ we had $T^k(T^i(x)) = T^i(x)$, i.e. $T^{k+i}(x) = T^i(x)$, that would imply $x = T^n(x) = T^{n-i}(T^i(x)) = T^{n-i}(T^{k+i}(x)) = T^k(T^n(x)) = T^k(x)$ contradicting the minimality of n. Thus $T^i(x) \in B(n)$ and $T^i(x) \neq T^j(x)$ for $0 \leq i < j \leq n-1$.

So indeed, T permutes the elements of $B(n)$ in (disjoint) cycles of length n and in particular one has $n \big| |B(n)|$.

Now let $P(x) = \sum_{i=0}^{k} a_i x^i$, $a_i \in \mathbb{Z}$, $k \geq 1$, $a_k \neq 0$ and suppose that $|A(n)| = P(n)$ for all $n \geq 1$. Let p be any prime. Then

$$p^2 \big| |B(p^2)| = |A(p^2)| - |A(p)| = a_1(p^2 - p) + a_2(p^4 - p^2) + \ldots$$

Hence $p|a_1$ and since this is true for all primes we must have $a_1 = 0$.

Now consider any two different primes p and q. Since $a_1 = 0$ we have that

$$|A(p^2q)| - |A(pq)| = a_2(p^4q^2 - p^2q^2) + a_3(p^6q^3 - p^3q^3) + \ldots$$

is a multiple of p^2q. But we also have

$$p^2q \big| |B(p^2q)| = |A(p^2q)| - |A(pq)| - |B(p^2)|.$$

This implies

$$p^2q \big| |B(p^2)| = |A(p^2)| - |A(p)| = a_2(p^4 - p^2) + a_3(p^6 - p^3) + \cdots + a_k(p^{2k} - p^k).$$

Since this is true for every prime q we must have $a_2(p^4-p^2)+a_3(p^6-p^3)+\cdots+a_k(p^{2k}-p^k) = 0$ for every prime p. Since this expression is a polynomial in p of degree $2k$ (because $a_k \neq 0$) this is a contradiction, as such a polynomial can have at most $2k$ zeros.

Comment. The last contradiction can also be reached via

$$a_k = \lim_{p \to \infty} \frac{1}{p^{2k}} \left(a_2(p^4 - p^2) + a_3(p^6 - p^3) + \cdots + a_k(p^{2k} - p^k) \right) = 0.$$

Solution 2. As in the first solution define $A(n)$ and $B(n)$ and assume that a polynomial P with the required property exists. This again implies that $|A(n)|$ and $|B(n)|$ is finite for all positive integers n and that

$$P(n) = |A(n)| = \sum_{d|n} |B(d)| \qquad \text{and} \qquad n \big| |B(n)|.$$

Now, for any two distinct primes p and q, we have

$$P(0) \equiv P(pq) \equiv |B(1)| + |B(p)| + |B(q)| + |B(pq)| \equiv |B(1)| + |B(p)| \mod q.$$

Thus, for any fixed p, the expression $P(0) - |B(1)| - |B(p)|$ is divisible by arbitrarily large primes q which means that $P(0) = |B(1)| + |B(p)| = P(p)$ for any prime p. This implies that the polynomial P is constant, a contradiction.

N6 TUR (Turkey)

Let k be a positive integer. Show that if there exists a sequence $a_0, a_1, \ldots$ of integers satisfying the condition

$$a_n = \frac{a_{n-1} + n^k}{n} \quad \text{for all } n \geq 1,$$

then $k-2$ is divisible by 3.

Solution 1. *Part A.* For each positive integer k, there exists a polynomial P_k of degree $k-1$ with integer coefficients, i. e., $P_k \in \mathbb{Z}[x]$, and an integer q_k such that the polynomial identity

$$xP_k(x) = x^k + P_k(x-1) + q_k \tag{I_k}$$

is satisfied. To prove this, for fixed k we write

$$P_k(x) = b_{k-1}x^{k-1} + \cdots + b_1 x + b_0$$

and determine the coefficients $b_{k-1}, b_{k-2}, \ldots, b_0$ and the number q_k successively. Obviously, we have $b_{k-1} = 1$. For $m = k-1, k-2, \ldots, 1$, comparing the coefficients of x^m in the identity (I_k) results in an expression of b_{m-1} as an integer linear combination of $b_{k-1}, \ldots, b_m$, and finally $q_k = -P_k(-1)$.

Part B. Let k be a positive integer, and let $a_0, a_1, \ldots$ be a sequence of real numbers satisfying the recursion given in the problem. This recursion can be written as

$$a_n - P_k(n) = \frac{a_{n-1} - P_k(n-1)}{n} - \frac{q_k}{n} \quad \text{for all } n \geq 1,$$

which by induction gives

$$a_n - P_k(n) = \frac{a_0 - P_k(0)}{n!} - q_k \sum_{i=0}^{n-1} \frac{i!}{n!} \quad \text{for all } n \geq 1.$$

Therefore, the numbers a_n are integers for all $n \geq 1$ only if

$$a_0 = P_k(0) \quad \text{and} \quad q_k = 0.$$

Part C. Multiplying the identity (I_k) by $x^2 + x$ and subtracting the identities (I_{k+1}), (I_{k+2}) and $q_k x^2 = q_k x^2$ therefrom, we obtain

$$xT_k(x) = T_k(x-1) + 2x\big(P_k(x-1) + q_k\big) - (q_{k+2} + q_{k+1} + q_k),$$

where the polynomials $T_k \in \mathbb{Z}[x]$ are defined by $T_k(x) = (x^2+x)P_k(x) - P_{k+1}(x) - P_{k+2}(x) - q_k x$. Thus

$$xT_k(x) \equiv T_k(x-1) + q_{k+2} + q_{k+1} + q_k \bmod 2, \quad k = 1, 2, \ldots$$

Comparing the degrees, we easily see that this is only possible if T_k is the zero polynomial modulo 2, and

$$q_{k+2} \equiv q_{k+1} + q_k \bmod 2 \quad \text{for } k = 1, 2, \ldots$$

Since $q_1 = -1$ and $q_2 = 0$, these congruences finish the proof.

Solution 2. *Part A and B.* Let k be a positive integer, and suppose there is a sequence $a_0, a_1, \ldots$ as required. We prove: There exists a polynomial $P \in \mathbb{Z}[x]$, i. e., with integer coefficients, such that $a_n = P(n)$, $n = 0, 1, \ldots$, and $xP(x) = x^k + P(x-1)$.
To prove this, we write $P(x) = b_{k-1}x^{k-1} + \cdots + b_1 x + b_0$ and determine the coefficients $b_{k-1}, b_{k-2}, \ldots, b_0$ successively such that

$$xP(x) - x^k - P(x-1) = q,$$

where $q = q_k$ is an integer. Comparing the coefficients of x^m results in an expression of b_{m-1} as an integer linear combination of $b_{k-1}, \ldots, b_m$.
Defining $c_n = a_n - P(n)$, we get

$$\begin{aligned} P(n) + c_n &= \frac{P(n-1) + c_{n-1} + n^k}{n}, \quad \text{i. e.,} \\ q + nc_n &= c_{n-1}, \end{aligned}$$

hence

$$c_n = \frac{c_0}{n!} - q \cdot \frac{0! + 1! + \cdots + (n-1)!}{n!}.$$

We conclude $\lim_{n\to\infty} c_n = 0$, which, using $c_n \in \mathbb{Z}$, implies $c_n = 0$ for sufficiently large n. Therefore, we get $q = 0$ and $c_n = 0$, $n = 0, 1, \ldots$.

Part C. Suppose that $q = q_k = 0$, i. e. $xP(x) = x^k + P(x-1)$. To consider this identity for arguments $x \in \mathbb{F}_4$, we write $\mathbb{F}_4 = \{0, 1, \alpha, \alpha+1\}$. Then we get

$$\begin{aligned} \alpha P_k(\alpha) &= \alpha^k + P_k(\alpha+1) \quad \text{and} \\ (\alpha+1)P_k(\alpha+1) &= (\alpha+1)^k + P_k(\alpha), \end{aligned}$$

hence

$$\begin{aligned} P_k(\alpha) = 1 \cdot P_k(\alpha) &= (\alpha+1)\alpha P_k(\alpha) \\ &= (\alpha+1)P_k(\alpha+1) + (\alpha+1)\alpha^k \\ &= P_k(\alpha) + (\alpha+1)^k + (\alpha+1)\alpha^k. \end{aligned}$$

Now, $(\alpha+1)^{k-1} = \alpha^k$ implies $k \equiv 2 \bmod 3$.

Comment 1. For $k = 2$, the sequence given by $a_n = n+1$, $n = 0, 1, \ldots$, satisfies the conditions of the problem.

Comment 2. The first few polynomials P_k and integers q_k are

$$\begin{aligned}
&P_1(x) = 1, \quad q_1 = -1,\\
&P_2(x) = x + 1, \quad q_2 = 0,\\
&P_3(x) = x^2 + x - 1, \quad q_3 = 1,\\
&P_4(x) = x^3 + x^2 - 2x - 1, \quad q_4 = -1,\\
&P_5(x) = x^4 + x^3 - 3x^2 + 5, \quad q_5 = -2,\\
&P_6(x) = x^5 + x^4 - 4x^3 + 2x^2 + 10x - 5, \quad q_6 = 9,\\
&q_7 = -9, \quad q_8 = -50, \quad q_9 = 267, \quad q_{10} = -413, \quad q_{11} = -2180.
\end{aligned}$$

A lookup in the On-Line Encyclopedia of Integer Sequences (A000587) reveals that the sequence $q_1, -q_2, q_3, -q_4, q_5, \ldots$ is known as Uppuluri-Carpenter numbers. The result that $q_k = 0$ implies $k \equiv 2 \bmod 3$ is contained in
Murty, Summer: On the p-adic series $\sum_{n=0}^{\infty} n^k \cdot n!$. CRM Proc. and Lecture Notes 36, 2004.
As shown by Alexander (Non-Vanishing of Uppuluri-Carpenter Numbers, Preprint 2006), Uppuluri-Carpenter numbers are zero at most twice.

Comment 3. The numbers q_k can be written in terms of the Stirling numbers of the second kind. To show this, we fix the notation such that

$$\begin{aligned}
x^k =& S_{k-1,k-1}x(x-1)\cdots(x-k+1)\\
&+ S_{k-1,k-2}x(x-1)\cdots(x-k+2)\\
&+ \cdots + S_{k-1,0}x,
\end{aligned} \tag{$*$}$$

e. g., $S_{2,2} = 1$, $S_{2,1} = 3$, $S_{2,0} = 1$, and we define

$$\Omega_k = S_{k-1,k-1} - S_{k-1,k-2} + - \cdots .$$

Replacing x by $-x$ in $(*)$ results in

$$\begin{aligned}
x^k =& S_{k-1,k-1}x(x+1)\cdots(x+k-1)\\
&- S_{k-1,k-2}x(x+1)\cdots(x+k-2)\\
&+ - \cdots \pm S_{k-1,0}x.
\end{aligned}$$

Defining

$$\begin{aligned}
P(x) =& S_{k-1,k-1}(x+1)\cdots(x+k-1)\\
&+ (S_{k-1,k-1} - S_{k-1,k-2})(x+1)\cdots(x+k-2)\\
&+ (S_{k-1,k-1} - S_{k-1,k-2} + S_{k-1,k-3})(x+1)\cdots(x+k-3)\\
&+ \cdots + \Omega_k,
\end{aligned}$$

we obtain

$$\begin{aligned}
xP(x) - P(x-1) &= S_{k-1,k-1}x(x+1)\cdots(x+k-1)\\
&\quad - S_{k-1,k-2}x(x+1)\cdots(x+k-2)\\
&\quad + - \cdots \pm S_{k-1,0}x - \Omega_k\\
&= x^k - \Omega_k,
\end{aligned}$$

hence $q_k = -\Omega_k$.

N7 MNG (Mongolia)

Let a and b be distinct integers greater than 1. Prove that there exists a positive integer n such that $(a^n-1)(b^n-1)$ is not a perfect square.

Solution 1. At first we notice that

$$(1-\alpha)^{\frac{1}{2}}(1-\beta)^{\frac{1}{2}} = \left(1-\tfrac{1}{2}\cdot\alpha-\tfrac{1}{8}\cdot\alpha^2-\cdots\right)\left(1-\tfrac{1}{2}\cdot\beta-\tfrac{1}{8}\cdot\beta^2-\cdots\right)$$
$$= \sum_{k,\ell\geq 0} c_{k,\ell}\cdot\alpha^k\beta^\ell \quad \text{for all } \alpha,\beta\in(0,1), \tag{1}$$

where $c_{0,0}=1$ and $c_{k,\ell}$ are certain coefficients.

For an indirect proof, we suppose that $x_n=\sqrt{(a^n-1)(b^n-1)}\in\mathbb{Z}$ for all positive integers n. Replacing a by a^2 and b by b^2 if necessary, we may assume that a and b are perfect squares, hence $\sqrt{ab}$ is an integer.

At first we shall assume that $a^\mu\neq b^\nu$ for all positive integers μ,ν. We have

$$x_n=(\sqrt{ab})^n\left(1-\frac{1}{a^n}\right)^{\frac{1}{2}}\left(1-\frac{1}{b^n}\right)^{\frac{1}{2}}=\sum_{k,\ell\geq 0}c_{k,\ell}\left(\frac{\sqrt{ab}}{a^kb^\ell}\right)^n. \tag{2}$$

Choosing k_0 and ℓ_0 such that $a^{k_0}>\sqrt{ab}$, $b^{\ell_0}>\sqrt{ab}$, we define the polynomial

$$P(x)=\prod_{k=0,\ell=0}^{k_0-1,\ell_0-1}(a^kb^\ell x-\sqrt{ab}) =: \sum_{i=0}^{k_0\cdot\ell_0} d_ix^i$$

with integer coefficients d_i. By our assumption, the zeros

$$\frac{\sqrt{ab}}{a^kb^\ell},\quad k=0,\dots,k_0-1,\quad \ell=0,\dots,\ell_0-1,$$

of P are pairwise distinct.

Furthermore, we consider the integer sequence

$$y_n=\sum_{i=0}^{k_0\cdot\ell_0}d_ix_{n+i},\quad n=1,2,\dots \tag{3}$$

By the theory of linear recursions, we obtain

$$y_n=\sum_{\substack{k,\ell\geq 0\\ k\geq k_0 \text{ or } \ell\geq\ell_0}} e_{k,\ell}\left(\frac{\sqrt{ab}}{a^kb^\ell}\right)^n,\quad n=1,2,\dots, \tag{4}$$

with real numbers $e_{k,\ell}$. We have

$$|y_n|\leq \sum_{\substack{k,\ell\geq 0\\ k\geq k_0 \text{ or } \ell\geq\ell_0}} |e_{k,\ell}|\left(\frac{\sqrt{ab}}{a^kb^\ell}\right)^n =: M_n.$$

Because the series in (4) is obtained by a finite linear combination of the absolutely convergent series (1), we conclude that in particular $M_1 < \infty$. Since

$$\frac{\sqrt{ab}}{a^k b^\ell} \le \lambda := \max\left\{\frac{\sqrt{ab}}{a^{k_0}}, \frac{\sqrt{ab}}{b^{\ell_0}}\right\} \quad \text{for all } k, \ell \ge 0 \text{ such that } k \ge k_0 \text{ or } \ell \ge \ell_0,$$

we get the estimates $M_{n+1} \le \lambda M_n$, $n = 1, 2, \ldots$ Our choice of k_0 and ℓ_0 ensures $\lambda < 1$, which implies $M_n \to 0$ and consequently $y_n \to 0$ as $n \to \infty$. It follows that $y_n = 0$ for all sufficiently large n.

So, equation (3) reduces to $\sum_{i=0}^{k_0 \cdot \ell_0} d_i x_{n+i} = 0$.

Using the theory of linear recursions again, for sufficiently large n we have

$$x_n = \sum_{k=0,\ell=0}^{k_0-1,\ell_0-1} f_{k,\ell} \left(\frac{\sqrt{ab}}{a^k b^\ell}\right)^n$$

for certain real numbers $f_{k,\ell}$.

Comparing with (2), we see that $f_{k,\ell} = c_{k,\ell}$ for all $k, \ell \ge 0$ with $k < k_0$, $\ell < \ell_0$, and $c_{k,\ell} = 0$ if $k \ge k_0$ or $\ell \ge \ell_0$, since we assumed that $a^\mu \ne b^\nu$ for all positive integers μ, ν.

In view of (1), this means

$$(1-\alpha)^{\frac{1}{2}}(1-\beta)^{\frac{1}{2}} = \sum_{k=0,\ell=0}^{k_0-1,\ell_0-1} c_{k,\ell} \cdot \alpha^k \beta^\ell \tag{5}$$

for all real numbers $\alpha, \beta \in (0, 1)$. We choose $k^* < k_0$ maximal such that there is some i with $c_{k^*,i} \ne 0$. Squaring (5) and comparing coefficients of $\alpha^{2k^*}\beta^{2i^*}$, where i^* is maximal with $c_{k^*,i^*} \ne 0$, we see that $k^* = 0$. This means that the right hand side of (5) is independent of α, which is clearly impossible.

We are left with the case that $a^\mu = b^\nu$ for some positive integers μ and ν. We may assume that μ and ν are relatively prime. Then there is some positive integer c such that $a = c^\nu$ and $b = c^\mu$. Now starting with the expansion (2), i. e.,

$$x_n = \sum_{j\ge 0} g_j \left(\frac{\sqrt{c^{\mu+\nu}}}{c^j}\right)^n$$

for certain coefficients g_j, and repeating the arguments above, we see that $g_j = 0$ for sufficiently large j, say $j > j_0$. But this means that

$$(1-x^\mu)^{\frac{1}{2}}(1-x^\nu)^{\frac{1}{2}} = \sum_{j=0}^{j_0} g_j x^j$$

for all real numbers $x \in (0, 1)$. Squaring, we see that

$$(1-x^\mu)(1-x^\nu)$$

is the square of a polynomial in x. In particular, all its zeros are of order at least 2, which implies $\mu = \nu$ by looking at roots of unity. So we obtain $\mu = \nu = 1$, i. e., $a = b$, a contradiction.

Solution 2. We set $a^2 = A$, $b^2 = B$, and $z_n = \sqrt{(A^n - 1)(B^n - 1)}$. Let us assume that z_n is an integer for $n = 1, 2, \dots$ Without loss of generality, we may suppose that $b < a$. We determine an integer $k \geq 2$ such that $b^{k-1} \leq a < b^k$, and define a sequence $\gamma_1, \gamma_2, \dots$ of rational numbers such that

$$2\gamma_1 = 1 \qquad \text{and} \qquad 2\gamma_{n+1} = \sum_{i=1}^{n} \gamma_i \gamma_{n-i} \quad \text{for} \quad n = 1, 2, \dots$$

It could easily be shown that $\gamma_n = \frac{1 \cdot 1 \cdot 3 \dots (2n-3)}{2 \cdot 4 \cdot 6 \dots 2n}$, for instance by reading VANDERMONDEs convolution as an equation between polynomials, but we shall have no use for this fact.

Using LANDAUs O–Notation in the usual way, we have

$$\begin{aligned}
&\left\{(ab)^n - \gamma_1 \left(\frac{a}{b}\right)^n - \gamma_2 \left(\frac{a}{b^3}\right)^n - \dots - \gamma_k \left(\frac{a}{b^{2k-1}}\right)^n + O\left(\frac{b}{a}\right)^n\right\}^2 \\
&= A^n B^n - 2\gamma_1 A^n - \sum_{i=2}^{k} \left(2\gamma_i - \sum_{j=1}^{i-1} \gamma_j \gamma_{i-j}\right) \left(\frac{A}{B^{i-1}}\right)^n + O\left(\frac{A}{B^k}\right)^n + O\left(B^n\right) \\
&= A^n B^n - A^n + O\left(B^n\right),
\end{aligned}$$

whence

$$z_n = (ab)^n - \gamma_1 \left(\frac{a}{b}\right)^n - \gamma_2 \left(\frac{a}{b^3}\right)^n - \dots - \gamma_k \left(\frac{a}{b^{2k-1}}\right)^n + O\left(\frac{b}{a}\right)^n.$$

Now choose rational numbers $r_1, r_2, \dots, r_{k+1}$ such that

$$(x - ab) \cdot (x - \tfrac{a}{b}) \dots (x - \tfrac{a}{b^{2k-1}}) = x^{k+1} - r_1 x^k + - \dots \pm r_{k+1},$$

and then a natural number M for which $Mr_1, Mr_2, \dots Mr_{k+1}$ are integers. For known reasons,

$$M(z_{n+k+1} - r_1 z_{n+k} + - \dots \pm r_{k+1} z_n) = O\left(\frac{b}{a}\right)^n$$

for all $n \in \mathbb{N}$ and thus there is a natural number N which is so large, that

$$z_{n+k+1} = r_1 z_{n+k} - r_2 z_{n+k-1} + - \dots \mp r_{k+1} z_n$$

holds for all $n \geqslant N$. Now the theory of linear recursions reveals that there are some rational numbers $\delta_0, \delta_1, \delta_2, \dots, \delta_k$ such that

$$z_n = \delta_0 (ab)^n - \delta_1 \left(\frac{a}{b}\right)^n - \delta_2 \left(\frac{a}{b^3}\right)^n - \dots - \delta_k \left(\frac{a}{b^{2k-1}}\right)^n$$

for sufficiently large n, where $\delta_0 > 0$ as $z_n > 0$. As before, one obtains

$$\begin{aligned}
&A^n B^n - A^n - B^n + 1 = z_n^2 \\
&= \left\{\delta_0 (ab)^n - \delta_1 \left(\frac{a}{b}\right)^n - \delta_2 \left(\frac{a}{b^3}\right)^n - \dots - \delta_k \left(\frac{a}{b^{2k-1}}\right)^n\right\}^2 \\
&= \delta_0^2 A^n B^n - 2\delta_0 \delta_1 A^n - \sum_{i=2}^{i=k} \left(2\delta_0 \delta_i - \sum_{j=1}^{j=i-1} \delta_j \delta_{i-j}\right) \left(\frac{A}{B^{i-1}}\right)^n + O\left(\frac{A}{B^k}\right)^n.
\end{aligned}$$

Easy asymptotic calculations yield $\delta_0 = 1$, $\delta_1 = \frac{1}{2}$, $\delta_i = \frac{1}{2} \sum_{j=1}^{j=i-1} \delta_j \delta_{i-j}$ for $i = 2, 3, \dots, k-2$, and then $a = b^{k-1}$. It follows that $k > 2$ and there is some $P \in \mathbb{Q}[X]$ for which $(X-1)(X^{k-1}-1) = P(X)^2$. But this cannot occur, for instance as $X^{k-1} - 1$ has no double zeros. Thus our assumption that z_n was an integer for $n = 1, 2, \ldots$ turned out to be wrong, which solves the problem.

Original formulation of the problem. a, b are positive integers such that $a \cdot b$ is not a square of an integer. Prove that there exists a (infinitely many) positive integer n such that $(a^n - 1)(b^n - 1)$ is not a square of an integer.

Solution. *Lemma.* Let c be a positive integer, which is not a perfect square. Then there exists an odd prime p such that c is not a quadratic residue modulo p.
Proof. Denoting the square-free part of c by c', we have the equality $\left(\frac{c'}{p}\right) = \left(\frac{c}{p}\right)$ of the corresponding LEGENDRE symbols. Suppose that $c' = q_1 \cdots q_m$, where $q_1 < \cdots < q_m$ are primes. Then we have

$$\left(\frac{c'}{p}\right) = \left(\frac{q_1}{p}\right) \cdots \left(\frac{q_m}{p}\right).$$

Case 1. Let q_1 be odd. We choose a quadratic nonresidue r_1 modulo q_1 and quadratic residues r_i modulo q_i for $i = 2, \ldots, m$. By the Chinese remainder theorem and the DIRICHLET theorem, there exists a (infinitely many) prime p such that

$$\begin{aligned} p &\equiv r_1 \bmod q_1, \\ p &\equiv r_2 \bmod q_2, \\ \vdots & \quad \vdots \\ p &\equiv r_m \bmod q_m, \\ p &\equiv 1 \bmod 4. \end{aligned}$$

By our choice of the residues, we obtain

$$\left(\frac{p}{q_i}\right) = \left(\frac{r_i}{q_i}\right) = \begin{cases} -1, & i = 1, \\ 1, & i = 2, \ldots, m. \end{cases}$$

The congruence $p \equiv 1 \bmod 4$ implies that $\left(\frac{q_i}{p}\right) = \left(\frac{p}{q_i}\right)$, $i = 1, \ldots, m$, by the law of quadratic reciprocity. Thus

$$\left(\frac{c'}{p}\right) = \left(\frac{q_1}{p}\right) \cdots \left(\frac{q_m}{p}\right) = -1.$$

Case 2. Suppose $q_1 = 2$. We choose quadratic residues r_i modulo q_i for $i = 2, \ldots, m$. Again, by the Chinese remainder theorem and the DIRICHLET theorem, there exists a prime p such that

$$\begin{aligned} p &\equiv r_2 \bmod q_2, \\ \vdots & \quad \vdots \\ p &\equiv r_m \bmod q_m, \\ p &\equiv 5 \bmod 8. \end{aligned}$$

By the choice of the residues, we obtain $\left(\frac{p}{q_i}\right) = \left(\frac{r_i}{q_i}\right) = 1$ for $i = 2, \ldots, m$. Since $p \equiv 1 \bmod 4$ we have $\left(\frac{q_i}{p}\right) = \left(\frac{p}{q_i}\right)$, $i = 2, \ldots, m$, by the law of quadratic reciprocity. The congruence $p \equiv 5 \bmod 8$

implies that $\left(\frac{2}{p}\right) = -1$. Thus

$$\left(\frac{c'}{p}\right) = \left(\frac{2}{p}\right)\left(\frac{q_2}{p}\right)\cdots\left(\frac{q_m}{p}\right) = -1,$$

and the lemma is proved.

Applying the lemma for $c = a \cdot b$, we find an odd prime p such that

$$\left(\frac{ab}{p}\right) = \left(\frac{a}{p}\right)\cdot\left(\frac{b}{p}\right) = -1.$$

This implies either

$$a^{\frac{p-1}{2}} \equiv 1 \bmod p, \quad b^{\frac{p-1}{2}} \equiv -1 \bmod p, \qquad \text{or} \qquad a^{\frac{p-1}{2}} \equiv -1 \bmod p, \quad b^{\frac{p-1}{2}} \equiv 1 \bmod p.$$

Without loss of generality, suppose that $a^{\frac{p-1}{2}} \equiv 1 \bmod p$ and $b^{\frac{p-1}{2}} \equiv -1 \bmod p$. The second congruence implies that $b^{\frac{p-1}{2}} - 1$ is not divisible by p. Hence, if the exponent $\nu_p(a^{\frac{p-1}{2}} - 1)$ of p in the prime decomposition of $(a^{\frac{p-1}{2}} - 1)$ is odd, then $(a^{\frac{p-1}{2}} - 1)(b^{\frac{p-1}{2}} - 1)$ is not a perfect square. If $\nu_p(a^{\frac{p-1}{2}} - 1)$ is even, then $\nu_p(a^{\frac{p-1}{2}p} - 1)$ is odd by the well-known power lifting property

$$\nu_p\left(a^{\frac{p-1}{2}p} - 1\right) = \nu_p\left(a^{\frac{p-1}{2}} - 1\right) + 1.$$

In this case, $(a^{\frac{p-1}{2}p} - 1)(b^{\frac{p-1}{2}p} - 1)$ is not a perfect square.

Comment 1. In 1998, the following problem appeared in Crux Mathematicorum:
Problem 2344. Find all positive integers N that are quadratic residues modulo all primes greater than N.
The published solution (Crux Mathematicorum, 25(1999)4) is the same as the proof of the lemma given above, see also `http://www.mathlinks.ro/viewtopic.php?t=150495`.

Comment 2. There is also an elementary proof of the lemma. We cite Theorem 3 of Chapter 5 and its proof from the book
Ireland, Rosen: A Classical Introduction to Modern Number Theory, Springer 1982.

Theorem. Let a be a nonsquare integer. Then there are infinitely many primes p for which a is a quadratic nonresidue.

Proof. It is easily seen that we may assume that a is square-free. Let $a = 2^e q_1 q_2 \cdots q_n$, where q_i are distinct odd primes and $e = 0$ or 1. The case $a = 2$ has to be dealt with separately. We shall assume to begin with that $n \geq 1$, i.e., that a is divisible by an odd prime.

Let $\ell_1, \ell_2, \ldots, \ell_k$ be a finite set of odd primes not including any q_i. Let s be any quadratic nonresidue modq_n, and find a simultaneous solution to the congruences

$$\begin{aligned} x &\equiv 1 \bmod \ell_i, \quad i = 1, \ldots, k, \\ x &\equiv 1 \bmod 8, \\ x &\equiv 1 \bmod q_i, \quad i = 1, \ldots, n-1, \\ x &\equiv s \bmod q_n. \end{aligned}$$

Call the solution b. b is odd. Suppose that $b = p_1 p_2 \cdots p_m$ is its prime decomposition. Since

$b \equiv 1 \bmod 8$ we have $\left(\frac{2}{b}\right) = 1$ and $\left(\frac{q_i}{b}\right) = \left(\frac{b}{q_i}\right)$ by a result on JACOBI symbols. Thus

$$\left(\frac{a}{b}\right) = \left(\frac{2}{b}\right)^e \left(\frac{q_1}{b}\right) \cdots \left(\frac{q_{n-1}}{b}\right)\left(\frac{q_n}{b}\right) = \left(\frac{b}{q_1}\right) \cdots \left(\frac{b}{q_{n-1}}\right)\left(\frac{b}{q_n}\right) = \left(\frac{1}{q_1}\right) \cdots \left(\frac{1}{q_{n-1}}\right)\left(\frac{s}{q_n}\right) = -1.$$

On the other hand, by the definition of $\left(\frac{a}{b}\right)$, we have $\left(\frac{a}{b}\right) = \left(\frac{a}{p_1}\right)\left(\frac{a}{p_2}\right) \cdots \left(\frac{a}{p_m}\right)$. It follows that $\left(\frac{a}{p_i}\right) = -1$ for some i.

Notice that ℓ_j does not divide b. Thus $p_i \notin \{\ell_1, \ell_2, \ldots, \ell_k\}$.

To summarize, if a is a nonsquare, divisible by an odd prime, we have found a prime p, outside of a given finite set of primes $\{2, \ell_1, \ell_2, \ldots, \ell_k\}$, such that $\left(\frac{a}{p}\right) = -1$. This proves the theorem in this case.

It remains to consider the case $a = 2$. Let $\ell_1, \ell_2, \ldots, \ell_k$ be a finite set of primes, excluding 3, for which $\left(\frac{2}{\ell_i}\right) = -1$. Let $b = 8\ell_1\ell_2 \cdots \ell_k + 3$. b is not divisible by 3 or any ℓ_i. Since $b \equiv 3 \bmod 8$ we have $\left(\frac{2}{b}\right) = (-1)^{\frac{b^2-1}{8}} = -1$. Suppose that $b = p_1p_2 \cdots p_m$ is the prime decomposition of b. Then, as before, we see that $\left(\frac{2}{p_i}\right) = -1$ for some i. $p_i \notin \{3, \ell_1, \ell_2, \ldots, \ell_k\}$. This proves the theorem for $a = 2$.

This proof has also been posted to `mathlinks`, see `http://www.mathlinks.ro/viewtopic.php?t=150495` again.

The IMO 2009 is organised by

Bildung & Begabung

in cooperation with

JACOBS UNIVERSITY

INTERNATIONAL MATHEMATICAL OLYMPIAD

51st IMO Shortlisted Problems with Solutions

Astana, Kazakhstan
July 2-14, 2010

51$^{\text{st}}$ International Mathematical Olympiad

Astana, Kazakhstan 2010

Shortlisted Problems with Solutions

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Note of Confidentiality

The Shortlisted Problems should be kept strictly confidential until IMO 2011.

Contributing Countries

The Organizing Committee and the Problem Selection Committee of IMO 2010 thank the following 42 countries for contributing 158 problem proposals.

Armenia, Australia, Austria, Bulgaria, Canada, Columbia, Croatia, Cyprus, Estonia, Finland, France, Georgia, Germany, Greece, Hong Kong, Hungary, India, Indonesia, Iran, Ireland, Japan, Korea (North), Korea (South), Luxembourg, Mongolia, Netherlands, Pakistan, Panama, Poland, Romania, Russia, Saudi Arabia, Serbia, Slovakia, Slovenia, Switzerland, Thailand, Turkey, Ukraine, United Kingdom, United States of America, Uzbekistan

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Algebra

A1. Determine all functions $f : \mathbb{R} \to \mathbb{R}$ such that the equality

$$f([x]y) = f(x)[f(y)]. \tag{1}$$

holds for all $x, y \in \mathbb{R}$. Here, by $[x]$ we denote the greatest integer not exceeding x.

(France)

Answer. $f(x) = \text{const} = C$, where $C = 0$ or $1 \le C < 2$.

Solution 1. First, setting $x = 0$ in (1) we get

$$f(0) = f(0)[f(y)] \tag{2}$$

for all $y \in \mathbb{R}$. Now, two cases are possible.

Case 1. Assume that $f(0) \neq 0$. Then from (2) we conclude that $[f(y)] = 1$ for all $y \in \mathbb{R}$. Therefore, equation (1) becomes $f([x]y) = f(x)$, and substituting $y = 0$ we have $f(x) = f(0) = C \neq 0$. Finally, from $[f(y)] = 1 = [C]$ we obtain that $1 \le C < 2$.

Case 2. Now we have $f(0) = 0$. Here we consider two subcases.

Subcase 2a. Suppose that there exists $0 < \alpha < 1$ such that $f(\alpha) \neq 0$. Then setting $x = \alpha$ in (1) we obtain $0 = f(0) = f(\alpha)[f(y)]$ for all $y \in \mathbb{R}$. Hence, $[f(y)] = 0$ for all $y \in \mathbb{R}$. Finally, substituting $x = 1$ in (1) provides $f(y) = 0$ for all $y \in \mathbb{R}$, thus contradicting the condition $f(\alpha) \neq 0$.

Subcase 2b. Conversely, we have $f(\alpha) = 0$ for all $0 \le \alpha < 1$. Consider any real z; there exists an integer N such that $\alpha = \dfrac{z}{N} \in [0, 1)$ (one may set $N = [z] + 1$ if $z \ge 0$ and $N = [z] - 1$ otherwise). Now, from (1) we get $f(z) = f([N]\alpha) = f(N)[f(\alpha)] = 0$ for all $z \in \mathbb{R}$.

Finally, a straightforward check shows that all the obtained functions satisfy (1).

Solution 2. Assume that $[f(y)] = 0$ for some y; then the substitution $x = 1$ provides $f(y) = f(1)[f(y)] = 0$. Hence, if $[f(y)] = 0$ for all y, then $f(y) = 0$ for all y. This function obviously satisfies the problem conditions.

So we are left to consider the case when $[f(a)] \neq 0$ for some a. Then we have

$$f([x]a) = f(x)[f(a)], \qquad \text{or} \quad f(x) = \frac{f([x]a)}{[f(a)]}. \tag{3}$$

This means that $f(x_1) = f(x_2)$ whenever $[x_1] = [x_2]$, hence $f(x) = f([x])$, and we may assume that a is an integer.

Now we have

$$f(a) = f\left(2a \cdot \tfrac{1}{2}\right) = f(2a)\left[f\left(\tfrac{1}{2}\right)\right] = f(2a)[f(0)];$$

this implies $[f(0)] \neq 0$, so we may even assume that $a = 0$. Therefore equation (3) provides

$$f(x) = \frac{f(0)}{[f(0)]} = C \neq 0$$

for each x. Now, condition (1) becomes equivalent to the equation $C = C[C]$ which holds exactly when $[C] = 1$.

A2. Let the real numbers a, b, c, d satisfy the relations $a+b+c+d = 6$ and $a^2+b^2+c^2+d^2 = 12$. Prove that
$$36 \leq 4(a^3 + b^3 + c^3 + d^3) - (a^4 + b^4 + c^4 + d^4) \leq 48.$$

(Ukraine)

Solution 1. Observe that

$$\begin{aligned} 4(a^3 + b^3 + c^3 + d^3) - (a^4 + b^4 + c^4 + d^4) &= -\left((a-1)^4 + (b-1)^4 + (c-1)^4 + (d-1)^4\right) \\ &\quad + 6(a^2 + b^2 + c^2 + d^2) - 4(a+b+c+d) + 4 \\ &= -\left((a-1)^4 + (b-1)^4 + (c-1)^4 + (d-1)^4\right) + 52. \end{aligned}$$

Now, introducing $x = a-1$, $y = b-1$, $z = c-1$, $t = d-1$, we need to prove the inequalities
$$16 \geq x^4 + y^4 + z^4 + t^4 \geq 4,$$
under the constraint
$$x^2 + y^2 + z^2 + t^2 = (a^2 + b^2 + c^2 + d^2) - 2(a+b+c+d) + 4 = 4 \tag{1}$$
(we will not use the value of $x+y+z+t$ though it can be found).

Now the rightmost inequality in (1) follows from the power mean inequality:
$$x^4 + y^4 + z^4 + t^4 \geq \frac{(x^2 + y^2 + z^2 + t^2)^2}{4} = 4.$$
For the other one, expanding the brackets we note that
$$(x^2 + y^2 + z^2 + t^2)^2 = (x^4 + y^4 + z^4 + t^4) + q,$$
where q is a nonnegative number, so
$$x^4 + y^4 + z^4 + t^4 \leq (x^2 + y^2 + z^2 + t^2)^2 = 16,$$
and we are done.

Comment 1. The estimates are sharp; the lower and upper bounds are attained at $(3, 1, 1, 1)$ and $(0, 2, 2, 2)$, respectively.

Comment 2. After the change of variables, one can finish the solution in several different ways. The latter estimate, for instance, it can be performed by moving the variables – since we need only the second of the two shifted conditions.

Solution 2. First, we claim that $0 \leq a, b, c, d \leq 3$. Actually, we have
$$a + b + c = 6 - d, \quad a^2 + b^2 + c^2 = 12 - d^2,$$
hence the power mean inequality
$$a^2 + b^2 + c^2 \geq \frac{(a+b+c)^2}{3}$$
rewrites as
$$12 - d^2 \geq \frac{(6-d)^2}{3} \quad \iff \quad 2d(d-3) \leq 0,$$

which implies the desired inequalities for d; since the conditions are symmetric, we also have the same estimate for the other variables.

Now, to prove the rightmost inequality, we use the obvious inequality $x^2(x-2)^2 \geq 0$ for each real x; this inequality rewrites as $4x^3 - x^4 \leq 4x^2$. It follows that

$$(4a^3 - a^4) + (4b^3 - b^4) + (4c^3 - c^4) + (4d^3 - d^4) \leq 4(a^2 + b^2 + c^2 + d^2) = 48,$$

as desired.

Now we prove the leftmost inequality in an analogous way. For each $x \in [0, 3]$, we have $(x+1)(x-1)^2(x-3) \leq 0$ which is equivalent to $4x^3 - x^4 \geq 2x^2 + 4x - 3$. This implies that

$$(4a^3 - a^4) + (4b^3 - b^4) + (4c^3 - c^4) + (4d^3 - d^4) \geq 2(a^2 + b^2 + c^2 + d^2) + 4(a+b+c+d) - 12 = 36,$$

as desired.

Comment. It is easy to guess the extremal points $(0, 2, 2, 2)$ and $(3, 1, 1, 1)$ for this inequality. This provides a method of finding the polynomials used in Solution 2. Namely, these polynomials should have the form $x^4 - 4x^3 + ax^2 + bx + c$; moreover, the former polynomial should have roots at 2 (with an even multiplicity) and 0, while the latter should have roots at 1 (with an even multiplicity) and 3. These conditions determine the polynomials uniquely.

Solution 3. First, expanding $48 = 4(a^2 + b^2 + c^2 + d^2)$ and applying the AM–GM inequality, we have

$$\begin{aligned} a^4 + b^4 + c^4 + d^4 + 48 &= (a^4 + 4a^2) + (b^4 + 4b^2) + (c^4 + 4c^2) + (d^4 + 4d^2) \\ &\geq 2\left(\sqrt{a^4 \cdot 4a^2} + \sqrt{b^4 \cdot 4b^2} + \sqrt{c^4 \cdot 4c^2} + \sqrt{d^4 \cdot 4d^2}\right) \\ &= 4(|a^3| + |b^3| + |c^3| + |d^3|) \geq 4(a^3 + b^3 + c^3 + d^3), \end{aligned}$$

which establishes the rightmost inequality.

To prove the leftmost inequality, we first show that $a, b, c, d \in [0, 3]$ as in the previous solution. Moreover, we can assume that $0 \leq a \leq b \leq c \leq d$. Then we have $a + b \leq b + c \leq \frac{2}{3}(b + c + d) \leq \frac{2}{3} \cdot 6 = 4$.

Next, we show that $4b - b^2 \leq 4c - c^2$. Actually, this inequality rewrites as $(c-b)(b+c-4) \leq 0$, which follows from the previous estimate. The inequality $4a - a^2 \leq 4b - b^2$ can be proved analogously.

Further, the inequalities $a \leq b \leq c$ together with $4a - a^2 \leq 4b - b^2 \leq 4c - c^2$ allow us to apply the Chebyshev inequality obtaining

$$\begin{aligned} a^2(4a - a^2) + b^2(4b - b^2) + c^2(4c - c^2) &\geq \frac{1}{3}(a^2 + b^2 + c^2)\left(4(a + b + c) - (a^2 + b^2 + c^2)\right) \\ &= \frac{(12 - d^2)(4(6 - d) - (12 - d^2))}{3}. \end{aligned}$$

This implies that

$$\begin{aligned} (4a^3 - a^4) + (4b^3 - b^4) + (4c^3 - c^4) + (4d^3 - d^4) &\geq \frac{(12 - d^2)(d^2 - 4d + 12)}{3} + 4d^3 - d^4 \\ = \frac{144 - 48d + 16d^3 - 4d^4}{3} &= 36 + \frac{4}{3}(3 - d)(d - 1)(d^2 - 3). \end{aligned} \tag{2}$$

Finally, we have $d^2 \geq \frac{1}{4}(a^2 + b^2 + c^2 + d^2) = 3$ (which implies $d > 1$); so, the expression $\frac{4}{3}(3 - d)(d - 1)(d^2 - 3)$ in the right-hand part of (2) is nonnegative, and the desired inequality is proved.

Comment. The rightmost inequality is easier than the leftmost one. In particular, Solutions 2 and 3 show that only the condition $a^2 + b^2 + c^2 + d^2 = 12$ is needed for the former one.

A3. Let $x_1, \ldots, x_{100}$ be nonnegative real numbers such that $x_i + x_{i+1} + x_{i+2} \leq 1$ for all $i = 1, \ldots, 100$ (we put $x_{101} = x_1$, $x_{102} = x_2$). Find the maximal possible value of the sum

$$S = \sum_{i=1}^{100} x_i x_{i+2}.$$

(Russia)

Answer. $\dfrac{25}{2}$.

Solution 1. Let $x_{2i} = 0$, $x_{2i-1} = \frac{1}{2}$ for all $i = 1, \ldots, 50$. Then we have $S = 50 \cdot \left(\frac{1}{2}\right)^2 = \frac{25}{2}$. So, we are left to show that $S \leq \frac{25}{2}$ for all values of x_i's satisfying the problem conditions.

Consider any $1 \leq i \leq 50$. By the problem condition, we get $x_{2i-1} \leq 1 - x_{2i} - x_{2i+1}$ and $x_{2i+2} \leq 1 - x_{2i} - x_{2i+1}$. Hence by the AM–GM inequality we get

$$\begin{aligned} x_{2i-1}x_{2i+1} + x_{2i}x_{2i+2} &\leq (1 - x_{2i} - x_{2i+1})x_{2i+1} + x_{2i}(1 - x_{2i} - x_{2i+1}) \\ &= (x_{2i} + x_{2i+1})(1 - x_{2i} - x_{2i+1}) \leq \left(\frac{(x_{2i} + x_{2i+1}) + (1 - x_{2i} - x_{2i+1})}{2}\right)^2 = \frac{1}{4}. \end{aligned}$$

Summing up these inequalities for $i = 1, 2, \ldots, 50$, we get the desired inequality

$$\sum_{i=1}^{50}(x_{2i-1}x_{2i+1} + x_{2i}x_{2i+2}) \leq 50 \cdot \frac{1}{4} = \frac{25}{2}.$$

Comment. This solution shows that a bit more general fact holds. Namely, consider $2n$ nonnegative numbers $x_1, \ldots, x_{2n}$ in a row (with no cyclic notation) and suppose that $x_i + x_{i+1} + x_{i+2} \leq 1$ for all $i = 1, 2, \ldots, 2n-2$. Then $\sum\limits_{i=1}^{2n-2} x_i x_{i+2} \leq \dfrac{n-1}{4}$.

The proof is the same as above, though if might be easier to find it (for instance, applying induction). The original estimate can be obtained from this version by considering the sequence $x_1, x_2, \ldots, x_{100}, x_1, x_2$.

Solution 2. We present another proof of the estimate. From the problem condition, we get

$$\begin{aligned} S = \sum_{i=1}^{100} x_i x_{i+2} \leq \sum_{i=1}^{100} x_i(1 - x_i - x_{i+1}) &= \sum_{i=1}^{100} x_i - \sum_{i=1}^{100} x_i^2 - \sum_{i=1}^{100} x_i x_{i+1} \\ &= \sum_{i=1}^{100} x_i - \frac{1}{2}\sum_{i=1}^{100}(x_i + x_{i+1})^2. \end{aligned}$$

By the AM–QM inequality, we have $\sum(x_i + x_{i+1})^2 \geq \frac{1}{100}\left(\sum(x_i + x_{i+1})\right)^2$, so

$$\begin{aligned} S \leq \sum_{i=1}^{100} x_i - \frac{1}{200}\left(\sum_{i=1}^{100}(x_i + x_{i+1})\right)^2 &= \sum_{i=1}^{100} x_i - \frac{2}{100}\left(\sum_{i=1}^{100} x_i\right)^2 \\ &= \frac{2}{100}\left(\sum_{i=1}^{100} x_i\right)\left(\frac{100}{2} - \sum_{i=1}^{100} x_i\right). \end{aligned}$$

And finally, by the AM–GM inequality

$$S \leq \frac{2}{100} \cdot \left(\frac{1}{2}\left(\sum_{i=1}^{100} x_i + \frac{100}{2} - \sum_{i=1}^{100} x_i\right)\right)^2 = \frac{2}{100} \cdot \left(\frac{100}{4}\right)^2 = \frac{25}{2}.$$

Comment. These solutions are not as easy as they may seem at the first sight. There are two different optimal configurations in which the variables have different values, and not all of sums of three consecutive numbers equal 1. Although it is easy to find the value $\frac{25}{2}$, the estimates must be done with care to preserve equality in the optimal configurations.

A4. A sequence $x_1, x_2, \dots$ is defined by $x_1 = 1$ and $x_{2k} = -x_k$, $x_{2k-1} = (-1)^{k+1}x_k$ for all $k \geq 1$. Prove that $x_1 + x_2 + \cdots + x_n \geq 0$ for all $n \geq 1$.

(Austria)

Solution 1. We start with some observations. First, from the definition of x_i it follows that for each positive integer k we have

$$x_{4k-3} = x_{2k-1} = -x_{4k-2} \quad \text{and} \quad x_{4k-1} = x_{4k} = -x_{2k} = x_k. \tag{1}$$

Hence, denoting $S_n = \sum_{i=1}^n x_i$, we have

$$S_{4k} = \sum_{i=1}^{k}\big((x_{4k-3} + x_{4k-2}) + (x_{4k-1} + x_{4k})\big) = \sum_{i=1}^{k}(0 + 2x_k) = 2S_k, \tag{2}$$

$$S_{4k+2} = S_{4k} + (x_{4k+1} + x_{4k+2}) = S_{4k}. \tag{3}$$

Observe also that $S_n = \sum_{i=1}^n x_i \equiv \sum_{i=1}^n 1 = n \pmod 2$.

Now we prove by induction on k that $S_i \geq 0$ for all $i \leq 4k$. The base case is valid since $x_1 = x_3 = x_4 = 1$, $x_2 = -1$. For the induction step, assume that $S_i \geq 0$ for all $i \leq 4k$. Using the relations (1)–(3), we obtain

$$S_{4k+4} = 2S_{k+1} \geq 0, \quad S_{4k+2} = S_{4k} \geq 0, \quad S_{4k+3} = S_{4k+2} + x_{4k+3} = \frac{S_{4k+2} + S_{4k+4}}{2} \geq 0.$$

So, we are left to prove that $S_{4k+1} \geq 0$. If k is odd, then $S_{4k} = 2S_k \geq 0$; since k is odd, S_k is odd as well, so we have $S_{4k} \geq 2$ and hence $S_{4k+1} = S_{4k} + x_{4k+1} \geq 1$.

Conversely, if k is even, then we have $x_{4k+1} = x_{2k+1} = x_{k+1}$, hence $S_{4k+1} = S_{4k} + x_{4k+1} = 2S_k + x_{k+1} = S_k + S_{k+1} \geq 0$. The step is proved.

Solution 2. We will use the notation of S_n and the relations (1)–(3) from the previous solution.

Assume the contrary and consider the minimal n such that $S_{n+1} < 0$; surely $n \geq 1$, and from $S_n \geq 0$ we get $S_n = 0$, $x_{n+1} = -1$. Hence, we are especially interested in the set $M = \{n : S_n = 0\}$; our aim is to prove that $x_{n+1} = 1$ whenever $n \in M$ thus coming to a contradiction.

For this purpose, we first describe the set M inductively. We claim that (i) M consists only of even numbers, (ii) $2 \in M$, and (iii) for every even $n \geq 4$ we have $n \in M \iff [n/4] \in M$. Actually, (i) holds since $S_n \equiv n \pmod 2$, (ii) is straightforward, while (iii) follows from the relations $S_{4k+2} = S_{4k} = 2S_k$.

Now, we are left to prove that $x_{n+1} = 1$ if $n \in M$. We use the induction on n. The base case is $n = 2$, that is, the minimal element of M; here we have $x_3 = 1$, as desired.

For the induction step, consider some $4 \leq n \in M$ and let $m = [n/4] \in M$; then m is even, and $x_{m+1} = 1$ by the induction hypothesis. We prove that $x_{n+1} = x_{m+1} = 1$. If $n = 4m$ then we have $x_{n+1} = x_{2m+1} = x_{m+1}$ since m is even; otherwise, $n = 4m+2$, and $x_{n+1} = -x_{2m+2} = x_{m+1}$, as desired. The proof is complete.

Comment. Using the inductive definition of set M, one can describe it explicitly. Namely, M consists exactly of all positive integers not containing digits 1 and 3 in their 4-base representation.

A5. Denote by $\mathbb{Q}^+$ the set of all positive rational numbers. Determine all functions $f : \mathbb{Q}^+ \to \mathbb{Q}^+$ which satisfy the following equation for all $x, y \in \mathbb{Q}^+$:

$$f\big(f(x)^2 y\big) = x^3 f(xy). \tag{1}$$

(Switzerland)

Answer. The only such function is $f(x) = \dfrac{1}{x}$.

Solution. By substituting $y = 1$, we get

$$f\big(f(x)^2\big) = x^3 f(x). \tag{2}$$

Then, whenever $f(x) = f(y)$, we have

$$x^3 = \frac{f\big(f(x)^2\big)}{f(x)} = \frac{f\big(f(y)^2\big)}{f(y)} = y^3$$

which implies $x = y$, so the function f is injective.

Now replace x by xy in (2), and apply (1) twice, second time to $\big(y, f(x)^2\big)$ instead of (x, y):

$$f\big(f(xy)^2\big) = (xy)^3 f(xy) = y^3 f\big(f(x)^2 y\big) = f\big(f(x)^2 f(y)^2\big).$$

Since f is injective, we get

$$f(xy)^2 = f(x)^2 f(y)^2,$$
$$f(xy) = f(x) f(y).$$

Therefore, f is multiplicative. This also implies $f(1) = 1$ and $f(x^n) = f(x)^n$ for all integers n.

Then the function equation (1) can be re-written as

$$f\big(f(x)\big)^2 f(y) = x^3 f(x) f(y),$$
$$f\big(f(x)\big) = \sqrt{x^3 f(x)}. \tag{3}$$

Let $g(x) = x f(x)$. Then, by (3), we have

$$g\big(g(x)\big) = g\big(x f(x)\big) = x f(x) \cdot f\big(x f(x)\big) = x f(x)^2 f\big(f(x)\big) =$$
$$= x f(x)^2 \sqrt{x^3 f(x)} = \big(x f(x)\big)^{5/2} = \big(g(x)\big)^{5/2},$$

and, by induction,

$$\underbrace{g\Big(g\big(\ldots g}_{n+1}(x)\ldots\big)\Big) = \big(g(x)\big)^{(5/2)^n} \tag{4}$$

for every positive integer n.

Consider (4) for a fixed x. The left-hand side is always rational, so $\big(g(x)\big)^{(5/2)^n}$ must be rational for every n. We show that this is possible only if $g(x) = 1$. Suppose that $g(x) \neq 1$, and let the prime factorization of $g(x)$ be $g(x) = p_1^{\alpha_1} \ldots p_k^{\alpha_k}$ where $p_1, \ldots, p_k$ are distinct primes and $\alpha_1, \ldots, \alpha_k$ are nonzero integers. Then the unique prime factorization of (4) is

$$\underbrace{g\Big(g\big(\ldots g}_{n+1}(x)\ldots\big)\Big) = \big(g(x)\big)^{(5/2)^n} = p_1^{(5/2)^n \alpha_1} \cdots p_k^{(5/2)^n \alpha_k}$$

where the exponents should be integers. But this is not true for large values of n, for example $(\frac{5}{2})^n \alpha_1$ cannot be a integer number when $2^n \not| \alpha_1$. Therefore, $g(x) \neq 1$ is impossible.

Hence, $g(x) = 1$ and thus $f(x) = \dfrac{1}{x}$ for all x.

The function $f(x) = \dfrac{1}{x}$ satisfies the equation (1):

$$f(f(x)^2 y) = \frac{1}{f(x)^2 y} = \frac{1}{\left(\frac{1}{x}\right)^2 y} = \frac{x^3}{xy} = x^3 f(xy).$$

Comment. Among $\mathbb{R}^+ \to \mathbb{R}^+$ functions, $f(x) = \dfrac{1}{x}$ is not the only solution. Another solution is $f_1(x) = x^{3/2}$. Using transfinite tools, infinitely many other solutions can be constructed.

A6. Suppose that f and g are two functions defined on the set of positive integers and taking positive integer values. Suppose also that the equations $f(g(n)) = f(n) + 1$ and $g(f(n)) = g(n) + 1$ hold for all positive integers. Prove that $f(n) = g(n)$ for all positive integer n.

(Germany)

Solution 1. Throughout the solution, by $\mathbb{N}$ we denote the set of all positive integers. For any function $h\colon \mathbb{N} \to \mathbb{N}$ and for any positive integer k, define $h^k(x) = \underbrace{h\big(h\big(\dots h}_{k}(x)\dots\big)\big)$ (in particular, $h^0(x) = x$).

Observe that $f\big(g^k(x)\big) = f\big(g^{k-1}(x)\big) + 1 = \dots = f(x) + k$ for any positive integer k, and similarly $g\big(f^k(x)\big) = g(x) + k$. Now let a and b are the minimal values attained by f and g, respectively; say $f(n_f) = a$, $g(n_g) = b$. Then we have $f\big(g^k(n_f)\big) = a + k$, $g\big(f^k(n_g)\big) = b + k$, so the function f attains all values from the set $N_f = \{a, a + 1, \dots\}$, while g attains all the values from the set $N_g = \{b, b + 1, \dots\}$.

Next, note that $f(x) = f(y)$ implies $g(x) = g\big(f(x)\big) - 1 = g\big(f(y)\big) - 1 = g(y)$; surely, the converse implication also holds. Now, we say that x and y are *similar* (and write $x \sim y$) if $f(x) = f(y)$ (equivalently, $g(x) = g(y)$). For every $x \in \mathbb{N}$, we define $[x] = \{y \in \mathbb{N} : x \sim y\}$; surely, $y_1 \sim y_2$ for all $y_1, y_2 \in [x]$, so $[x] = [y]$ whenever $y \in [x]$.

Now we investigate the structure of the sets $[x]$.

Claim 1. Suppose that $f(x) \sim f(y)$; then $x \sim y$, that is, $f(x) = f(y)$. Consequently, each class $[x]$ contains at most one element from N_f, as well as at most one element from N_g.

Proof. If $f(x) \sim f(y)$, then we have $g(x) = g\big(f(x)\big) - 1 = g\big(f(y)\big) - 1 = g(y)$, so $x \sim y$. The second statement follows now from the sets of values of f and g. $\square$

Next, we clarify which classes do not contain large elements.

Claim 2. For any $x \in \mathbb{N}$, we have $[x] \subseteq \{1, 2, \dots, b - 1\}$ if and only if $f(x) = a$. Analogously, $[x] \subseteq \{1, 2, \dots, a - 1\}$ if and only if $g(x) = b$.

Proof. We will prove that $[x] \nsubseteq \{1, 2, \dots, b - 1\} \iff f(x) > a$; the proof of the second statement is similar.

Note that $f(x) > a$ implies that there exists some y satisfying $f(y) = f(x) - 1$, so $f\big(g(y)\big) = f(y) + 1 = f(x)$, and hence $x \sim g(y) \geq b$. Conversely, if $b \leq c \sim x$ then $c = g(y)$ for some $y \in \mathbb{N}$, which in turn follows $f(x) = f\big(g(y)\big) = f(y) + 1 \geq a + 1$, and hence $f(x) > a$. $\square$

Claim 2 implies that there exists exactly one class contained in $\{1, \dots, a - 1\}$ (that is, the class $[n_g]$), as well as exactly one class contained in $\{1, \dots, b - 1\}$ (the class $[n_f]$). Assume for a moment that $a \leq b$; then $[n_g]$ is contained in $\{1, \dots, b - 1\}$ as well, hence it coincides with $[n_g]$. So, we get that

$$f(x) = a \iff g(x) = b \iff x \sim n_f \sim n_g. \tag{1}$$

Claim 3. $a = b$.

Proof. By Claim 2, we have $[a] \neq [n_f]$, so $[a]$ should contain some element $a' \geq b$ by Claim 2 again. If $a \neq a'$, then $[a]$ contains two elements $\geq a$ which is impossible by Claim 1. Therefore, $a = a' \geq b$. Similarly, $b \geq a$. $\square$

Now we are ready to prove the problem statement. First, we establish the following

Claim 4. For every integer $d \geq 0$, $f^{d+1}(n_f) = g^{d+1}(n_f) = a + d$.

Proof. Induction on d. For $d = 0$, the statement follows from (1) and Claim 3. Next, for $d > 1$ from the induction hypothesis we have $f^{d+1}(n_f) = f\big(f^d(n_f)\big) = f\big(g^d(n_f)\big) = f(n_f) + d = a + d$. The equality $g^{d+1}(n_f) = a + d$ is analogous. $\square$

Finally, for each $x \in \mathbb{N}$, we have $f(x) = a + d$ for some $d \geq 0$, so $f(x) = f\big(g^d(n_f)\big)$ and hence $x \sim g^d(n_f)$. It follows that $g(x) = g\big(g^d(n_f)\big) = g^{d+1}(n_f) = a + d = f(x)$ by Claim 4.

Solution 2. We start with the same observations, introducing the relation $\sim$ and proving Claim 1 from the previous solution.

Note that $f(a) > a$ since otherwise we have $f(a) = a$ and hence $g(a) = g\big(f(a)\big) = g(a) + 1$, which is false.

Claim 2′. $a = b$.

Proof. We can assume that $a \leq b$. Since $f(a) \geq a + 1$, there exists some $x \in \mathbb{N}$ such that $f(a) = f(x) + 1$, which is equivalent to $f(a) = f\big(g(x)\big)$ and $a \sim g(x)$. Since $g(x) \geq b \geq a$, by Claim 1 we have $a = g(x) \geq b$, which together with $a \leq b$ proves the Claim. $\square$

Now, almost the same method allows to find the values $f(a)$ and $g(a)$.

Claim 3′. $f(a) = g(a) = a + 1$.

Proof. Assume the contrary; then $f(a) \geq a + 2$, hence there exist some $x, y \in \mathbb{N}$ such that $f(x) = f(a) - 2$ and $f(y) = g(x)$ (as $g(x) \geq a = b$). Now we get $f(a) = f(x) + 2 = f\big(g^2(x)\big)$, so $a \sim g^2(x) \geq a$, and by Claim 1 we get $a = g^2(x) = g\big(f(y)\big) = 1 + g(y) \geq 1 + a$; this is impossible. The equality $g(a) = a + 1$ is similar.

Now, we are prepared for the proof of the problem statement. First, we prove it for $n \geq a$.

Claim 4′. For each integer $x \geq a$, we have $f(x) = g(x) = x + 1$.

Proof. Induction on x. The base case $x = a$ is provided by Claim 3′, while the induction step follows from $f(x + 1) = f\big(g(x)\big) = f(x) + 1 = (x + 1) + 1$ and the similar computation for $g(x + 1)$.

Finally, for an arbitrary $n \in \mathbb{N}$ we have $g(n) \geq a$, so by Claim 4′ we have $f(n) + 1 = f\big(g(n)\big) = g(n) + 1$, hence $f(n) = g(n)$.

Comment. It is not hard now to describe all the functions $f : \mathbb{N} \to \mathbb{N}$ satisfying the property $f(f(n)) = f(n) + 1$. For each such function, there exists $n_0 \in \mathbb{N}$ such that $f(n) = n + 1$ for all $n \geq n_0$, while for each $n < n_0$, $f(n)$ is an arbitrary number greater than of equal to n_0 (these numbers may be different for different $n < n_0$).

A7. Let $a_1, \dots, a_r$ be positive real numbers. For $n > r$, we inductively define

$$a_n = \max_{1 \le k \le n-1} (a_k + a_{n-k}). \tag{1}$$

Prove that there exist positive integers $\ell \le r$ and N such that $a_n = a_{n-\ell} + a_\ell$ for all $n \ge N$.

(Iran)

Solution 1. First, from the problem conditions we have that each a_n $(n > r)$ can be expressed as $a_n = a_{j_1} + a_{j_2}$ with $j_1, j_2 < n$, $j_1 + j_2 = n$. If, say, $j_1 > r$ then we can proceed in the same way with a_{j_1}, and so on. Finally, we represent a_n in a form

$$a_n = a_{i_1} + \dots + a_{i_k}, \tag{2}$$

$$1 \le i_j \le r, \quad i_1 + \dots + i_k = n. \tag{3}$$

Moreover, if a_{i_1} and a_{i_2} are the numbers in (2) obtained on the last step, then $i_1 + i_2 > r$. Hence we can adjust (3) as

$$1 \le i_j \le r, \quad i_1 + \dots + i_k = n, \quad i_1 + i_2 > r. \tag{4}$$

On the other hand, suppose that the indices $i_1, \dots, i_k$ satisfy the conditions (4). Then, denoting $s_j = i_1 + \dots + i_j$, from (1) we have

$$a_n = a_{s_k} \ge a_{s_{k-1}} + a_{i_k} \ge a_{s_{k-2}} + a_{i_{k-1}} + a_{i_k} \ge \dots \ge a_{i_1} + \dots + a_{i_k}.$$

Summarizing these observations we get the following

Claim. For every $n > r$, we have

$$a_n = \max\{a_{i_1} + \dots + a_{i_k} : \text{the collection } (i_1, \dots, i_k) \text{ satisfies } (4)\}. \qquad \square$$

Now we denote

$$s = \max_{1 \le i \le r} \frac{a_i}{i}$$

and fix some index $\ell \le r$ such that $s = \dfrac{a_\ell}{\ell}$.

Consider some $n \ge r^2\ell + 2r$ and choose an expansion of a_n in the form (2), (4). Then we have $n = i_1 + \dots + i_k \le rk$, so $k \ge n/r \ge r\ell + 2$. Suppose that none of the numbers $i_3, \dots, i_k$ equals ℓ. Then by the pigeonhole principle there is an index $1 \le j \le r$ which appears among $i_3, \dots, i_k$ at least ℓ times, and surely $j \ne \ell$. Let us delete these ℓ occurrences of j from $(i_1, \dots, i_k)$, and add j occurrences of ℓ instead, obtaining a sequence $(i_1, i_2, i'_3, \dots, i'_{k'})$ also satisfying (4). By Claim, we have

$$a_{i_1} + \dots + a_{i_k} = a_n \ge a_{i_1} + a_{i_2} + a_{i'_3} + \dots + a_{i'_{k'}},$$

or, after removing the coinciding terms, $\ell a_j \ge j a_\ell$, so $\dfrac{a_\ell}{\ell} \le \dfrac{a_j}{j}$. By the definition of ℓ, this means that $\ell a_j = j a_\ell$, hence

$$a_n = a_{i_1} + a_{i_2} + a_{i'_3} + \dots + a_{i'_{k'}}.$$

Thus, for every $n \ge r^2\ell + 2r$ we have found a representation of the form (2), (4) with $i_j = \ell$ for some $j \ge 3$. Rearranging the indices we may assume that $i_k = \ell$.

Finally, observe that in this representation, the indices $(i_1, \dots, i_{k-1})$ satisfy the conditions (4) with n replaced by $n - \ell$. Thus, from the Claim we get

$$a_{n-\ell} + a_\ell \ge (a_{i_1} + \dots + a_{i_{k-1}}) + a_\ell = a_n,$$

which by (1) implies

$$a_n = a_{n-\ell} + a_\ell \qquad \text{for each } n \ge r^2\ell + 2r,$$

as desired.

Solution 2. As in the previous solution, we involve the expansion (2), (3), and we fix some index $1 \le \ell \le r$ such that

$$\frac{a_\ell}{\ell} = s = \max_{1\le i\le r} \frac{a_i}{i}.$$

Now, we introduce the sequence (b_n) as $b_n = a_n - sn$; then $b_\ell = 0$.

We prove by induction on n that $b_n \le 0$, and (b_n) satisfies the same recurrence relation as (a_n). The base cases $n \le r$ follow from the definition of s. Now, for $n > r$ from the induction hypothesis we have

$$b_n = \max_{1\le k\le n-1}(a_k + a_{n-k}) - ns = \max_{1\le k\le n-1}(b_k + b_{n-k} + ns) - ns = \max_{1\le k\le n-1}(b_k + b_{n-k}) \le 0,$$

as required.

Now, if $b_k = 0$ for all $1 \le k \le r$, then $b_n = 0$ for all n, hence $a_n = sn$, and the statement is trivial. Otherwise, define

$$M = \max_{1\le i\le r} |b_i|, \quad \varepsilon = \min\{|b_i| : 1 \le i \le r,\ b_i < 0\}.$$

Then for $n > r$ we obtain

$$b_n = \max_{1\le k\le n-1}(b_k + b_{n-k}) \ge b_\ell + b_{n-\ell} = b_{n-\ell},$$

so

$$0 \ge b_n \ge b_{n-\ell} \ge b_{n-2\ell} \ge \dots \ge -M.$$

Thus, in view of the expansion (2), (3) applied to the sequence (b_n), we get that each b_n is contained in a set

$$T = \{b_{i_1} + b_{i_2} + \dots + b_{i_k} :\ i_1, \dots, i_k \le r\} \cap [-M, 0]$$

We claim that this set is finite. Actually, for any $x \in T$, let $x = b_{i_1} + \dots + b_{i_k}$ $(i_1, \dots, i_k \le r)$. Then among b_{i_j}'s there are at most $\frac{M}{\varepsilon}$ nonzero terms (otherwise $x < \frac{M}{\varepsilon} \cdot (-\varepsilon) < -M$). Thus x can be expressed in the same way with $k \le \frac{M}{\varepsilon}$, and there is only a finite number of such sums.

Finally, for every $t = 1, 2, \dots, \ell$ we get that the sequence

$$b_{r+t}, b_{r+t+\ell}, b_{r+t+2\ell}, \dots$$

is non-decreasing and attains the finite number of values; therefore it is constant from some index. Thus, the sequence (b_n) is periodic with period ℓ from some index N, which means that

$$b_n = b_{n-\ell} = b_{n-\ell} + b_\ell \qquad \text{for all } n > N + \ell,$$

and hence

$$a_n = b_n + ns = (b_{n-\ell} + (n-\ell)s) + (b_\ell + \ell s) = a_{n-\ell} + a_\ell \qquad \text{for all } n > N + \ell,$$

as desired.

A8. Given six positive numbers a, b, c, d, e, f such that $a < b < c < d < e < f$. Let $a+c+e = S$ and $b + d + f = T$. Prove that

$$2ST > \sqrt{3(S+T)\big(S(bd+bf+df)+T(ac+ae+ce)\big)}. \qquad (1)$$

(South Korea)

Solution 1. We define also $\sigma = ac + ce + ae$, $\tau = bd + bf + df$. The idea of the solution is to interpret (1) as a natural inequality on the roots of an appropriate polynomial.

Actually, consider the polynomial

$$\begin{aligned} P(x) &= (b+d+f)(x-a)(x-c)(x-e) + (a+c+e)(x-b)(x-d)(x-f) \\ &= T(x^3 - Sx^2 + \sigma x - ace) + S(x^3 - Tx^2 + \tau x - bdf). \qquad (2) \end{aligned}$$

Surely, P is cubic with leading coefficient $S + T > 0$. Moreover, we have

$$\begin{aligned} &P(a) = S(a-b)(a-d)(a-f) < 0, \qquad &&P(c) = S(c-b)(c-d)(c-f) > 0, \\ &P(e) = S(e-b)(e-d)(e-f) < 0, \qquad &&P(f) = T(f-a)(f-c)(f-e) > 0. \end{aligned}$$

Hence, each of the intervals (a, c), (c, e), (e, f) contains at least one root of $P(x)$. Since there are at most three roots at all, we obtain that there is exactly one root in each interval (denote them by $\alpha \in (a, c)$, $\beta \in (c, e)$, $\gamma \in (e, f)$). Moreover, the polynomial P can be factorized as

$$P(x) = (T+S)(x-\alpha)(x-\beta)(x-\gamma). \qquad (3)$$

Equating the coefficients in the two representations (2) and (3) of $P(x)$ provides

$$\alpha + \beta + \gamma = \frac{2TS}{T+S}, \qquad \alpha\beta + \alpha\gamma + \beta\gamma = \frac{S\tau + T\sigma}{T+S}.$$

Now, since the numbers α, β, γ are distinct, we have

$$0 < (\alpha-\beta)^2 + (\alpha-\gamma)^2 + (\beta-\gamma)^2 = 2(\alpha+\beta+\gamma)^2 - 6(\alpha\beta+\alpha\gamma+\beta\gamma),$$

which implies

$$\frac{4S^2T^2}{(T+S)^2} = (\alpha+\beta+\gamma)^2 > 3(\alpha\beta+\alpha\gamma+\beta\gamma) = \frac{3(S\tau+T\sigma)}{T+S},$$

or

$$4S^2T^2 > 3(T+S)(T\sigma+S\tau),$$

which is exactly what we need.

Comment 1. In fact, one can locate the roots of $P(x)$ more narrowly: they should lie in the intervals (a, b), (c, d), (e, f).

Surely, if we change all inequality signs in the problem statement to non-strict ones, the (non-strict) inequality will also hold by continuity. One can also find when the equality is achieved. This happens in that case when $P(x)$ is a perfect cube, which immediately implies that $b = c = d = e (= \alpha = \beta = \gamma)$, together with the additional condition that $P''(b) = 0$. Algebraically,

$$\begin{aligned} 6(T+S)b - 4TS = 0 \qquad &\Longleftrightarrow \qquad 3b(a+4b+f) = 2(a+2b)(2b+f) \\ &\Longleftrightarrow \qquad f = \frac{b(4b-a)}{2a+b} = b\left(1 + \frac{3(b-a)}{2a+b}\right) > b. \end{aligned}$$

This means that for every pair of numbers a, b such that $0 < a < b$, there exists $f > b$ such that the point (a, b, b, b, b, f) is a point of equality.

Solution 2. Let
$$U=\frac{1}{2}\big((e-a)^2+(c-a)^2+(e-c)^2\big)=S^2-3(ac+ae+ce)$$
and
$$V=\frac{1}{2}\big((f-b)^2+(f-d)^2+(d-b)^2\big)=T^2-3(bd+bf+df).$$
Then
$$\begin{aligned}(\text{L.H.S.})^2-(\text{R.H.S.})^2&=(2ST)^2-(S+T)\big(S\cdot 3(bd+bf+df)+T\cdot 3(ac+ae+ce)\big)=\\ &=4S^2T^2-(S+T)\big(S(T^2-V)+T(S^2-U)\big)=(S+T)(SV+TU)-ST(T-S)^2,\end{aligned}$$
and the statement is equivalent with
$$(S+T)(SV+TU)>ST(T-S)^2. \tag{4}$$
By the Cauchy-Schwarz inequality,
$$(S+T)(TU+SV)\ge\big(\sqrt{S\cdot TU}+\sqrt{T\cdot SV}\big)^2=ST\big(\sqrt{U}+\sqrt{V}\big)^2. \tag{5}$$
Estimate the quantities $\sqrt{U}$ and $\sqrt{V}$ by the QM–AM inequality with the positive terms $(e-c)^2$ and $(d-b)^2$ being omitted:
$$\begin{aligned}\sqrt{U}+\sqrt{V}&>\sqrt{\frac{(e-a)^2+(c-a)^2}{2}}+\sqrt{\frac{(f-b)^2+(f-d)^2}{2}}\\ &>\frac{(e-a)+(c-a)}{2}+\frac{(f-b)+(f-d)}{2}=\left(f-\frac{d}{2}-\frac{b}{2}\right)+\left(\frac{e}{2}+\frac{c}{2}-a\right)\\ &=(T-S)+\frac{3}{2}(e-d)+\frac{3}{2}(c-b)>T-S.\end{aligned} \tag{6}$$
The estimates (5) and (6) prove (4) and hence the statement.

Solution 3. We keep using the notations σ and τ from Solution 1. Moreover, let $s=c+e$. Note that
$$(c-b)(c-d)+(e-f)(e-d)+(e-f)(c-b)<0,$$
since each summand is negative. This rewrites as
$$(bd+bf+df)-(ac+ce+ae)<(c+e)(b+d+f-a-c-e),\ \text{or}$$
$$\tau-\sigma<s(T-S). \tag{7}$$
Then we have
$$\begin{aligned}S\tau+T\sigma&=S(\tau-\sigma)+(S+T)\sigma<Ss(T-S)+(S+T)(ce+as)\\ &\le Ss(T-S)+(S+T)\left(\frac{s^2}{4}+(S-s)s\right)=s\left(2ST-\frac{3}{4}(S+T)s\right).\end{aligned}$$
Using this inequality together with the AM–GM inequality we get
$$\begin{aligned}\sqrt{\frac{3}{4}(S+T)(S\tau+T\sigma)}&<\sqrt{\frac{3}{4}(S+T)s\left(2ST-\frac{3}{4}(S+T)s\right)}\\ &\le\frac{\frac{3}{4}(S+T)s+2ST-\frac{3}{4}(S+T)s}{2}=ST.\end{aligned}$$
Hence,
$$2ST>\sqrt{3(S+T)\big(S(bd+bf+df)+T(ac+ae+ce)\big)}.$$

Comment 2. The expression (7) can be found by considering the sum of the roots of the quadratic polynomial $q(x) = (x-b)(x-d)(x-f) - (x-a)(x-c)(x-e)$.

Solution 4. We introduce the expressions σ and τ as in the previous solutions. The idea of the solution is to change the values of variables $a, \dots, f$ keeping the left-hand side unchanged and increasing the right-hand side; it will lead to a simpler inequality which can be proved in a direct way.

Namely, we change the variables (i) keeping the (non-strict) inequalities $a \le b \le c \le d \le e \le f$; (ii) keeping the values of sums S and T unchanged; and finally (iii) increasing the values of σ and τ. Then the left-hand side of (1) remains unchanged, while the right-hand side increases. Hence, the inequality (1) (and even a non-strict version of (1)) for the changed values would imply the same (strict) inequality for the original values.

First, we find the sufficient conditions for (ii) and (iii) to be satisfied.

Lemma. Let $x, y, z > 0$; denote $U(x,y,z) = x+y+z$, $\upsilon(x,y,z) = xy+xz+yz$. Suppose that $x'+y' = x+y$ but $|x-y| \ge |x'-y'|$; then we have $U(x',y',z) = U(x,y,z)$ and $\upsilon(x',y',z) \ge \upsilon(x,y,z)$ with equality achieved only when $|x-y| = |x'-y'|$.

Proof. The first equality is obvious. For the second, we have

$$\begin{aligned}\upsilon(x',y',z) = z(x'+y') + x'y' &= z(x'+y') + \frac{(x'+y')^2 - (x'-y')^2}{4} \\ &\ge z(x+y) + \frac{(x+y)^2 - (x-y)^2}{4} = \upsilon(x,y,z),\end{aligned}$$

with the equality achieved only for $(x'-y')^2 = (x-y)^2 \iff |x'-y'| = |x-y|$, as desired. □

Now, we apply Lemma several times making the following changes. For each change, we denote the new values by the same letters to avoid cumbersome notations.

1. Let $k = \dfrac{d-c}{2}$. Replace (b,c,d,e) by $(b+k, c+k, d-k, e-k)$. After the change we have $a < b < c = d < e < f$, the values of S, T remain unchanged, but σ, τ strictly increase by Lemma.

2. Let $\ell = \dfrac{e-d}{2}$. Replace (c,d,e,f) by $(c+\ell, d+\ell, e-\ell, f-\ell)$. After the change we have $a < b < c = d = e < f$, the values of S, T remain unchanged, but σ, τ strictly increase by the Lemma.

3. Finally, let $m = \dfrac{c-b}{3}$. Replace (a,b,c,d,e,f) by $(a+2m, b+2m, c-m, d-m, e-m, f-m)$. After the change, we have $a < b = c = d = e < f$ and S, T are unchanged. To check (iii), we observe that our change can be considered as a composition of two changes: $(a,b,c,d) \to (a+m, b+m, c-m, d-m)$ and $(a,b,e,f) \to (a+m, b+m, e-m, f-m)$. It is easy to see that each of these two consecutive changes satisfy the conditions of the Lemma, hence the values of σ and τ increase.

Finally, we come to the situation when $a < b = c = d = e < f$, and we need to prove the inequality

$$\begin{aligned}2(a+2b)(2b+f) &\ge \sqrt{3(a+4b+f)\big((a+2b)(b^2+2bf) + (2b+f)(2ab+b^2)\big)} \\ &= \sqrt{3b(a+4b+f)\cdot\big((a+2b)(b+2f) + (2b+f)(2a+b)\big)}. \qquad (8)\end{aligned}$$

Now, observe that

$$2 \cdot 2(a+2b)(2b+f) = 3b(a+4b+f) + \big((a+2b)(b+2f) + (2a+b)(2b+f)\big).$$

Hence (4) rewrites as

$$\begin{aligned}3b(a+4b+f)&+\big((a+2b)(b+2f)+(2a+b)(2b+f)\big)\\&\geq 2\sqrt{3b(a+4b+f)\cdot\big((a+2b)(b+2f)+(2b+f)(2a+b)\big)},\end{aligned}$$

which is simply the AM–GM inequality.

Comment 3. Here, we also can find all the cases of equality. Actually, it is easy to see that if some two numbers among b, c, d, e are distinct then one can use Lemma to increase the right-hand side of (1). Further, if $b = c = d = e$, then we need equality in (4); this means that we apply AM–GM to equal numbers, that is,

$$3b(a+4b+f)=(a+2b)(b+2f)+(2a+b)(2b+f),$$

which leads to the same equality as in Comment 1.

Combinatorics

C1. In a concert, 20 singers will perform. For each singer, there is a (possibly empty) set of other singers such that he wishes to perform later than all the singers from that set. Can it happen that there are exactly 2010 orders of the singers such that all their wishes are satisfied?

(Austria)

Answer. Yes, such an example exists.

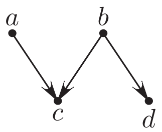
Solution. We say that an order of singers is *good* if it satisfied all their wishes. Next, we say that a number N is *realizable* by k singers (or k*-realizable*) if for some set of wishes of these singers there are exactly N good orders. Thus, we have to prove that a number 2010 is 20-realizable.

We start with the following simple

Lemma. Suppose that numbers n_1, n_2 are realizable by respectively k_1 and k_2 singers. Then the number $n_1 n_2$ is $(k_1 + k_2)$-realizable.

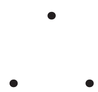
Proof. Let the singers $A_1, \ldots, A_{k_1}$ (with some wishes among them) realize n_1, and the singers B_1, $\ldots$, B_{k_2} (with some wishes among them) realize n_2. Add to each singer B_i the wish to perform later than all the singers A_j. Then, each good order of the obtained set of singers has the form $(A_{i_1}, \ldots, A_{i_{k_1}}, B_{j_1}, \ldots, B_{j_{k_2}})$, where $(A_{i_1}, \ldots, A_{i_{k_1}})$ is a good order for A_i's and $(B_{j_1}, \ldots, B_{j_{k_2}})$ is a good order for B_j's. Conversely, each order of this form is obviously good. Hence, the number of good orders is $n_1 n_2$. $\square$

In view of Lemma, we show how to construct sets of singers containing 4, 3 and 13 singers and realizing the numbers 5, 6 and 67, respectively. Thus the number $2010 = 6 \cdot 5 \cdot 67$ will be realizable by $4 + 3 + 13 = 20$ singers. These companies of singers are shown in Figs. 1–3; the wishes are denoted by arrows, and the number of good orders for each Figure stands below in the brackets.



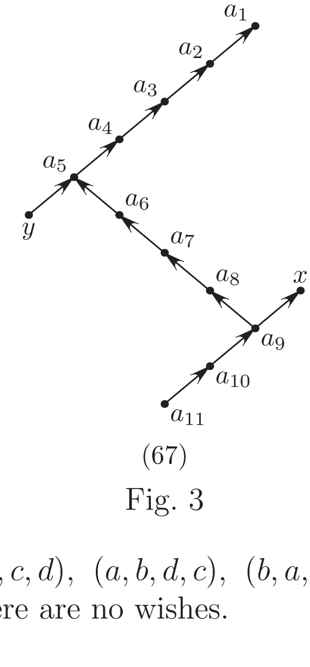
(5)

Fig. 1



(3)

Fig. 2



(67)

Fig. 3

For Fig. 1, there are exactly 5 good orders (a, b, c, d), (a, b, d, c), (b, a, c, d), (b, a, d, c), (b, d, a, c). For Fig. 2, each of 6 orders is good since there are no wishes.

Finally, for Fig. 3, the order of $a_1, \ldots, a_{11}$ is fixed; in this line, singer x can stand before each of a_i $(i \le 9)$, and singer y can stand after each of a_j $(j \ge 5)$, thus resulting in $9 \cdot 7 = 63$ cases. Further, the positions of x and y in this line determine the whole order uniquely unless both of them come between the same pair (a_i, a_{i+1}) (thus $5 \le i \le 8$); in the latter cases, there are two orders instead of 1 due to the order of x and y. Hence, the total number of good orders is $63 + 4 = 67$, as desired.

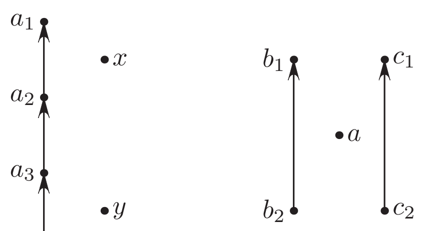
Comment. The number 20 in the problem statement is not sharp and is put there to respect the original formulation. So, if necessary, the difficulty level of this problem may be adjusted by replacing 20 by a smaller number. Here we present some improvements of the example leading to a smaller number of singers.

Surely, each example with < 20 singers can be filled with some "super-stars" who should perform at the very end in a fixed order. Hence each of these improvements provides a different solution for the problem. Moreover, the large variety of ideas standing behind these examples allows to suggest that there are many other examples.

1. Instead of building the examples realizing 5 and 6, it is more economic to make an example realizing 30; it may seem even simpler. Two possible examples consisting of 5 and 6 singers are shown in Fig. 4; hence the number 20 can be decreased to 19 or 18.

For Fig. 4a, the order of $a_1, \ldots, a_4$ is fixed, there are 5 ways to add x into this order, and there are 6 ways to add y into the resulting order of $a_1, \ldots, a_4, x$. Hence there are $5 \cdot 6 = 30$ good orders.

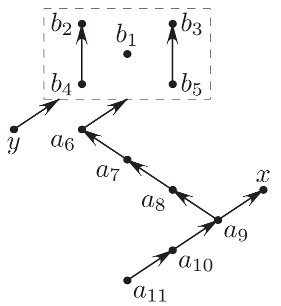
On Fig. 4b, for 5 singers a, b_1, b_2, c_1, c_2 there are $5! = 120$ orders at all. Obviously, exactly one half of them satisfies the wish $b_1 \leftarrow b_2$, and exactly one half of these orders satisfies another wish $c_1 \leftarrow c_2$; hence, there are exactly $5!/4 = 30$ good orders.



(30) (30)

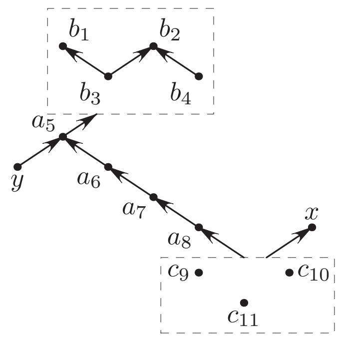
a) b)

Fig. 4



(2010)

Fig. 5



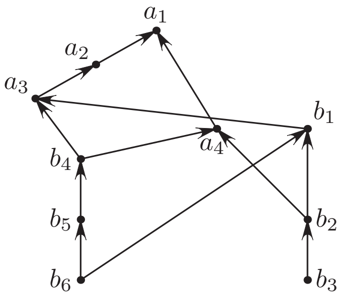
(2010)

Fig. 6

2. One can merge the examples for 30 and 67 shown in Figs. 4b and 3 in a smarter way, obtaining a set of 13 singers representing 2010. This example is shown in Fig. 5; an arrow from/to group $\{b_1, \ldots, b_5\}$ means that there exists such arrow from each member of this group.

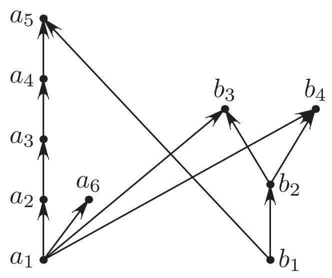
Here, as in Fig. 4b, one can see that there are exactly 30 orders of $b_1, \ldots, b_5, a_6, \ldots, a_{11}$ satisfying all their wishes among themselves. Moreover, one can prove in the same way as for Fig. 3 that each of these orders can be complemented by x and y in exactly 67 ways, hence obtaining $30 \cdot 67 = 2010$ good orders at all.

Analogously, one can merge the examples in Figs. 1–3 to represent 2010 by 13 singers as is shown in Fig. 6.



(67)

Fig. 7



(2010)

Fig. 8

3. Finally, we will present two other improvements; the proofs are left to the reader. The graph in Fig. 7 shows how 10 singers can represent 67. Moreover, even a company of 10 singers representing 2010 can be found; this company is shown in Fig. 8.

C2. On some planet, there are 2^N countries ($N \geq 4$). Each country has a flag N units wide and one unit high composed of N fields of size 1×1, each field being either yellow or blue. No two countries have the same flag.

We say that a set of N flags is *diverse* if these flags can be arranged into an $N \times N$ square so that all N fields on its main diagonal will have the same color. Determine the smallest positive integer M such that among any M distinct flags, there exist N flags forming a diverse set.

(Croatia)

Answer. $M = 2^{N-2} + 1$.

Solution. When speaking about the diagonal of a square, we will always mean the main diagonal.

Let M_N be the smallest positive integer satisfying the problem condition. First, we show that $M_N > 2^{N-2}$. Consider the collection of all 2^{N-2} flags having yellow first squares and blue second ones. Obviously, both colors appear on the diagonal of each $N \times N$ square formed by these flags.

We are left to show that $M_N \leq 2^{N-2} + 1$, thus obtaining the desired answer. We start with establishing this statement for $N = 4$.

Suppose that we have 5 flags of length 4. We decompose each flag into two parts of 2 squares each; thereby, we denote each flag as LR, where the 2×1 flags $L, R \in \mathcal{S} = \{\text{BB}, \text{BY}, \text{YB}, \text{YY}\}$ are its left and right parts, respectively. First, we make two easy observations on the flags 2×1 which can be checked manually.

(i) For each $A \in \mathcal{S}$, there exists only one 2×1 flag $C \in \mathcal{S}$ (possibly $C = A$) such that A and C cannot form a 2×2 square with monochrome diagonal (for part BB, that is YY, and for BY that is YB).

(ii) Let $A_1, A_2, A_3 \in \mathcal{S}$ be three distinct elements; then two of them can form a 2×2 square with yellow diagonal, and two of them can form a 2×2 square with blue diagonal (for all parts but BB, a pair (BY, YB) fits for both statements, while for all parts but BY, these pairs are (YB, YY) and (BB, YB)).

Now, let ℓ and r be the numbers of distinct left and right parts of our 5 flags, respectively. The total number of flags is $5 \leq r\ell$, hence one of the factors (say, r) should be at least 3. On the other hand, $\ell, r \leq 4$, so there are two flags with coinciding right part; let them be L_1R_1 and L_2R_1 ($L_1 \neq L_2$).

Next, since $r \geq 3$, there exist some flags L_3R_3 and L_4R_4 such that R_1, R_3, R_4 are distinct. Let $L'R'$ be the remaining flag. By (i), one of the pairs (L', L_1) and (L', L_2) can form a 2×2 square with monochrome diagonal; we can assume that L', L_2 form a square with a blue diagonal. Finally, the right parts of two of the flags L_1R_1, L_3R_3, L_4R_4 can also form a 2×2 square with a blue diagonal by (ii). Putting these 2×2 squares on the diagonal of a 4×4 square, we find a desired arrangement of four flags.

We are ready to prove the problem statement by induction on N; actually, above we have proved the base case $N = 4$. For the induction step, assume that $N > 4$, consider any $2^{N-2} + 1$ flags of length N, and arrange them into a large flag of size $(2^{N-2} + 1) \times N$. This flag contains a non-monochrome column since the flags are distinct; we may assume that this column is the first one. By the pigeonhole principle, this column contains at least $\left\lceil \frac{2^{N-2}+1}{2} \right\rceil = 2^{N-3} + 1$ squares of one color (say, blue). We call the flags with a blue first square *good*.

Consider all the good flags and remove the first square from each of them. We obtain at least $2^{N-3} + 1 \geq M_{N-1}$ flags of length $N - 1$; by the induction hypothesis, $N - 1$ of them can form a square Q with the monochrome diagonal. Now, returning the removed squares, we obtain a rectangle $(N-1) \times N$, and our aim is to supplement it on the top by one more flag.

If Q has a yellow diagonal, then we can take each flag with a yellow first square (it exists by a choice of the first column; moreover, it is not used in Q). Conversely, if the diagonal of Q is blue then we can take any of the $\geq 2^{N-3}+1-(N-1) > 0$ remaining good flags. So, in both cases we get a desired $N \times N$ square.

Solution 2. We present a different proof of the estimate $M_N \leq 2^{N-2}+1$. We do not use the induction, involving Hall's lemma on matchings instead.

Consider arbitrary $2^{N-2}+1$ distinct flags and arrange them into a large $(2^{N-2}+1) \times N$ flag. Construct two bipartite graphs $G_{\mathrm{y}} = (V \cup V', E_{\mathrm{y}})$ and $G_{\mathrm{b}} = (V \cup V', E_{\mathrm{b}})$ with the common set of vertices as follows. Let V and V' be the set of columns and the set of flags under consideration, respectively. Next, let the edge (c, f) appear in E_{y} if the intersection of column c and flag f is yellow, and $(c, f) \in E_{\mathrm{b}}$ otherwise. Then we have to prove exactly that one of the graphs G_{y} and G_{b} contains a matching with all the vertices of V involved.

Assume that these matchings do not exist. By Hall's lemma, it means that there exist two sets of columns $S_{\mathrm{y}}, S_{\mathrm{b}} \subset V$ such that $|E_{\mathrm{y}}(S_{\mathrm{y}})| \leq |S_{\mathrm{y}}|-1$ and $|E_{\mathrm{b}}(S_{\mathrm{b}})| \leq |S_{\mathrm{b}}|-1$ (in the left-hand sides, $E_{\mathrm{y}}(S_{\mathrm{y}})$ and $E_{\mathrm{b}}(S_{\mathrm{b}})$ denote respectively the sets of all vertices connected to S_{y} and S_{b} in the corresponding graphs). Our aim is to prove that this is impossible. Note that $S_{\mathrm{y}}, S_{\mathrm{b}} \neq V$ since $N \leq 2^{N-2}+1$.

First, suppose that $S_{\mathrm{y}} \cap S_{\mathrm{b}} \neq \varnothing$, so there exists some $c \in S_{\mathrm{y}} \cap S_{\mathrm{b}}$. Note that each flag is connected to c either in G_{y} or in G_{b}, hence $E_{\mathrm{y}}(S_{\mathrm{y}}) \cup E_{\mathrm{b}}(S_{\mathrm{b}}) = V'$. Hence we have $2^{N-2}+1 = |V'| \leq |E_{\mathrm{y}}(S_{\mathrm{y}})| + |E_{\mathrm{b}}(S_{\mathrm{b}})| \leq |S_{\mathrm{y}}| + |S_{\mathrm{b}}| - 2 \leq 2N-4$; this is impossible for $N \geq 4$.

So, we have $S_{\mathrm{y}} \cap S_{\mathrm{b}} = \varnothing$. Let $y = |S_{\mathrm{y}}|$, $b = |S_{\mathrm{b}}|$. From the construction of our graph, we have that all the flags in the set $V'' = V' \setminus \big(E_{\mathrm{y}}(S_{\mathrm{y}}) \cup E_{\mathrm{b}}(S_{\mathrm{b}})\big)$ have blue squares in the columns of S_{y} and yellow squares in the columns of S_{b}. Hence the only undetermined positions in these flags are the remaining $N-y-b$ ones, so $2^{N-y-b} \geq |V''| \geq |V'| - |E_{\mathrm{y}}(S_{\mathrm{y}})| - |E_{\mathrm{b}}(S_{\mathrm{b}})| \geq 2^{N-2}+1-(y-1)-(b-1)$, or, denoting $c = y+b$, $2^{N-c}+c > 2^{N-2}+2$. This is impossible since $N \geq c \geq 2$.

C3. 2500 chess kings have to be placed on a 100×100 chessboard so that

(i) no king can capture any other one (i.e. no two kings are placed in two squares sharing a common vertex);

(ii) each row and each column contains exactly 25 kings.

Find the number of such arrangements. (Two arrangements differing by rotation or symmetry are supposed to be different.)

(Russia)

Answer. There are two such arrangements.

Solution. Suppose that we have an arrangement satisfying the problem conditions. Divide the board into 2×2 pieces; we call these pieces *blocks*. Each block can contain not more than one king (otherwise these two kings would attack each other); hence, by the pigeonhole principle each block must contain exactly one king.

Now assign to each block a letter T or B if a king is placed in its top or bottom half, respectively. Similarly, assign to each block a letter L or R if a king stands in its left or right half. So we define *T-blocks*, *B-blocks*, *L-blocks*, and *R-blocks*. We also combine the letters; we call a block *a TL-block* if it is simultaneously T-block and L-block. Similarly we define *TR-blocks*, *BL-blocks*, and *BR-blocks*. The arrangement of blocks determines uniquely the arrangement of kings; so in the rest of the solution we consider the 50×50 system of blocks (see Fig. 1). We identify the blocks by their coordinate pairs; the pair (i, j), where $1 \leq i, j \leq 50$, refers to the jth block in the ith row (or the ith block in the jth column). The upper-left block is $(1, 1)$.

The system of blocks has the following properties..

(i′) If (i, j) is a B-block then $(i + 1, j)$ is a B-block: otherwise the kings in these two blocks can take each other. Similarly: if (i, j) is a T-block then $(i - 1, j)$ is a T-block; if (i, j) is an L-block then $(i, j - 1)$ is an L-block; if (i, j) is an R-block then $(i, j + 1)$ is an R-block.

(ii′) Each column contains exactly 25 L-blocks and 25 R-blocks, and each row contains exactly 25 T-blocks and 25 B-blocks. In particular, the total number of L-blocks (or R-blocks, or T-blocks, or B-blocks) is equal to $25 \cdot 50 = 1250$.

Consider any B-block of the form $(1, j)$. By (i′), all blocks in the jth column are B-blocks; so we call such a column *B-column*. By (ii′), we have 25 B-blocks in the first row, so we obtain 25 B-columns. These 25 B-columns contain 1250 B-blocks, hence all blocks in the remaining columns are T-blocks, and we obtain 25 *T-columns*. Similarly, there are exactly 25 *L-rows* and exactly 25 *R-rows*.

Now consider an arbitrary pair of a T-column and a neighboring B-column (columns with numbers j and $j + 1$).

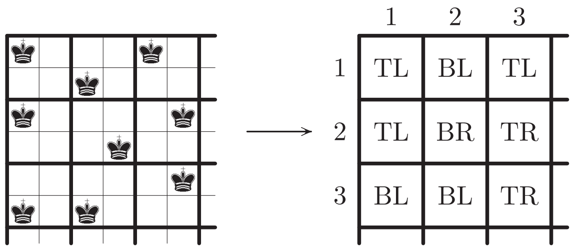


Fig. 1

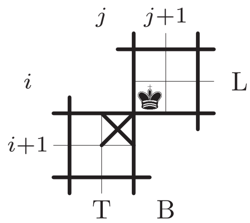


Fig. 2

Case 1. Suppose that the jth column is a T-column, and the $(j + 1)$th column is a B-column. Consider some index i such that the ith row is an L-row; then $(i, j + 1)$ is a BL-block. Therefore, $(i + 1, j)$ cannot be a TR-block (see Fig. 2), hence $(i + 1, j)$ is a TL-block, thus the

$(i+1)$th row is an L-row. Now, choosing the ith row to be the topmost L-row, we successively obtain that all rows from the ith to the 50th are L-rows. Since we have exactly 25 L-rows, it follows that the rows from the 1st to the 25th are R-rows, and the rows from the 26th to the 50th are L-rows.

Now consider the neighboring R-row and L-row (that are the rows with numbers 25 and 26). Replacing in the previous reasoning rows by columns and vice versa, the columns from the 1st to the 25th are T-columns, and the columns from the 26th to the 50th are B-columns. So we have a unique arrangement of blocks that leads to the arrangement of kings satisfying the condition of the problem (see Fig. 3).

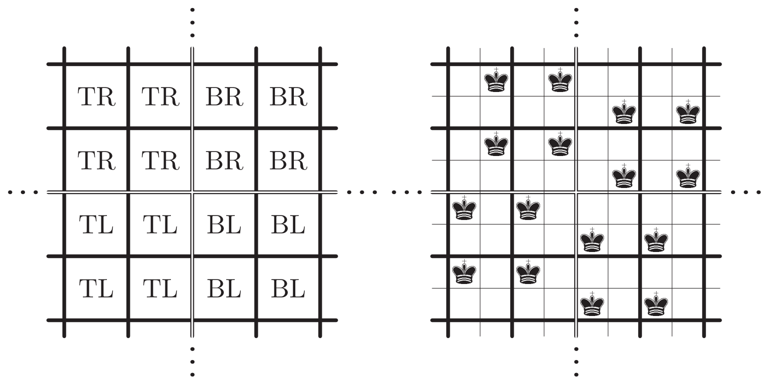


Fig. 3

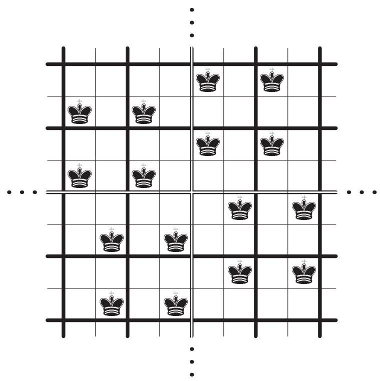


Fig. 4

Case 2. Suppose that the jth column is a B-column, and the $(j+1)$th column is a T-column. Repeating the arguments from Case 1, we obtain that the rows from the 1st to the 25th are L-rows (and all other rows are R-rows), the columns from the 1st to the 25th are B-columns (and all other columns are T-columns), so we find exactly one more arrangement of kings (see Fig. 4).

C4. Six stacks $S_1, \dots, S_6$ of coins are standing in a row. In the beginning every stack contains a single coin. There are two types of allowed moves:

Move 1: If stack S_k with $1 \le k \le 5$ contains at least one coin, you may remove one coin from S_k and add two coins to S_{k+1}.

Move 2: If stack S_k with $1 \le k \le 4$ contains at least one coin, then you may remove one coin from S_k and exchange stacks S_{k+1} and S_{k+2}.

Decide whether it is possible to achieve by a sequence of such moves that the first five stacks are empty, whereas the sixth stack S_6 contains exactly $2010^{2010^{2010}}$ coins.

C4′. Same as Problem C4, but the constant $2010^{2010^{2010}}$ is replaced by 2010^{2010}.

(Netherlands)

Answer. Yes (in both variants of the problem). There exists such a sequence of moves.

Solution. Denote by $(a_1, a_2, \dots, a_n) \to (a'_1, a'_2, \dots, a'_n)$ the following: if some consecutive stacks contain $a_1, \dots, a_n$ coins, then it is possible to perform several allowed moves such that the stacks contain $a'_1, \dots, a'_n$ coins respectively, whereas the contents of the other stacks remain unchanged.

Let $A = 2010^{2010}$ or $A = 2010^{2010^{2010}}$, respectively. Our goal is to show that

$$(1,1,1,1,1,1) \to (0,0,0,0,0,A).$$

First we prove two auxiliary observations.

Lemma 1. $(a,0,0) \to (0,2^a,0)$ for every $a \ge 1$.

Proof. We prove by induction that $(a,0,0) \to (a-k,2^k,0)$ for every $1 \le k \le a$. For $k=1$, apply Move 1 to the first stack:

$$(a,0,0) \to (a-1,2,0) = (a-1,2^1,0).$$

Now assume that $k < a$ and the statement holds for some $k < a$. Starting from $(a-k,2^k,0)$, apply Move 1 to the middle stack 2^k times, until it becomes empty. Then apply Move 2 to the first stack:

$$(a-k,2^k,0) \to (a-k,2^k-1,2) \to \cdots \to (a-k,0,2^{k+1}) \to (a-k-1,2^{k+1},0).$$

Hence,

$$(a,0,0) \to (a-k,2^k,0) \to (a-k-1,2^{k+1},0). \qquad \square$$

Lemma 2. For every positive integer n, let $P_n = \underbrace{2^{2^{\cdot^{\cdot^{2}}}}}_{n}$ (e.g. $P_3 = 2^{2^2} = 16$). Then $(a,0,0,0) \to (0,P_a,0,0)$ for every $a \ge 1$.

Proof. Similarly to Lemma 1, we prove that $(a,0,0,0) \to (a-k,P_k,0,0)$ for every $1 \le k \le a$.

For $k=1$, apply Move 1 to the first stack:

$$(a,0,0,0) \to (a-1,2,0,0) = (a-1,P_1,0,0).$$

Now assume that the lemma holds for some $k < a$. Starting from $(a-k,P_k,0,0)$, apply Lemma 1, then apply Move 1 to the first stack:

$$(a-k,P_k,0,0) \to (a-k,0,2^{P_k},0) = (a-k,0,P_{k+1},0) \to (a-k-1,P_{k+1},0,0).$$

Therefore,

$$(a,0,0,0) \to (a-k,P_k,0,0) \to (a-k-1,P_{k+1},0,0). \qquad \square$$

Now we prove the statement of the problem.

First apply Move 1 to stack 5, then apply Move 2 to stacks S_4, S_3, S_2 and S_1 in this order. Then apply Lemma 2 twice:

$$(1,1,1,1,1,1) \to (1,1,1,1,0,3) \to (1,1,1,0,3,0) \to (1,1,0,3,0,0) \to (1,0,3,0,0,0) \to \\ \to (0,3,0,0,0,0) \to (0,0,P_3,0,0,0) = (0,0,16,0,0,0) \to (0,0,0,P_{16},0,0).$$

We already have more than A coins in stack S_4, since

$$A \le 2010^{2010^{2010}} < (2^{11})^{2010^{2010}} = 2^{11\cdot 2010^{2010}} < 2^{2010^{2011}} < 2^{(2^{11})^{2011}} = 2^{2^{11\cdot 2011}} < 2^{2^{2^{15}}} < P_{16}.$$

To decrease the number of coins in stack S_4, apply Move 2 to this stack repeatedly until its size decreases to $A/4$. (In every step, we remove a coin from S_4 and exchange the empty stacks S_5 and S_6.)

$$(0,0,0,P_{16},0,0) \to (0,0,0,P_{16}-1,0,0) \to (0,0,0,P_{16}-2,0,0) \to \\ \to \cdots \to (0,0,0,A/4,0,0).$$

Finally, apply Move 1 repeatedly to empty stacks S_4 and S_5:

$$(0,0,0,A/4,0,0) \to \cdots \to (0,0,0,0,A/2,0) \to \cdots \to (0,0,0,0,0,A).$$

Comment 1. Starting with only 4 stack, it is not hard to check manually that we can achieve at most 28 coins in the last position. However, around 5 and 6 stacks the maximal number of coins explodes. With 5 stacks it is possible to achieve more than $2^{2^{14}}$ coins. With 6 stacks the maximum is greater than $P_{P_{2^{14}}}$.

It is not hard to show that the numbers 2010^{2010} and $2010^{2010^{2010}}$ in the problem can be replaced by any nonnegative integer up to $P_{P_{2^{14}}}$.

Comment 2. The simpler variant C4′ of the problem can be solved without Lemma 2:

$$(1,1,1,1,1,1) \to (0,3,1,1,1,1) \to (0,1,5,1,1,1) \to (0,1,1,9,1,1) \to \\ \to (0,1,1,1,17,1) \to (0,1,1,1,0,35) \to (0,1,1,0,35,0) \to (0,1,0,35,0,0) \to \\ \to (0,0,35,0,0,0) \to (0,0,1,2^{34},0,0) \to (0,0,1,0,2^{2^{34}},0) \to (0,0,0,2^{2^{34}},0,0) \\ \to (0,0,0,2^{2^{34}}-1,0,0) \to \ldots \to (0,0,0,A/4,0,0) \to (0,0,0,0,A/2,0) \to (0,0,0,0,0,A).$$

For this reason, the PSC suggests to consider the problem C4 as well. Problem C4 requires more invention and technical care. On the other hand, the problem statement in C4′ hides the fact that the resulting amount of coins can be such incredibly huge and leaves this discovery to the students.

C5. $n \geq 4$ players participated in a tennis tournament. Any two players have played exactly one game, and there was no tie game. We call a company of four players *bad* if one player was defeated by the other three players, and each of these three players won a game and lost another game among themselves. Suppose that there is no bad company in this tournament. Let w_i and ℓ_i be respectively the number of wins and losses of the ith player. Prove that

$$\sum_{i=1}^{n}(w_i - \ell_i)^3 \geq 0. \tag{1}$$

(South Korea)

Solution. For any tournament T satisfying the problem condition, denote by $S(T)$ sum under consideration, namely

$$S(T) = \sum_{i=1}^{n}(w_i - \ell_i)^3.$$

First, we show that the statement holds if a tournament T has only 4 players. Actually, let $A = (a_1, a_2, a_3, a_4)$ be the number of wins of the players; we may assume that $a_1 \geq a_2 \geq a_3 \geq a_4$. We have $a_1 + a_2 + a_3 + a_4 = \binom{4}{2} = 6$, hence $a_4 \leq 1$. If $a_4 = 0$, then we cannot have $a_1 = a_2 = a_3 = 2$, otherwise the company of all players is bad. Hence we should have $A = (3, 2, 1, 0)$, and $S(T) = 3^3 + 1^3 + (-1)^3 + (-3)^3 = 0$. On the other hand, if $a_4 = 1$, then only two possibilities, $A = (3, 1, 1, 1)$ and $A = (2, 2, 1, 1)$ can take place. In the former case we have $S(T) = 3^3 + 3 \cdot (-2)^3 > 0$, while in the latter one $S(T) = 1^3 + 1^3 + (-1)^3 + (-1)^3 = 0$, as desired.

Now we turn to the general problem. Consider a tournament T with no bad companies and enumerate the players by the numbers from 1 to n. For every 4 players i_1, i_2, i_3, i_4 consider a "sub-tournament" $T_{i_1 i_2 i_3 i_4}$ consisting of only these players and the games which they performed with each other. By the abovementioned, we have $S(T_{i_1 i_2 i_3 i_4}) \geq 0$. Our aim is to prove that

$$S(T) = \sum_{i_1, i_2, i_3, i_4} S(T_{i_1 i_2 i_3 i_4}), \tag{2}$$

where the sum is taken over all 4-tuples of distinct numbers from the set $\{1, \dots, n\}$. This way the problem statement will be established.

We interpret the number $(w_i - \ell_i)^3$ as following. For $i \neq j$, let $\varepsilon_{ij} = 1$ if the ith player wins against the jth one, and $\varepsilon_{ij} = -1$ otherwise. Then

$$(w_i - \ell_i)^3 = \left(\sum_{j \neq i} \varepsilon_{ij}\right)^3 = \sum_{j_1, j_2, j_3 \neq i} \varepsilon_{ij_1} \varepsilon_{ij_2} \varepsilon_{ij_3}.$$

Hence,

$$S(T) = \sum_{i \notin \{j_1, j_2, j_3\}} \varepsilon_{ij_1} \varepsilon_{ij_2} \varepsilon_{ij_3}.$$

To simplify this expression, consider all the terms in this sum where two indices are equal. If, for instance, $j_1 = j_2$, then the term contains $\varepsilon_{ij_1}^2 = 1$, so we can replace this term by ε_{ij_3}. Make such replacements for each such term; obviously, after this change each term of the form ε_{ij_3} will appear $P(T)$ times, hence

$$S(T) = \sum_{|\{i, j_1, j_2, j_3\}| = 4} \varepsilon_{ij_1} \varepsilon_{ij_2} \varepsilon_{ij_3} + P(T) \sum_{i \neq j} \varepsilon_{ij} = S_1(T) + P(T) S_2(T).$$

We show that $S_2(T)=0$ and hence $S(T)=S_1(T)$ for each tournament. Actually, note that $\varepsilon_{ij}=-\varepsilon_{ji}$, and the whole sum can be split into such pairs. Since the sum in each pair is 0, so is $S_2(T)$.

Thus the desired equality (2) rewrites as

$$S_1(T)=\sum_{i_1,i_2,i_3,i_4} S_1(T_{i_1i_2i_3i_4}). \tag{3}$$

Now, if all the numbers j_1,j_2,j_3 are distinct, then the set $\{i,j_1,j_2,j_3\}$ is contained in exactly one 4-tuple, hence the term $\varepsilon_{ij_1}\varepsilon_{ij_2}\varepsilon_{ij_3}$ appears in the right-hand part of (3) exactly once, as well as in the left-hand part. Clearly, there are no other terms in both parts, so the equality is established.

Solution 2. Similarly to the first solution, we call the subsets of players as *companies*, and the k-element subsets will be called as k*-companies*.

In any company of the players, call a player *the local champion of the company* if he defeated all other members of the company. Similarly, if a player lost all his games against the others in the company then call him *the local loser of the company*. Obviously every company has at most one local champion and at most one local loser. By the condition of the problem, whenever a 4-company has a local loser, then this company has a local champion as well.

Suppose that k is some positive integer, and let us count all cases when a player is the local champion of some k-company. The ith player won against w_i other player. To be the local champion of a k-company, he must be a member of the company, and the other $k-1$ members must be chosen from those whom he defeated. Therefore, the ith player is the local champion of $\binom{w_i}{k-1}$ k-companies. Hence, the total number of local champions of all k-companies is $\sum_{i=1}^{n}\binom{w_i}{k-1}$.

Similarly, the total number of local losers of the k-companies is $\sum_{i=1}^{n}\binom{\ell_i}{k-1}$.

Now apply this for $k=2,3$ and 4.

Since every game has a winner and a loser, we have $\sum_{i=1}^{n} w_i=\sum_{i=1}^{n}\ell_i=\binom{n}{2}$, and hence

$$\sum_{i=1}^{n}(w_i-\ell_i)=0. \tag{4}$$

In every 3-company, either the players defeated one another in a cycle or the company has both a local champion and a local loser. Therefore, the total number of local champions and local losers in the 3-companies is the same, $\sum_{i=1}^{n}\binom{w_i}{2}=\sum_{i=1}^{n}\binom{\ell_i}{2}$. So we have

$$\sum_{i=1}^{n}\left(\binom{w_i}{2}-\binom{\ell_i}{2}\right)=0. \tag{5}$$

In every 4-company, by the problem's condition, the number of local losers is less than or equal to the number of local champions. Then the same holds for the total numbers of local champions and local losers in all 4-companies, so $\sum_{i=1}^{n}\binom{w_i}{3} \geq \sum_{i=1}^{n}\binom{\ell_i}{3}$. Hence,

$$\sum_{i=1}^{n}\left(\binom{w_i}{3}-\binom{\ell_i}{3}\right) \geq 0. \tag{6}$$

Now we establish the problem statement (1) as a linear combination of (4), (5) and (6). It is easy check that

$$(x-y)^3 = 24\left(\binom{x}{3}-\binom{y}{3}\right)+24\left(\binom{x}{2}-\binom{y}{2}\right)-\big(3(x+y)^2-4\big)(x-y).$$

Apply this identity to $x = w_1$ and $y = \ell_i$. Since every player played $n-1$ games, we have $w_i + \ell_i = n-1$, and thus

$$(w_i-\ell_i)^3 = 24\left(\binom{w_i}{3}-\binom{\ell_i}{3}\right)+24\left(\binom{w_i}{2}-\binom{\ell_i}{2}\right)-\big(3(n-1)^2-4\big)\Big(w_i-\ell_i\Big).$$

Then

$$\sum_{i=1}^{n}(w_i-\ell_i)^3 = 24\underbrace{\sum_{i=1}^{n}\left(\binom{w_i}{3}-\binom{\ell_i}{3}\right)}_{\geq 0}+24\underbrace{\sum_{i=1}^{n}\left(\binom{w_i}{2}-\binom{\ell_i}{2}\right)}_{0}-\big(3(n-1)^2-4\big)\underbrace{\sum_{i=1}^{n}\Big(w_i-\ell_i\Big)}_{0} \geq 0.$$

C6. Given a positive integer k and other two integers $b > w > 1$. There are two strings of pearls, a string of b black pearls and a string of w white pearls. The *length* of a string is the number of pearls on it.

One cuts these strings in some steps by the following rules. In each step:

(i) The strings are ordered by their lengths in a non-increasing order. If there are some strings of equal lengths, then the white ones precede the black ones. Then k first ones (if they consist of more than one pearl) are chosen; if there are less than k strings longer than 1, then one chooses all of them.

(ii) Next, one cuts each chosen string into two parts differing in length by at most one.

(For instance, if there are strings of 5, 4, 4, 2 black pearls, strings of 8, 4, 3 white pearls and $k = 4$, then the strings of 8 white, 5 black, 4 white and 4 black pearls are cut into the parts $(4, 4)$, $(3, 2)$, $(2, 2)$ and $(2, 2)$, respectively.)

The process stops immediately after the step when a first isolated white pearl appears. Prove that at this stage, there will still exist a string of at least two black pearls.

(Canada)

Solution 1. Denote the situation after the ith step by A_i; hence A_0 is the initial situation, and $A_{i-1} \to A_i$ is the ith step. We call a string containing m pearls an m-string; it is an *m-w-string* or a *m-b-string* if it is white or black, respectively.

We continue the process until every string consists of a single pearl. We will focus on three moments of the process: (a) the first stage A_s when the first 1-string (no matter black or white) appears; (b) the first stage A_t where the total number of strings is greater than k (if such moment does not appear then we put $t = \infty$); and (c) the first stage A_f when all black pearls are isolated. It is sufficient to prove that in A_{f-1} (or earlier), a 1-w-string appears.

We start with some easy properties of the situations under consideration. Obviously, we have $s \leq f$. Moreover, all b-strings from A_{f-1} become single pearls in the fth step, hence all of them are 1- or 2-b-strings.

Next, observe that in each step $A_i \to A_{i+1}$ with $i \leq t-1$, all (>1)-strings were cut since there are not more than k strings at all; if, in addition, $i < s$, then there were no 1-string, so all the strings were cut in this step.

Now, let B_i and b_i be the lengths of the longest and the shortest b-strings in A_i, and let W_i and w_i be the same for w-strings. We show by induction on $i \leq \min\{s, t\}$ that (i) the situation A_i contains exactly 2^i black and 2^i white strings, (ii) $B_i \geq W_i$, and (iii) $b_i \geq w_i$. The base case $i = 0$ is obvious. For the induction step, if $i \leq \min\{s, t\}$ then in the ith step, each string is cut, thus the claim (i) follows from the induction hypothesis; next, we have $B_i = \lceil B_{i-1}/2 \rceil \geq \lceil W_{i-1}/2 \rceil = W_i$ and $b_i = \lfloor b_{i-1}/2 \rfloor \geq \lfloor w_{i-1}/2 \rfloor = w_i$, thus establishing (ii) and (iii).

For the numbers s, t, f, two cases are possible.

Case 1. Suppose that $s \leq t$ or $f \leq t+1$ (and hence $s \leq t+1$); in particular, this is true when $t = \infty$. Then in A_{s-1} we have $B_{s-1} \geq W_{s-1}$, $b_{s-1} \geq w_{s-1} > 1$ as $s - 1 \leq \min\{s, t\}$. Now, if $s = f$, then in A_{s-1}, there is no 1-w-string as well as no (>2)-b-string. That is, $2 = B_{s-1} \geq W_{s-1} \geq b_{s-1} \geq w_{s-1} > 1$, hence all these numbers equal 2. This means that in A_{s-1}, all strings contain 2 pearls, and there are 2^{s-1} black and 2^{s-1} white strings, which means $b = 2 \cdot 2^{s-1} = w$. This contradicts the problem conditions.

Hence we have $s \leq f - 1$ and thus $s \leq t$. Therefore, in the sth step each string is cut into two parts. Now, if a 1-b-string appears in this step, then from $w_{s-1} \leq b_{s-1}$ we see that a

1-w-string appears as well; so, in each case in the sth step a 1-w-string appears, while not all black pearls become single, as desired.

Case 2. Now assume that $t+1 \leq s$ and $t+2 \leq f$. Then in A_t we have exactly 2^t white and 2^t black strings, all being larger than 1, and $2^{t+1} > k \geq 2^t$ (the latter holds since 2^t is the total number of strings in A_{t-1}). Now, in the $(t+1)$st step, exactly k strings are cut, not more than 2^t of them being black; so the number of w-strings in A_{t+1} is at least $2^t + (k - 2^t) = k$. Since the number of w-strings does not decrease in our process, in A_{f-1} we have at least k white strings as well.

Finally, in A_{f-1}, all b-strings are not larger than 2, and at least one 2-b-string is cut in the fth step. Therefore, at most $k-1$ white strings are cut in this step, hence there exists a w-string $\mathcal{W}$ which is not cut in the fth step. On the other hand, since a 2-b-string is cut, all (≥ 2)-w-strings should also be cut in the fth step; hence $\mathcal{W}$ should be a single pearl. This is exactly what we needed.

Comment. In this solution, we used the condition $b \neq w$ only to avoid the case $b = w = 2^t$. Hence, if a number $b = w$ is not a power of 2, then the problem statement is also valid.

Solution 2. We use the same notations as introduced in the first paragraph of the previous solution. We claim that at every stage, there exist a u-b-string and a v-w-string such that either

(i) $u > v \geq 1$, or

(ii) $2 \leq u \leq v < 2u$, and there also exist $k-1$ of $(>v/2)$-strings other than considered above.

First, we notice that this statement implies the problem statement. Actually, in both cases (i) and (ii) we have $u > 1$, so at each stage there exists a (≥ 2)-b-string, and for the last stage it is exactly what we need.

Now, we prove the claim by induction on the number of the stage. Obviously, for A_0 the condition (i) holds since $b > w$. Further, we suppose that the statement holds for A_i, and prove it for A_{i+1}. Two cases are possible.

Case 1. Assume that in A_i, there are a u-b-string and a v-w-string with $u > v$. We can assume that v is the length of the shortest w-string in A_i; since we are not at the final stage, we have $v \geq 2$. Now, in the $(i+1)$st step, two subcases may occur.

Subcase 1a. Suppose that either no u-b-string is cut, or both some u-b-string and some v-w-string are cut. Then in A_{i+1}, we have either a u-b-string and a $(\leq v)$-w-string (and (i) is valid), or we have a $\lceil u/2 \rceil$-b-string and a $\lfloor v/2 \rfloor$-w-string. In the latter case, from $u > v$ we get $\lceil u/2 \rceil > \lfloor v/2 \rfloor$, and (i) is valid again.

Subcase 1b. Now, some u-b-string is cut, and no v-w-string is cut (and hence all the strings which are cut are longer than v). If $u' = \lceil u/2 \rceil > v$, then the condition (i) is satisfied since we have a u'-b-string and a v-w-string in A_{i+1}. Otherwise, notice that the inequality $u > v \geq 2$ implies $u' \geq 2$. Furthermore, besides a fixed u-b-string, other $k-1$ of $(\geq v+1)$-strings should be cut in the $(i+1)$st step, hence providing at least $k-1$ of $(\geq \lceil (v+1)/2 \rceil)$-strings, and $\lceil (v+1)/2 \rceil > v/2$. So, we can put $v' = v$, and we have $u' \leq v < u \leq 2u'$, so the condition (ii) holds for A_{i+1}.

Case 2. Conversely, assume that in A_i there exist a u-b-string, a v-w-string $(2 \leq u \leq v < 2u)$ and a set S of $k-1$ other strings larger than $v/2$ (and hence larger than 1). In the $(i+1)$st step, three subcases may occur.

Subcase 2a. Suppose that some u-b-string is not cut, and some v-w-string is cut. The latter one results in a $\lfloor v/2 \rfloor$-w-string, we have $v' = \lfloor v/2 \rfloor < u$, and the condition (i) is valid.

Subcase 2b. Next, suppose that no v-w-string is cut (and therefore no u-b-string is cut as $u \leq v$). Then all k strings which are cut have the length $> v$, so each one results in a $(>v/2)$-string. Hence in A_{i+1}, there exist $k \geq k-1$ of $(>v/2)$-strings other than the considered u- and v-strings, and the condition (ii) is satisfied.

Subcase 2c. In the remaining case, all u-b-strings are cut. This means that all $(\geq u)$-strings are cut as well, hence our v-w-string is cut. Therefore in A_{i+1} there exists a $\lceil u/2 \rceil$-b-string together with a $\lfloor v/2 \rfloor$-w-string. Now, if $u' = \lceil u/2 \rceil > \lfloor v/2 \rfloor = v'$ then the condition (i) is fulfilled. Otherwise, we have $u' \leq v' < u \leq 2u'$. In this case, we show that $u' \geq 2$. If, to the contrary, $u' = 1$ (and hence $u = 2$), then all black and white (≥ 2)-strings should be cut in the $(i+1)$st step, and among these strings there are at least a u-b-string, a v-w-string, and $k-1$ strings in S ($k+1$ strings altogether). This is impossible.

Hence, we get $2 \leq u' \leq v' < 2u'$. To reach (ii), it remains to check that in A_{i+1}, there exists a set S' of $k-1$ other strings larger than $v'/2$. These will be exactly the strings obtained from the elements of S. Namely, each $s \in S$ was either cut in the $(i+1)$st step, or not. In the former case, let us include into S' the largest of the strings obtained from s; otherwise we include s itself into S'. All $k-1$ strings in S' are greater than $v/2 \geq v'$, as desired.

C7. Let $P_1, \dots, P_s$ be arithmetic progressions of integers, the following conditions being satisfied:

(i) each integer belongs to at least one of them;

(ii) each progression contains a number which does not belong to other progressions.

Denote by n the least common multiple of steps of these progressions; let $n = p_1^{\alpha_1} \dots p_k^{\alpha_k}$ be its prime factorization. Prove that

$$s \geq 1 + \sum_{i=1}^{k} \alpha_i (p_i - 1).$$

(Germany)

Solution 1. First, we prove the key lemma, and then we show how to apply it to finish the solution.

Let $n_1, \dots, n_k$ be positive integers. By an $n_1 \times n_2 \times \cdots \times n_k$ *grid* we mean the set $N = \{(a_1, \dots, a_k) : a_i \in \mathbb{Z},\ 0 \leq a_i \leq n_i - 1\}$; the elements of N will be referred to as *points*. In this grid, we define *a subgrid* as a subset of the form

$$L = \{(b_1, \dots, b_k) \in N : b_{i_1} = x_{i_1}, \dots, b_{i_t} = x_{i_t}\}, \tag{1}$$

where $I = \{i_1, \dots, i_t\}$ is an arbitrary nonempty set of indices, and $x_{i_j} \in [0, n_{i_j} - 1]$ $(1 \leq j \leq t)$ are fixed integer numbers. Further, we say that a subgrid (1) is *orthogonal* to the ith coordinate axis if $i \in I$, and that it is *parallel* to the ith coordinate axis otherwise.

Lemma. Assume that the grid N is covered by subgrids $L_1, L_2, \dots, L_s$ (this means $N = \bigcup_{i=1}^{s} L_i$) so that

(ii′) each subgrid contains a point which is not covered by other subgrids;

(iii) for each coordinate axis, there exists a subgrid L_i orthogonal to this axis.

Then

$$s \geq 1 + \sum_{i=1}^{k} (n_i - 1).$$

Proof. Assume to the contrary that $s \leq \sum_i (n_i - 1) = s'$. Our aim is to find a point that is not covered by $L_1, \dots, L_s$.

The idea of the proof is the following. Imagine that we expand each subgrid to some maximal subgrid so that for the ith axis, there will be at most $n_i - 1$ maximal subgrids orthogonal to this axis. Then the desired point can be found easily: its ith coordinate should be that not covered by the maximal subgrids orthogonal to the ith axis. Surely, the conditions for existence of such expansion are provided by Hall's lemma on matchings. So, we will follow this direction, although we will apply Hall's lemma to some subgraph instead of the whole graph.

Construct a bipartite graph $G = (V \cup V', E)$ as follows. Let $V = \{L_1, \dots, L_s\}$, and let $V' = \{v_{ij} : 1 \leq i \leq s,\ 1 \leq j \leq n_i - 1\}$ be some set of s' elements. Further, let the edge (L_m, v_{ij}) appear iff L_m is orthogonal to the ith axis.

For each subset $W \subset V$, denote

$$f(W) = \{v \in V' : (L, v) \in E \text{ for some } L \in W\}.$$

Notice that $f(V) = V'$ by (iii).

Now, consider the set $W \subset V$ containing the maximal number of elements such that $|W| > |f(W)|$; if there is no such set then we set $W = \varnothing$. Denote $W' = f(W)$, $U = V \backslash W$, $U' = V' \backslash W'$.

By our assumption and the Lemma condition, $|f(V)| = |V'| \ge |V|$, hence $W \ne V$ and $U \ne \varnothing$. Permuting the coordinates, we can assume that $U' = \{v_{ij} : 1 \le i \le \ell\}$, $W' = \{v_{ij} : \ell{+}1 \le i \le k\}$.

Consider the induced subgraph G' of G on the vertices $U \cup U'$. We claim that for every $X \subset U$, we get $|f(X) \cap U'| \ge |X|$ (so G' satisfies the conditions of Hall's lemma). Actually, we have $|W| \ge |f(W)|$, so if $|X| > |f(X) \cap U'|$ for some $X \subset U$, then we have

$$|W \cup X| = |W| + |X| > |f(W)| + |f(X) \cap U'| = |f(W) \cup (f(X) \cap U')| = |f(W \cup X)|.$$

This contradicts the maximality of $|W|$.

Thus, applying Hall's lemma, we can assign to each $L \in U$ some vertex $v_{ij} \in U'$ so that to distinct elements of U, distinct vertices of U' are assigned. In this situation, we say that $L \in U$ corresponds to the ith axis, and write $g(L) = i$. Since there are $n_i - 1$ vertices of the form v_{ij}, we get that for each $1 \le i \le \ell$, not more than $n_i - 1$ subgrids correspond to the ith axis.

Finally, we are ready to present the desired point. Since $W \ne V$, there exists a point $b = (b_1, b_2, \dots, b_k) \in N \backslash (\cup_{L \in W} L)$. On the other hand, for every $1 \le i \le \ell$, consider any subgrid $L \in U$ with $g(L) = i$. This means exactly that L is orthogonal to the ith axis, and hence all its elements have the same ith coordinate c_L. Since there are at most $n_i - 1$ such subgrids, there exists a number $0 \le a_i \le n_i - 1$ which is not contained in a set $\{c_L : g(L) = i\}$. Choose such number for every $1 \le i \le \ell$. Now we claim that point $a = (a_1, \dots, a_\ell, b_{\ell+1}, \dots, b_k)$ is not covered, hence contradicting the Lemma condition.

Surely, point a cannot lie in some $L \in U$, since all the points in L have $g(L)$th coordinate $c_L \ne a_{g(L)}$. On the other hand, suppose that $a \in L$ for some $L \in W$; recall that $b \notin L$. But the points a and b differ only at first ℓ coordinates, so L should be orthogonal to at least one of the first ℓ axes, and hence our graph contains some edge (L, v_{ij}) for $i \le \ell$. It contradicts the definition of W'. The Lemma is proved. $\square$

Now we turn to the problem. Let d_j be the step of the progression P_j. Note that since $n = \text{l.c.m.}(d_1, \dots, d_s)$, for each $1 \le i \le k$ there exists an index $j(i)$ such that $p_i^{\alpha_i} \mid d_{j(i)}$. We assume that $n > 1$; otherwise the problem statement is trivial.

For each $0 \le m \le n-1$ and $1 \le i \le k$, let m_i be the residue of m modulo $p_i^{\alpha_i}$, and let $m_i = \overline{r_{i\alpha_i} \dots r_{i1}}$ be the base p_i representation of m_i (possibly, with some leading zeroes). Now, we put into correspondence to m the sequence $r(m) = (r_{11}, \dots, r_{1\alpha_1}, r_{21}, \dots, r_{k\alpha_k})$. Hence $r(m)$ lies in a $\underbrace{p_1 \times \dots \times p_1}_{\alpha_1 \text{ times}} \times \dots \times \underbrace{p_k \times \dots \times p_k}_{\alpha_k \text{ times}}$ grid N.

Surely, if $r(m) = r(m')$ then $p_i^{\alpha_i} \mid m_i - m_i'$, which follows $p_i^{\alpha_i} \mid m - m'$ for all $1 \le i \le k$; consequently, $n \mid m - m'$. So, when m runs over the set $\{0, \dots, n-1\}$, the sequences $r(m)$ do not repeat; since $|N| = n$, this means that r is a bijection between $\{0, \dots, n-1\}$ and N. Now we will show that for each $1 \le i \le s$, the set $L_i = \{r(m) : m \in P_i\}$ is a subgrid, and that for each axis there exists a subgrid orthogonal to this axis. Obviously, these subgrids cover N, and the condition (ii′) follows directly from (ii). Hence the Lemma provides exactly the estimate we need.

Consider some $1 \le j \le s$ and let $d_j = p_1^{\gamma_1} \dots p_k^{\gamma_k}$. Consider some $q \in P_j$ and let $r(q) = (r_{11}, \dots, r_{k\alpha_k})$. Then for an arbitrary q', setting $r(q') = (r'_{11}, \dots, r'_{k\alpha_k})$ we have

$$q' \in P_j \quad\iff\quad p_i^{\gamma_i} \mid q - q' \text{ for each } 1 \le i \le k \quad\iff\quad r_{i,t} = r'_{i,t} \text{ for all } t \le \gamma_i.$$

Hence $L_j = \{(r'_{11}, \dots, r'_{k\alpha_k}) \in N : r_{i,t} = r'_{i,t} \text{ for all } t \le \gamma_i\}$ which means that L_j is a subgrid containing $r(q)$. Moreover, in $L_{j(i)}$, all the coordinates corresponding to p_i are fixed, so it is orthogonal to all of their axes, as desired.

Comment 1. The estimate in the problem is sharp for every n. One of the possible examples is the following one. For each $1 \le i \le k$, $0 \le j \le \alpha_i - 1$, $1 \le k \le p-1$, let

$$P_{i,j,k} = kp_i^j + p_i^{j+1}\mathbb{Z},$$

and add the progression $P_0 = n\mathbb{Z}$. One can easily check that this set satisfies all the problem conditions. There also exist other examples.

On the other hand, the estimate can be adjusted in the following sense. For every $1 \le i \le k$, let $0 = \alpha_{i0}, \alpha_{i1}, \dots, \alpha_{ih_i}$ be all the numbers of the form $\operatorname{ord}_{p_i}(d_j)$ in an increasing order (we delete the repeating occurences of a number, and add a number $0 = \alpha_{i0}$ if it does not occur). Then, repeating the arguments from the solution one can obtain that

$$s \ge 1 + \sum_{i=1}^{k}\sum_{j=1}^{h_i}(p^{\alpha_j - \alpha_{j-1}} - 1).$$

Note that $p^\alpha - 1 \ge \alpha(p-1)$, and the equality is achieved only for $\alpha = 1$. Hence, for reaching the minimal number of the progressions, one should have $\alpha_{i,j} = j$ for all i, j. In other words, for each $1 \le j \le \alpha_i$, there should be an index t such that $\operatorname{ord}_{p_i}(d_t) = j$.

Solution 2. We start with introducing some notation. For positive integer r, we denote $[r] = \{1, 2, \dots, r\}$. Next, we say that a set of progressions $\mathcal{P} = \{P_1, \dots, P_s\}$ *cover* $\mathbb{Z}$ if each integer belongs to some of them; we say that this covering is *minimal* if no proper subset of $\mathcal{P}$ covers $\mathbb{Z}$. Obviously, each covering contains a minimal subcovering.

Next, for a minimal covering $\{P_1, \dots, P_s\}$ and for every $1 \le i \le s$, let d_i be the step of progression P_i, and h_i be some number which is contained in P_i but in none of the other progressions. We assume that $n > 1$, otherwise the problem is trivial. This implies $d_i > 1$, otherwise the progression P_i covers all the numbers, and $n = 1$.

We will prove a more general statement, namely the following

Claim. Assume that the progressions $P_1, \dots, P_s$ and number $n = p_1^{\alpha_1} \dots p_k^{\alpha_k} > 1$ are chosen as in the problem statement. Moreover, choose some nonempty set of indices $I = \{i_1, \dots, i_t\} \subseteq [k]$ and some positive integer $\beta_i \le \alpha_i$ for every $i \in I$. Consider the set of indices

$$T = \left\{ j : 1 \le j \le s, \text{and } p_i^{\alpha_i - \beta_i + 1} \mid d_j \text{ for some } i \in I \right\}.$$

Then

$$|T| \ge 1 + \sum_{i \in I} \beta_i(p_i - 1). \tag{2}$$

Observe that the Claim for $I = [k]$ and $\beta_i = \alpha_i$ implies the problem statement, since the left-hand side in (2) is not greater than s. Hence, it suffices to prove the Claim.

1. First, we prove the Claim assuming that all d_j's are prime numbers. If for some $1 \le i \le k$ we have at least p_i progressions with the step p_i, then they do not intersect and hence cover all the integers; it means that there are no other progressions, and $n = p_i$; the Claim is trivial in this case.

Now assume that for every $1 \le i \le k$, there are not more than $p_i - 1$ progressions with step p_i; each such progression covers the numbers with a fixed residue modulo p_i, therefore there exists a residue $q_i \mod p_i$ which is not touched by these progressions. By the Chinese Remainder Theorem, there exists a number q such that $q \equiv q_i \pmod{p_i}$ for all $1 \le i \le k$; this number cannot be covered by any progression with step p_i, hence it is not covered at all. A contradiction.

2. Now, we assume that the general Claim is not valid, and hence we consider a counterexample $\{P_1, \dots, P_s\}$ for the Claim; we can choose it to be minimal in the following sense:

• the number n is minimal possible among all the counterexamples;

• the sum $\sum_i d_i$ is minimal possible among all the counterexamples having the chosen value of n.

As was mentioned above, not all numbers d_i are primes; hence we can assume that d_1 is composite, say $p_1 \mid d_1$ and $d_1' = \frac{d_1}{p_1} > 1$. Consider a progression P_1' having the step d_1', and containing P_1. We will focus on two coverings constructed as follows.

(i) Surely, the progressions $P_1', P_2, \dots, P_s$ cover $\mathbb{Z}$, though this covering in not necessarily minimal. So, choose some minimal subcovering $\mathcal{P}'$ in it; surely $P_1' \in \mathcal{P}'$ since h_1 is not covered by $P_2, \dots, P_s$, so we may assume that $\mathcal{P}' = \{P_1', P_2, \dots, P_{s'}\}$ for some $s' \le s$. Furthermore, the period of the covering $\mathcal{P}'$ can appear to be less than n; so we denote this period by

$$n' = p_1^{\alpha_1 - \sigma_1} \dots p_k^{\alpha_k - \sigma_k} = \text{l.c.m.}\big(d_1', d_2, \dots, d_{s'}\big).$$

Observe that for each $P_j \notin \mathcal{P}'$, we have $h_j \in P_1'$, otherwise h_j would not be covered by $\mathcal{P}$.

(ii) On the other hand, each nonempty set of the form $R_i = P_i \cap P_1'$ $(1 \le i \le s)$ is also a progression with a step $r_i = \text{l.c.m.}(d_i, d_1')$, and such sets cover P_1'. Scaling these progressions with the ratio $1/d_1'$, we obtain the progressions Q_i with steps $q_i = r_i/d_1'$ which cover $\mathbb{Z}$. Now we choose a minimal subcovering $\mathcal{Q}$ of this covering; again we should have $Q_1 \in \mathcal{Q}$ by the reasons of h_1. Now, denote the period of $\mathcal{Q}$ by

$$n'' = \text{l.c.m.}\{q_i : Q_i \in \mathcal{Q}\} = \frac{\text{l.c.m.}\{r_i : Q_i \in \mathcal{Q}\}}{d_1'} = \frac{p_1^{\gamma_1} \dots p_k^{\gamma_k}}{d_1'}.$$

Note that if $h_j \in P_1'$, then the image of h_j under the scaling can be covered by Q_j only; so, in this case we have $Q_j \in \mathcal{Q}$.

Our aim is to find the desired number of progressions in coverings $\mathcal{P}$ and $\mathcal{Q}$. First, we have $n \ge n'$, and the sum of the steps in $\mathcal{P}'$ is less than that in $\mathcal{P}$; hence the Claim is valid for $\mathcal{P}'$. We apply it to the set of indices $I' = \{i \in I : \beta_i > \sigma_i\}$ and the exponents $\beta_i' = \beta_i - \sigma_i$; hence the set under consideration is

$$T' = \Big\{j : 1 \le j \le s', \text{and } p_i^{(\alpha_i - \sigma_i) - \beta_i' + 1} = p_i^{\alpha_i - \beta_i + 1} \mid d_j \text{ for some } i \in I'\Big\} \subseteq T \cap [s'],$$

and we obtain that

$$|T \cap [s']| \ge |T'| \ge 1 + \sum_{i \in I'} (\beta_i - \sigma_i)(p_i - 1) = 1 + \sum_{i \in I} (\beta_i - \sigma_i)_+ (p_i - 1),$$

where $(x)_+ = \max\{x, 0\}$; the latter equality holds as for $i \notin I'$ we have $\beta_i \le \sigma_i$.

Observe that $x = (x - y)_+ + \min\{x, y\}$ for all x, y. So, if we find at least

$$G = \sum_{i \in I} \min\{\beta_i, \sigma_i\}(p_i - 1)$$

indices in $T \cap \{s' + 1, \dots, s\}$, then we would have

$$|T| = |T \cap [s']| + |T \cap \{s' + 1, \dots, s\}| \ge 1 + \sum_{i \in I} \big((\beta_i - \sigma_i)_+ + \min\{\beta_i, \sigma_i\}\big)(p_i - 1) = 1 + \sum_{i \in I} \beta_i (p_i - 1),$$

thus leading to a contradiction with the choice of $\mathcal{P}$. We will find those indices among the indices of progressions in $\mathcal{Q}$.

3. Now denote $I'' = \{i \in I : \sigma_i > 0\}$ and consider some $i \in I''$; then $p_i^{\alpha_i} \nmid n'$. On the other hand, there exists an index $j(i)$ such that $p_i^{\alpha_i} \mid d_{j(i)}$; this means that $d_{j(i)} \nmid n'$ and hence $P_{j(i)}$ cannot appear in $\mathcal{P}'$, so $j(i) > s'$. Moreover, we have observed before that in this case $h_{j(i)} \in P_1'$, hence $Q_{j(i)} \in \mathcal{Q}$. This means that $q_{j(i)} \mid n''$, therefore $\gamma_i = \alpha_i$ for each $i \in I''$ (recall here that $q_i = r_i/d_1'$ and hence $d_{j(i)} \mid r_{j(i)} \mid d_1'n''$).

Let $d_1' = p_1^{\tau_1} \dots p_k^{\tau_k}$. Then $n'' = p_1^{\gamma_1-\tau_1} \dots p_k^{\gamma_i-\tau_i}$. Now, if $i \in I''$, then for every β the condition $p_i^{(\gamma_i-\tau_i)-\beta+1} \mid q_j$ is equivalent to $p_i^{\alpha_i-\beta+1} \mid r_j$.

Note that $n'' \le n/d_1' < n$, hence we can apply the Claim to the covering $\mathcal{Q}$. We perform this with the set of indices I'' and the exponents $\beta_i'' = \min\{\beta_i, \sigma_i\} > 0$. So, the set under consideration is

$$T'' = \left\{ j : Q_j \in \mathcal{Q}, \text{and } p_i^{(\gamma_i-\tau_i)-\min\{\beta_i,\sigma_i\}+1} \mid q_j \text{ for some } i \in I'' \right\}$$
$$= \left\{ j : Q_j \in \mathcal{Q}, \text{and } p_i^{\alpha_i-\min\{\beta_i,\sigma_i\}+1} \mid r_j \text{ for some } i \in I'' \right\},$$

and we obtain $|T''| \ge 1 + G$. Finally, we claim that $T'' \subseteq T \cap \big(\{1\} \cup \{s'+1, \dots, s\}\big)$; then we will obtain $|T \cap \{s'+1, \dots, s\}| \ge G$, which is exactly what we need.

To prove this, consider any $j \in T''$. Observe first that $\alpha_i - \min\{\beta_i, \sigma_i\} + 1 > \alpha_i - \sigma_i \ge \tau_i$, hence from $p_i^{\alpha_i-\min\{\beta_i,\sigma_i\}+1} \mid r_j = \text{l.c.m.}(d_1', d_j)$ we have $p_i^{\alpha_i-\min\{\beta_i,\sigma_i\}+1} \mid d_j$, which means that $j \in T$. Next, the exponent of p_i in d_j is greater than that in n', which means that $P_j \notin \mathcal{P}'$. This may appear only if $j = 1$ or $j > s'$, as desired. This completes the proof.

Comment 2. A grid analogue of the Claim is also valid. It reads as following.

Claim. Assume that the grid N is covered by subgrids $L_1, L_2, \dots, L_s$ so that

(ii′) each subgrid contains a point which is not covered by other subgrids;

(iii) for each coordinate axis, there exists a subgrid L_i orthogonal to this axis.

Choose some set of indices $I = \{i_1, \dots, i_t\} \subset [k]$, and consider the set of indices

$$T = \{j : 1 \le j \le s, \text{and } L_j \text{ is orthogonal to the } i\text{th axis for some } i \in I\}.$$

Then
$$|T| \ge 1 + \sum_{i \in I}(n_i - 1).$$

This Claim may be proved almost in the same way as in Solution 1.

Geometry

G1. Let ABC be an acute triangle with D, E, F the feet of the altitudes lying on BC, CA, AB respectively. One of the intersection points of the line EF and the circumcircle is P. The lines BP and DF meet at point Q. Prove that $AP = AQ$.

(United Kingdom)

Solution 1. The line EF intersects the circumcircle at two points. Depending on the choice of P, there are two different cases to consider.

Case 1: The point P lies on the ray EF (see Fig. 1).

Let $\angle CAB = \alpha$, $\angle ABC = \beta$ and $\angle BCA = \gamma$. The quadrilaterals $BCEF$ and $CAFD$ are cyclic due to the right angles at D, E and F. So,

$$\angle BDF = 180^\circ - \angle FDC = \angle CAF = \alpha,$$
$$\angle AFE = 180^\circ - \angle EFB = \angle BCE = \gamma,$$
$$\angle DFB = 180^\circ - \angle AFD = \angle DCA = \gamma.$$

Since P lies on the arc AB of the circumcircle, $\angle PBA < \angle BCA = \gamma$. Hence, we have

$$\angle PBD + \angle BDF = \angle PBA + \angle ABD + \angle BDF < \gamma + \beta + \alpha = 180^\circ,$$

and the point Q must lie on the extensions of BP and DF beyond the points P and F, respectively.

From the cyclic quadrilateral $APBC$ we get

$$\angle QPA = 180^\circ - \angle APB = \angle BCA = \gamma = \angle DFB = \angle QFA.$$

Hence, the quadrilateral $AQPF$ is cyclic. Then $\angle AQP = 180^\circ - \angle PFA = \angle AFE = \gamma$.

We obtained that $\angle AQP = \angle QPA = \gamma$, so the triangle AQP is isosceles, $AP = AQ$.

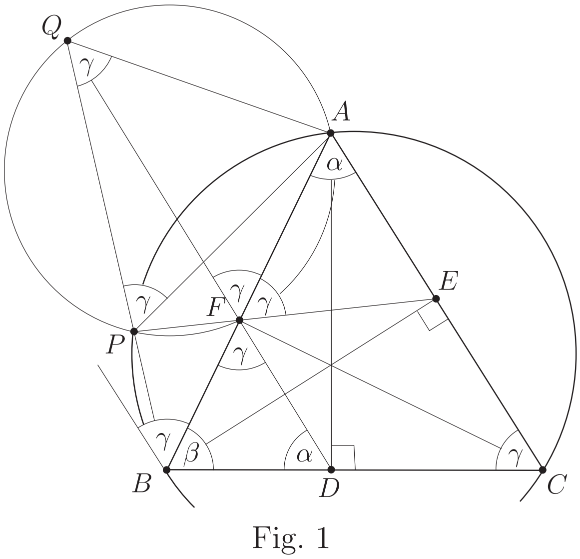


Fig. 1

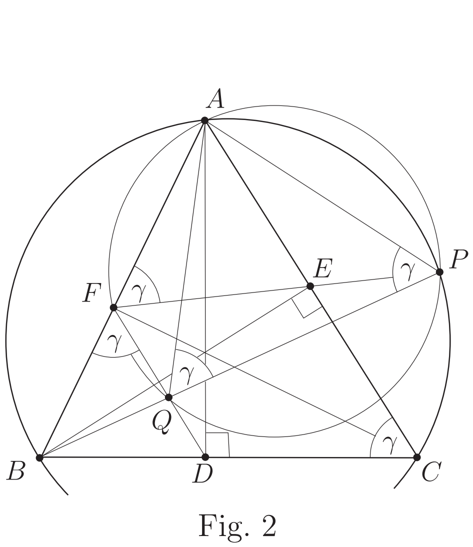


Fig. 2

Case 2: The point P lies on the ray FE (see Fig. 2). In this case the point Q lies inside the segment FD.

Similarly to the first case, we have

$$\angle QPA = \angle BCA = \gamma = \angle DFB = 180^\circ - \angle AFQ.$$

Hence, the quadrilateral $AFQP$ is cyclic.

Then $\angle AQP = \angle AFP = \angle AFE = \gamma = \angle QPA$. The triangle AQP is isosceles again, $\angle AQP = \angle QPA$ and thus $AP = AQ$.

Comment. Using signed angles, the two possible configurations can be handled simultaneously, without investigating the possible locations of P and Q.

Solution 2. For arbitrary points X, Y on the circumcircle, denote by $\widehat{XY}$ the central angle of the arc XY.

Let P and P' be the two points where the line EF meets the circumcircle; let P lie on the arc AB and let P' lie on the arc CA. Let BP and BP' meet the line DF and Q and Q', respectively (see Fig. 3). We will prove that $AP = AP' = AQ = AQ'$.

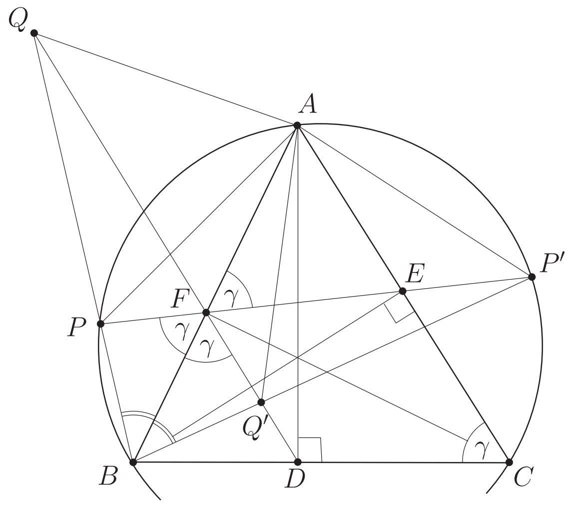


Fig. 3

Like in the first solution, we have $\angle AFE = \angle BFP = \angle DFB = \angle BCA = \gamma$ from the cyclic quadrilaterals $BCEF$ and $CAFD$.

By $\widehat{PB} + \widehat{P'A} = 2\angle AFP' = 2\gamma = 2\angle BCA = \widehat{AP} + \widehat{PB}$, we have

$$\widehat{AP} = \widehat{P'A}, \quad \angle PBA = \angle ABP' \qquad \text{and} \quad AP = AP'. \tag{1}$$

Due to $\widehat{AP} = \widehat{P'A}$, the lines BP and BQ' are symmetrical about line AB.

Similarly, by $\angle BFP = \angle Q'FB$, the lines FP and FQ' are symmetrical about AB. It follows that also the points P and P' are symmetrical to Q' and Q, respectively. Therefore,

$$AP = AQ' \quad \text{and} \quad AP' = AQ. \tag{2}$$

The relations (1) and (2) together prove $AP = AP' = AQ = AQ'$.

G2. Point P lies inside triangle ABC. Lines AP, BP, CP meet the circumcircle of ABC again at points K, L, M, respectively. The tangent to the circumcircle at C meets line AB at S. Prove that $SC = SP$ if and only if $MK = ML$.

(Poland)

Solution 1. We assume that $CA > CB$, so point S lies on the ray AB.

From the similar triangles $\triangle PKM \sim \triangle PCA$ and $\triangle PLM \sim \triangle PCB$ we get $\dfrac{PM}{KM} = \dfrac{PA}{CA}$ and $\dfrac{LM}{PM} = \dfrac{CB}{PB}$. Multiplying these two equalities, we get

$$\frac{LM}{KM} = \frac{CB}{CA} \cdot \frac{PA}{PB}.$$

Hence, the relation $MK = ML$ is equivalent to $\dfrac{CB}{CA} = \dfrac{PB}{PA}$.

Denote by E the foot of the bisector of angle B in triangle ABC. Recall that the locus of points X for which $\dfrac{XA}{XB} = \dfrac{CA}{CB}$ is the Apollonius circle Ω with the center Q on the line AB, and this circle passes through C and E. Hence, we have $MK = ML$ if and only if P lies on Ω, that is $QP = QC$.

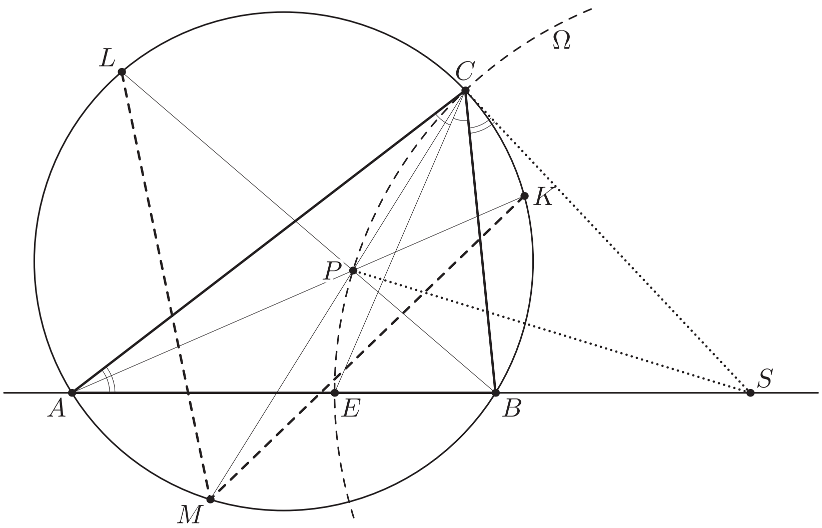


Fig. 1

Now we prove that $S = Q$, thus establishing the problem statement. We have $\angle CES = \angle CAE + \angle ACE = \angle BCS + \angle ECB = \angle ECS$, so $SC = SE$. Hence, the point S lies on AB as well as on the perpendicular bisector of CE and therefore coincides with Q.

Solution 2. As in the previous solution, we assume that S lies on the ray AB.

1. Let P be an arbitrary point inside both the circumcircle ω of the triangle ABC and the angle ASC, the points K, L, M defined as in the problem. We claim that $SP = SC$ implies $MK = ML$.

Let E and F be the points of intersection of the line SP with ω, point E lying on the segment SP (see Fig. 2).

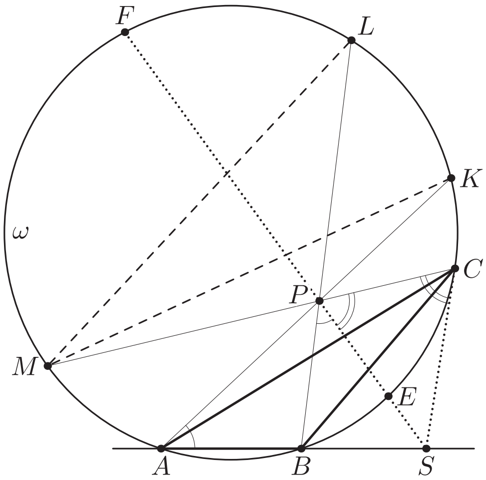


Fig. 2

We have $SP^2 = SC^2 = SA \cdot SB$, so $\dfrac{SP}{SB} = \dfrac{SA}{SP}$, and hence $\triangle PSA \sim \triangle BSP$. Then $\angle BPS = \angle SAP$. Since $2\angle BPS = \widehat{BE} + \widehat{LF}$ and $2\angle SAP = \widehat{BE} + \widehat{EK}$ we have

$$\widehat{LF} = \widehat{EK}. \tag{1}$$

On the other hand, from $\angle SPC = \angle SCP$ we have $\widehat{EC} + \widehat{MF} = \widehat{EC} + \widehat{EM}$, or

$$\widehat{MF} = \widehat{EM}. \tag{2}$$

From (1) and (2) we get $\widehat{MFL} = \widehat{MF} + \widehat{FL} = \widehat{ME} + \widehat{EK} = \widehat{MEK}$ and hence $MK = ML$. The claim is proved.

2. We are left to prove the converse. So, assume that $MK = ML$, and introduce the points E and F as above. We have $SC^2 = SE \cdot SF$; hence, there exists a point P' lying on the segment EF such that $SP' = SC$ (see Fig. 3).

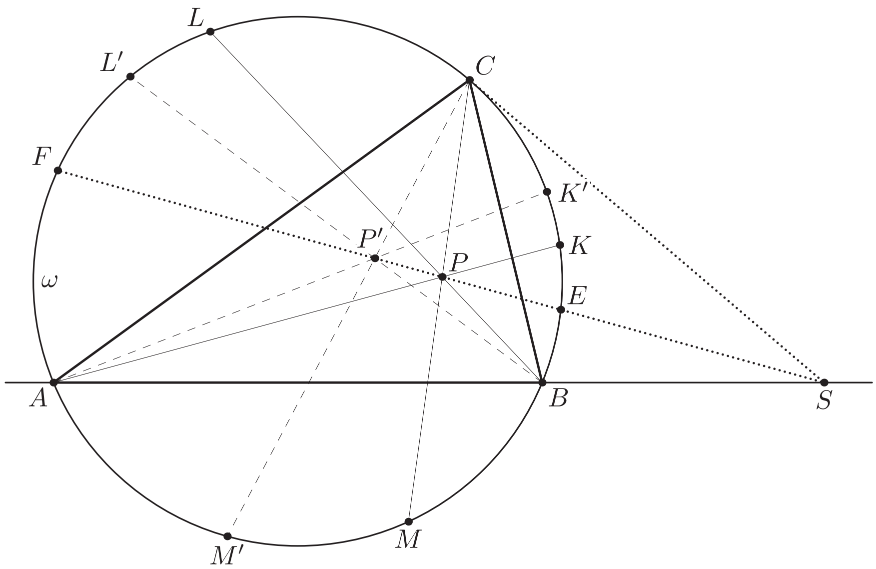


Fig. 3

Assume that $P \neq P'$. Let the lines AP', BP', CP' meet ω again at points K', L', M' respectively. Now, if P' lies on the segment PF then by the first part of the solution we have $\widehat{M'FL'} = \widehat{M'EK'}$. On the other hand, we have $\widehat{MFL} > \widehat{M'FL'} = \widehat{M'EK'} > \widehat{MEK}$, therefore $\widehat{MFL} > \widehat{MEK}$ which contradicts $MK = ML$.

Similarly, if point P' lies on the segment EP then we get $\widehat{MFL} < \widehat{MEK}$ which is impossible. Therefore, the points P and P' coincide and hence $SP = SP' = SC$.

Solution 3. We present a different proof of the converse direction, that is, $MK = ML \Rightarrow SP = SC$. As in the previous solutions we assume that $CA > CB$, and the line SP meets ω at E and F.

From $ML = MK$ we get $\widehat{MEK} = \widehat{MFL}$. Now we claim that $\widehat{ME} = \widehat{MF}$ and $\widehat{EK} = \widehat{FL}$.

To the contrary, suppose first that $\widehat{ME} > \widehat{MF}$; then $\widehat{EK} = \widehat{MEK} - \widehat{ME} < \widehat{MFL} - \widehat{MF} = \widehat{FL}$. Now, the inequality $\widehat{ME} > \widehat{MF}$ implies $2\angle SCM = \widehat{EC} + \widehat{ME} > \widehat{EC} + \widehat{MF} = 2\angle SPC$ and hence $SP > SC$. On the other hand, the inequality $\widehat{EK} < \widehat{FL}$ implies $2\angle SPK = \widehat{EK} + \widehat{AF} < \widehat{FL} + \widehat{AF} = 2\angle ABL$, hence

$$\angle SPA = 180^\circ - \angle SPK > 180^\circ - \angle ABL = \angle SBP.$$

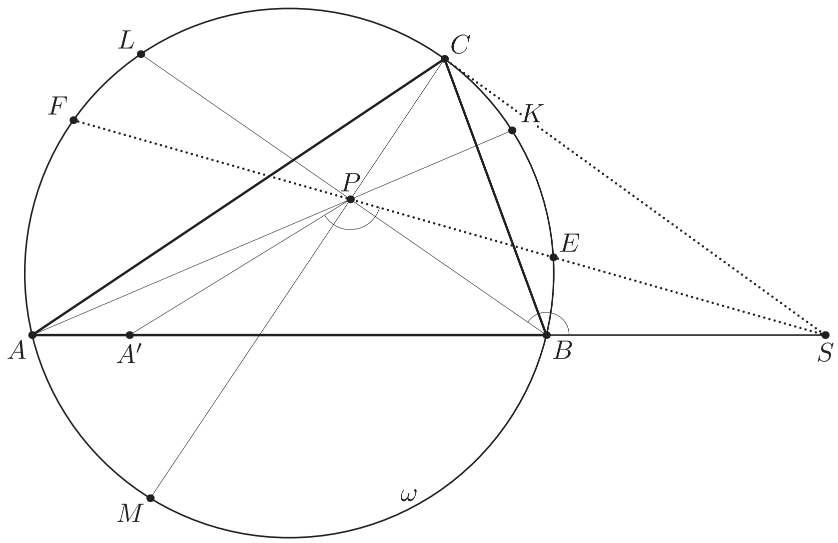


Fig. 4

Consider the point A' on the ray SA for which $\angle SPA' = \angle SBP$; in our case, this point lies on the segment SA (see Fig. 4). Then $\triangle SBP \sim \triangle SPA'$ and $SP^2 = SB \cdot SA' < SB \cdot SA = SC^2$. Therefore, $SP < SC$ which contradicts $SP > SC$.

Similarly, one can prove that the inequality $\widehat{ME} < \widehat{MF}$ is also impossible. So, we get $\widehat{ME} = \widehat{MF}$ and therefore $2\angle SCM = \widehat{EC} + \widehat{ME} = \widehat{EC} + \widehat{MF} = 2\angle SPC$, which implies $SC = SP$.

G3. Let $A_1A_2\dots A_n$ be a convex polygon. Point P inside this polygon is chosen so that its projections $P_1, \dots, P_n$ onto lines $A_1A_2, \dots, A_nA_1$ respectively lie on the sides of the polygon. Prove that for arbitrary points $X_1, \dots, X_n$ on sides $A_1A_2, \dots, A_nA_1$ respectively,

$$\max\left\{\frac{X_1X_2}{P_1P_2}, \dots, \frac{X_nX_1}{P_nP_1}\right\} \geq 1.$$

(Armenia)

Solution 1. Denote $P_{n+1} = P_1$, $X_{n+1} = X_1$, $A_{n+1} = A_1$.

Lemma. Let point Q lies inside $A_1A_2\dots A_n$. Then it is contained in at least one of the circumcircles of triangles $X_1A_2X_2, \dots, X_nA_1X_1$.

Proof. If Q lies in one of the triangles $X_1A_2X_2, \dots, X_nA_1X_1$, the claim is obvious. Otherwise Q lies inside the polygon $X_1X_2\dots X_n$ (see Fig. 1). Then we have

$$\begin{aligned}(\angle X_1A_2X_2 + \angle X_1QX_2) + \dots + (\angle X_nA_1X_1 + \angle X_nQX_1)\\ = (\angle X_1A_1X_2 + \dots + \angle X_nA_1X_1) + (\angle X_1QX_2 + \dots + \angle X_nQX_1) = (n-2)\pi + 2\pi = n\pi,\end{aligned}$$

hence there exists an index i such that $\angle X_iA_{i+1}X_{i+1} + \angle X_iQX_{i+1} \geq \frac{\pi n}{n} = \pi$. Since the quadrilateral $QX_iA_{i+1}X_{i+1}$ is convex, this means exactly that Q is contained the circumcircle of $\triangle X_iA_{i+1}X_{i+1}$, as desired. $\square$

Now we turn to the solution. Applying lemma, we get that P lies inside the circumcircle of triangle $X_iA_{i+1}X_{i+1}$ for some i. Consider the circumcircles ω and Ω of triangles $P_iA_{i+1}P_{i+1}$ and $X_iA_{i+1}X_{i+1}$ respectively (see Fig. 2); let r and R be their radii. Then we get $2r = A_{i+1}P \leq 2R$ (since P lies inside Ω), hence

$$P_iP_{i+1} = 2r\sin\angle P_iA_{i+1}P_{i+1} \leq 2R\sin\angle X_iA_{i+1}X_{i+1} = X_iX_{i+1},$$

QED.

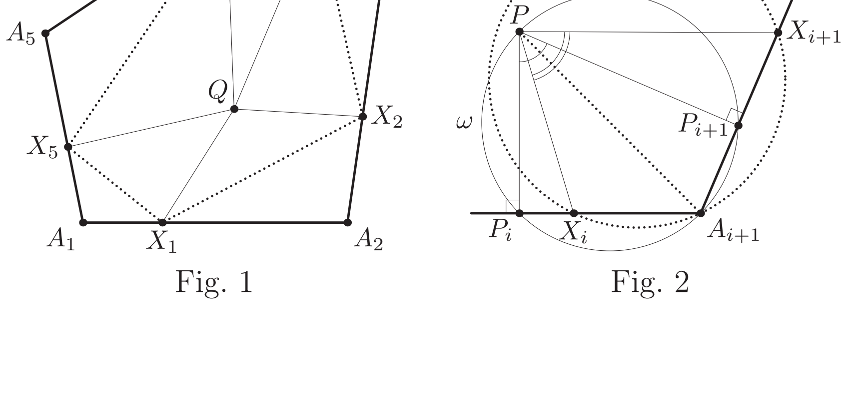


Fig. 1 Fig. 2

Solution 2. As in Solution 1, we assume that all indices of points are considered modulo n.

We will prove a bit stronger inequality, namely

$$\max\left\{\frac{X_1X_2}{P_1P_2}\cos\alpha_1, \ldots, \frac{X_nX_1}{P_nP_1}\cos\alpha_n\right\} \geq 1,$$

where α_i $(1 \leq i \leq n)$ is the angle between lines X_iX_{i+1} and P_iP_{i+1}. We denote $\beta_i = \angle A_iP_iP_{i-1}$ and $\gamma_i = \angle A_{i+1}P_iP_{i+1}$ for all $1 \leq i \leq n$.

Suppose that for some $1 \leq i \leq n$, point X_i lies on the segment A_iP_i, while point X_{i+1} lies on the segment $P_{i+1}A_{i+2}$. Then the projection of the segment X_iX_{i+1} onto the line P_iP_{i+1} contains segment P_iP_{i+1}, since γ_i and β_{i+1} are acute angles (see Fig. 3). Therefore, $X_iX_{i+1}\cos\alpha_i \geq P_iP_{i+1}$, and in this case the statement is proved.

So, the only case left is when point X_i lies on segment P_iA_{i+1} for all $1 \leq i \leq n$ (the case when each X_i lies on segment A_iP_i is completely analogous).

Now, assume to the contrary that the inequality

$$X_iX_{i+1}\cos\alpha_i < P_iP_{i+1} \tag{1}$$

holds for every $1 \leq i \leq n$. Let Y_i and Y'_{i+1} be the projections of X_i and X_{i+1} onto P_iP_{i+1}. Then inequality (1) means exactly that $Y_iY'_{i+1} < P_iP_{i+1}$, or $P_iY_i > P_{i+1}Y'_{i+1}$ (again since γ_i and β_{i+1} are acute; see Fig. 4). Hence, we have

$$X_iP_i\cos\gamma_i > X_{i+1}P_{i+1}\cos\beta_{i+1}, \qquad 1 \leq i \leq n.$$

Multiplying these inequalities, we get

$$\cos\gamma_1\cos\gamma_2\cdots\cos\gamma_n > \cos\beta_1\cos\beta_2\cdots\cos\beta_n. \tag{2}$$

On the other hand, the sines theorem applied to triangle PP_iP_{i+1} provides

$$\frac{PP_i}{PP_{i+1}} = \frac{\sin\left(\frac{\pi}{2} - \beta_{i+1}\right)}{\sin\left(\frac{\pi}{2} - \gamma_i\right)} = \frac{\cos\beta_{i+1}}{\cos\gamma_i}.$$

Multiplying these equalities we get

$$1 = \frac{\cos\beta_2}{\cos\gamma_1} \cdot \frac{\cos\beta_3}{\cos\gamma_2} \cdots \frac{\cos\beta_1}{\cos\gamma_n}$$

which contradicts (2).

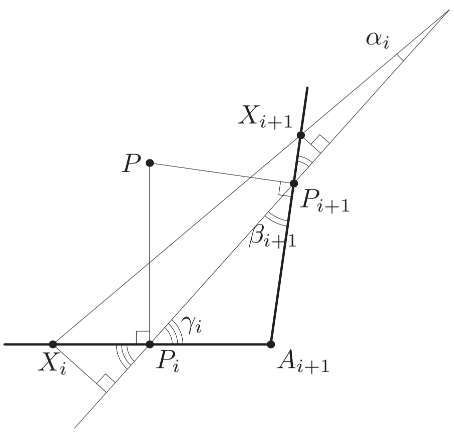


Fig. 3

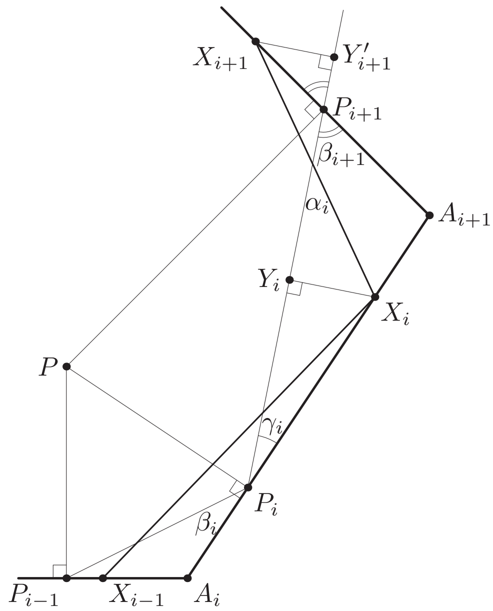


Fig. 4

G4. Let I be the incenter of a triangle ABC and Γ be its circumcircle. Let the line AI intersect Γ at a point $D \neq A$. Let F and E be points on side BC and arc BDC respectively such that $\angle BAF = \angle CAE < \frac{1}{2}\angle BAC$. Finally, let G be the midpoint of the segment IF. Prove that the lines DG and EI intersect on Γ.

(Hong Kong)

Solution 1. Let X be the second point of intersection of line EI with Γ, and L be the foot of the bisector of angle BAC. Let G' and T be the points of intersection of segment DX with lines IF and AF, respectively. We are to prove that $G = G'$, or $IG' = G'F$. By the Menelaus theorem applied to triangle AIF and line DX, it means that we need the relation

$$1 = \frac{G'F}{IG'} = \frac{TF}{AT} \cdot \frac{AD}{ID}, \qquad \text{or} \quad \frac{TF}{AT} = \frac{ID}{AD}.$$

Let the line AF intersect Γ at point $K \neq A$ (see Fig. 1); since $\angle BAK = \angle CAE$ we have $\widehat{BK} = \widehat{CE}$, hence $KE \parallel BC$. Notice that $\angle IAT = \angle DAK = \angle EAD = \angle EXD = \angle IXT$, so the points I, A, X, T are concyclic. Hence we have $\angle ITA = \angle IXA = \angle EXA = \angle EKA$, so $IT \parallel KE \parallel BC$. Therefore we obtain $\dfrac{TF}{AT} = \dfrac{IL}{AI}$.

Since CI is the bisector of $\angle ACL$, we get $\dfrac{IL}{AI} = \dfrac{CL}{AC}$. Furthermore, $\angle DCL = \angle DCB = \angle DAB = \angle CAD = \frac{1}{2}\angle BAC$, hence the triangles DCL and DAC are similar; therefore we get $\dfrac{CL}{AC} = \dfrac{DC}{AD}$. Finally, it is known that the midpoint D of arc BC is equidistant from points I, B, C, hence $\dfrac{DC}{AD} = \dfrac{ID}{AD}$.

Summarizing all these equalities, we get

$$\frac{TF}{AT} = \frac{IL}{AI} = \frac{CL}{AC} = \frac{DC}{AD} = \frac{ID}{AD},$$

as desired.

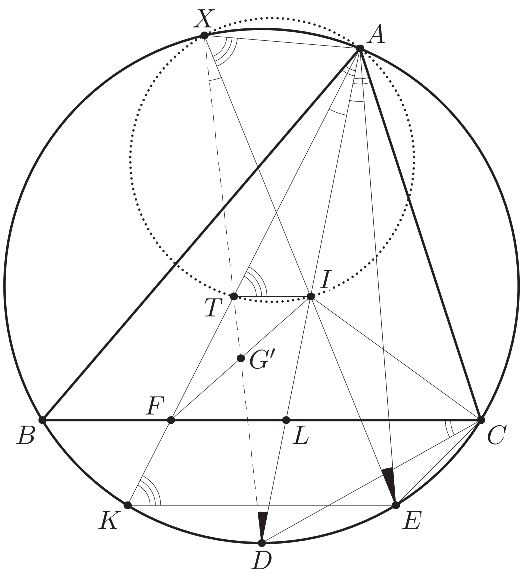


Fig. 1

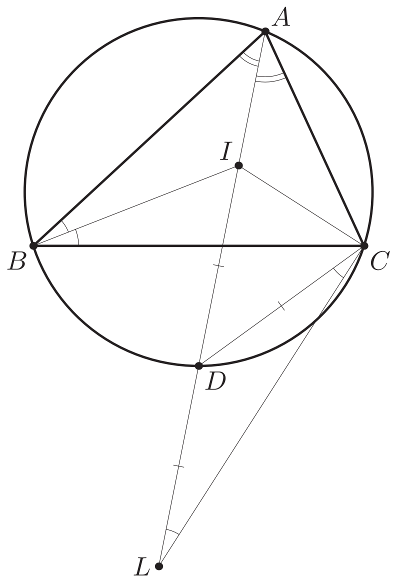


Fig. 2

Comment. The equality $\dfrac{AI}{IL}=\dfrac{AD}{DI}$ is known and can be obtained in many different ways. For instance, one can consider the inversion with center D and radius $DC=DI$. This inversion takes $\widehat{BAC}$ to the segment BC, so point A goes to L. Hence $\dfrac{IL}{DI}=\dfrac{AI}{AD}$, which is the desired equality.

Solution 2. As in the previous solution, we introduce the points X, T and K and note that it suffice to prove the equality

$$\frac{TF}{AT}=\frac{DI}{AD} \quad\Longleftrightarrow\quad \frac{TF+AT}{AT}=\frac{DI+AD}{AD} \quad\Longleftrightarrow\quad \frac{AT}{AD}=\frac{AF}{DI+AD}.$$

Since $\angle FAD=\angle EAI$ and $\angle TDA=\angle XDA=\angle XEA=\angle IEA$, we get that the triangles ATD and AIE are similar, therefore $\dfrac{AT}{AD}=\dfrac{AI}{AE}$.

Next, we also use the relation $DB=DC=DI$. Let J be the point on the extension of segment AD over point D such that $DJ=DI=DC$ (see Fig. 2). Then $\angle DJC=\angle JCD=\frac{1}{2}(\pi-\angle JDC)=\frac{1}{2}\angle ADC=\frac{1}{2}\angle ABC=\angle ABI$. Moreover, $\angle BAI=\angle JAC$, hence triangles ABI and AJC are similar, so $\dfrac{AB}{AJ}=\dfrac{AI}{AC}$, or $AB\cdot AC=AJ\cdot AI=(DI+AD)\cdot AI$.

On the other hand, we get $\angle ABF=\angle ABC=\angle AEC$ and $\angle BAF=\angle CAE$, so triangles ABF and AEC are also similar, which implies $\dfrac{AF}{AC}=\dfrac{AB}{AE}$, or $AB\cdot AC=AF\cdot AE$.

Summarizing we get

$$(DI+AD)\cdot AI=AB\cdot AC=AF\cdot AE \quad\Rightarrow\quad \frac{AI}{AE}=\frac{AF}{AD+DI} \quad\Rightarrow\quad \frac{AT}{AD}=\frac{AF}{AD+DI},$$

as desired.

Comment. In fact, point J is an excenter of triangle ABC.

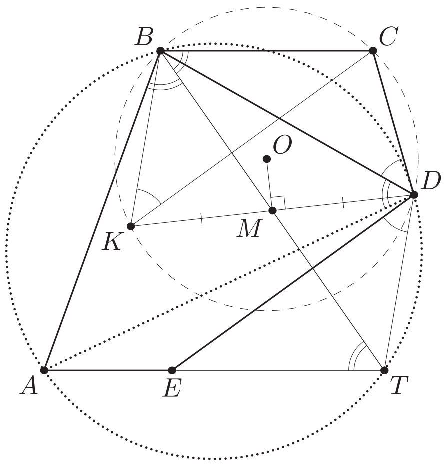
G5. Let $ABCDE$ be a convex pentagon such that $BC \parallel AE$, $AB = BC + AE$, and $\angle ABC = \angle CDE$. Let M be the midpoint of CE, and let O be the circumcenter of triangle BCD. Given that $\angle DMO = 90^\circ$, prove that $2\angle BDA = \angle CDE$.

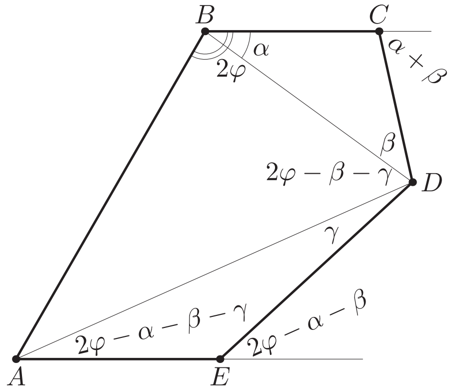
(Ukraine)

Solution 1. Choose point T on ray AE such that $AT = AB$; then from $AE \parallel BC$ we have $\angle CBT = \angle ATB = \angle ABT$, so BT is the bisector of $\angle ABC$. On the other hand, we have $ET = AT - AE = AB - AE = BC$, hence quadrilateral $BCTE$ is a parallelogram, and the midpoint M of its diagonal CE is also the midpoint of the other diagonal BT.

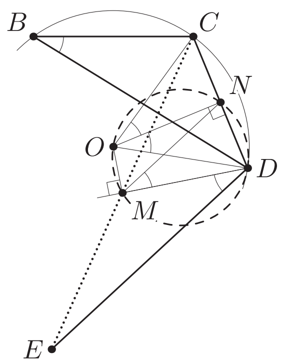
Next, let point K be symmetrical to D with respect to M. Then OM is the perpendicular bisector of segment DK, and hence $OD = OK$, which means that point K lies on the circumcircle of triangle BCD. Hence we have $\angle BDC = \angle BKC$. On the other hand, the angles BKC and TDE are symmetrical with respect to M, so $\angle TDE = \angle BKC = \angle BDC$.

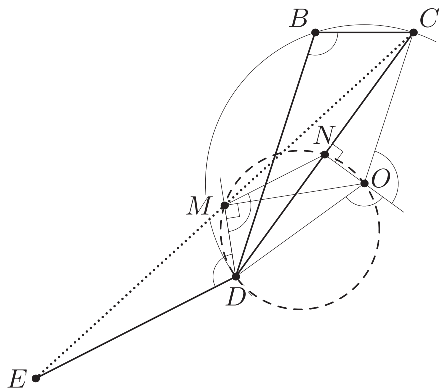
Therefore, $\angle BDT = \angle BDE + \angle EDT = \angle BDE + \angle BDC = \angle CDE = \angle ABC = 180^\circ - \angle BAT$. This means that the points A, B, D, T are concyclic, and hence $\angle ADB = \angle ATB = \frac{1}{2}\angle ABC = \frac{1}{2}\angle CDE$, as desired.





Solution 2. Let$\angle CBD = \alpha$, $\angle BDC = \beta$, $\angle ADE = \gamma$, and $\angle ABC = \angle CDE = 2\varphi$. Then we have $\angle ADB = 2\varphi - \beta - \gamma$, $\angle BCD = 180^\circ - \alpha - \beta$, $\angle AED = 360^\circ - \angle BCD - \angle CDE = 180^\circ - 2\varphi + \alpha + \beta$, and finally $\angle DAE = 180^\circ - \angle ADE - \angle AED = 2\varphi - \alpha - \beta - \gamma$.





Let N be the midpoint of CD; then $\angle DNO = 90^\circ = \angle DMO$, hence points M, N lie on the circle with diameter OD. Now, if points O and M lie on the same side of CD, we have $\angle DMN = \angle DON = \frac{1}{2}\angle DOC = \alpha$; in the other case, we have $\angle DMN = 180^\circ - \angle DON = \alpha$;

so, in both cases $\angle DMN = \alpha$ (see Figures). Next, since MN is a midline in triangle CDE, we have $\angle MDE = \angle DMN = \alpha$ and $\angle NDM = 2\varphi - \alpha$.

Now we apply the sine rule to the triangles ABD, ADE (twice), BCD and MND obtaining

$$\frac{AB}{AD} = \frac{\sin(2\varphi-\beta-\gamma)}{\sin(2\varphi-\alpha)}, \quad \frac{AE}{AD} = \frac{\sin\gamma}{\sin(2\varphi-\alpha-\beta)}, \quad \frac{DE}{AD} = \frac{\sin(2\varphi-\alpha-\beta-\gamma)}{\sin(2\varphi-\alpha-\beta)},$$
$$\frac{BC}{CD} = \frac{\sin\beta}{\sin\alpha}, \quad \frac{CD}{DE} = \frac{CD/2}{DE/2} = \frac{ND}{NM} = \frac{\sin\alpha}{\sin(2\varphi-\alpha)},$$

which implies
$$\frac{BC}{AD} = \frac{BC}{CD}\cdot\frac{CD}{DE}\cdot\frac{DE}{AD} = \frac{\sin\beta\cdot\sin(2\varphi-\alpha-\beta-\gamma)}{\sin(2\varphi-\alpha)\cdot\sin(2\varphi-\alpha-\beta)}.$$

Hence, the condition $AB = AE + BC$, or equivalently $\dfrac{AB}{AD} = \dfrac{AE+BC}{AD}$, after multiplying by the common denominator rewrites as

$$\begin{aligned}
&\sin(2\varphi-\alpha-\beta)\cdot\sin(2\varphi-\beta-\gamma) = \sin\gamma\cdot\sin(2\varphi-\alpha) + \sin\beta\cdot\sin(2\varphi-\alpha-\beta-\gamma)\\
\iff\ &\cos(\gamma-\alpha) - \cos(4\varphi-2\beta-\alpha-\gamma) = \cos(2\varphi-\alpha-2\beta-\gamma) - \cos(2\varphi+\gamma-\alpha)\\
\iff\ &\cos(\gamma-\alpha) + \cos(2\varphi+\gamma-\alpha) = \cos(2\varphi-\alpha-2\beta-\gamma) + \cos(4\varphi-2\beta-\alpha-\gamma)\\
\iff\ &\cos\varphi\cdot\cos(\varphi+\gamma-\alpha) = \cos\varphi\cdot\cos(3\varphi-2\beta-\alpha-\gamma)\\
\iff\ &\cos\varphi\cdot\big(\cos(\varphi+\gamma-\alpha) - \cos(3\varphi-2\beta-\alpha-\gamma)\big) = 0\\
\iff\ &\cos\varphi\cdot\sin(2\varphi-\beta-\alpha)\cdot\sin(\varphi-\beta-\gamma) = 0.
\end{aligned}$$

Since $2\varphi-\beta-\alpha = 180^\circ - \angle AED < 180^\circ$ and $\varphi = \frac{1}{2}\angle ABC < 90^\circ$, it follows that $\varphi = \beta+\gamma$, hence $\angle BDA = 2\varphi-\beta-\gamma = \varphi = \frac{1}{2}\angle CDE$, as desired.

G6. The vertices X, Y, Z of an equilateral triangle XYZ lie respectively on the sides BC, CA, AB of an acute-angled triangle ABC. Prove that the incenter of triangle ABC lies inside triangle XYZ.

G6′. The vertices X, Y, Z of an equilateral triangle XYZ lie respectively on the sides BC, CA, AB of a triangle ABC. Prove that if the incenter of triangle ABC lies outside triangle XYZ, then one of the angles of triangle ABC is greater than 120°.

(Bulgaria)

Solution 1 for G6. We will prove a stronger fact; namely, we will show that the incenter I of triangle ABC lies inside the incircle of triangle XYZ (and hence surely inside triangle XYZ itself). We denote by $d(U, VW)$ the distance between point U and line VW.

Denote by O the incenter of $\triangle XYZ$ and by r, r' and R' the inradii of triangles ABC, XYZ and the circumradius of XYZ, respectively. Then we have $R' = 2r'$, and the desired inequality is $OI \le r'$. We assume that $O \ne I$; otherwise the claim is trivial.

Let the incircle of $\triangle ABC$ touch its sides BC, AC, AB at points A_1, B_1, C_1 respectively. The lines IA_1, IB_1, IC_1 cut the plane into 6 acute angles, each one containing one of the points A_1, B_1, C_1 on its border. We may assume that O lies in an angle defined by lines IA_1, IC_1 and containing point C_1 (see Fig. 1). Let A' and C' be the projections of O onto lines IA_1 and IC_1, respectively.

Since $OX = R'$, we have $d(O, BC) \le R'$. Since $OA' \parallel BC$, it follows that $d(A', BC) = A'I + r \le R'$, or $A'I \le R' - r$. On the other hand, the incircle of $\triangle XYZ$ lies inside $\triangle ABC$, hence $d(O, AB) \ge r'$, and analogously we get $d(O, AB) = C'C_1 = r - IC' \ge r'$, or $IC' \le r - r'$.

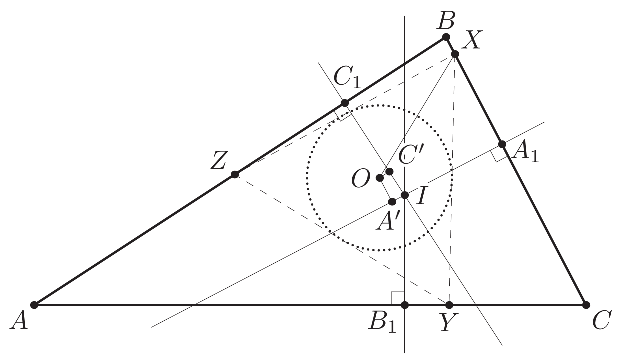


Fig. 1

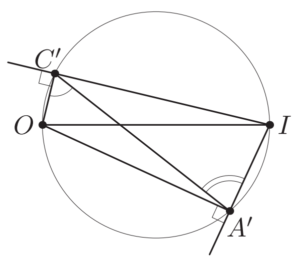


Fig. 2

Finally, the quadrilateral $IA'OC'$ is circumscribed due to the right angles at A' and C' (see Fig. 2). On its circumcircle, we have $\widehat{A'OC'} = 2\angle A'IC' < 180^\circ = \widehat{OC'I}$, hence $180^\circ \ge \widehat{IC'} > \widehat{A'O}$. This means that $IC' > A'O$. Finally, we have $OI \le IA' + A'O < IA' + IC' \le (R' - r) + (r - r') = R' - r' = r'$, as desired.

Solution 2 for G6. Assume the contrary. Then the incenter I should lie in one of triangles AYZ, BXZ, CXY — assume that it lies in $\triangle AYZ$. Let the incircle ω of $\triangle ABC$ touch sides BC, AC at point A_1, B_1 respectively. Without loss of generality, assume that point A_1 lies on segment CX. In this case we will show that $\angle C > 90^\circ$ thus leading to a contradiction.

Note that ω intersects each of the segments XY and YZ at two points; let U, U' and V, V' be the points of intersection of ω with XY and YZ, respectively ($UY > U'Y$, $VY > V'Y$; see Figs. 3 and 4). Note that $60^\circ = \angle XYZ = \frac{1}{2}(\widehat{UV} - \widehat{U'V'}) \le \frac{1}{2}\widehat{UV}$, hence $\widehat{UV} \ge 120^\circ$.

On the other hand, since I lies in $\triangle AYZ$, we get $\widehat{VUV'} < 180°$, hence $\widehat{UA_1U'} \leq \widehat{UA_1V'} < 180° - \widehat{UV} \leq 60°$.

Now, two cases are possible due to the order of points Y, B_1 on segment AC.

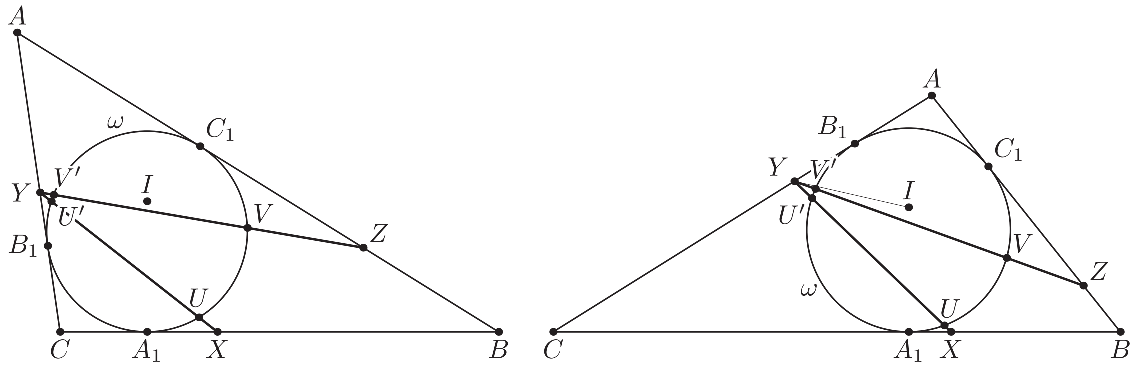


Fig. 3 Fig. 4

Case 1. Let point Y lie on the segment AB_1 (see Fig. 3). Then we have $\angle YXC = \frac{1}{2}\left(\widehat{A_1U'} - \widehat{A_1U}\right) \leq \frac{1}{2}\widehat{UA_1U'} < 30°$; analogously, we get $\angle XYC \leq \frac{1}{2}\widehat{UA_1U'} < 30°$. Therefore, $\angle YCX = 180° - \angle YXC - \angle XYC > 120°$, as desired.

Case 2. Now let point Y lie on the segment CB_1 (see Fig. 4). Analogously, we obtain $\angle YXC < 30°$. Next, $\angle IYX > \angle ZYX = 60°$, but $\angle IYX < \angle IYB_1$, since YB_1 is a tangent and YX is a secant line to circle ω from point Y. Hence, we get $120° < \angle IYB_1 + \angle IYX = \angle B_1YX = \angle YXC + \angle YCX < 30° + \angle YCX$, hence $\angle YCX > 120° - 30° = 90°$, as desired.

Comment. In the same way, one can prove a more general

Claim. Let the vertices X, Y, Z of a triangle XYZ lie respectively on the sides BC, CA, AB of a triangle ABC. Suppose that the incenter of triangle ABC lies outside triangle XYZ, and α is the least angle of $\triangle XYZ$. Then one of the angles of triangle ABC is greater than $3\alpha - 90°$.

Solution for G6′. Assume the contrary. As in Solution 2, we assume that the incenter I of $\triangle ABC$ lies in $\triangle AYZ$, and the tangency point A_1 of ω and BC lies on segment CX. Surely, $\angle YZA \leq 180° - \angle YZX = 120°$, hence points I and Y lie on one side of the perpendicular bisector to XY; therefore $IX > IY$. Moreover, ω intersects segment XY at two points, and therefore the projection M of I onto XY lies on the segment XY. In this case, we will prove that $\angle C > 120°$.

Let YK, YL be two tangents from point Y to ω (points K and A_1 lie on one side of XY; if Y lies on ω, we say $K = L = Y$); one of the points K and L is in fact a tangency point B_1 of ω and AC. From symmetry, we have $\angle YIK = \angle YIL$. On the other hand, since $IX > IY$, we get $XM < XY$ which implies $\angle A_1XY < \angle KYX$.

Next, we have $\angle MIY = 90° - \angle IYX < 90° - \angle ZYX = 30°$. Since $IA_1 \perp A_1X$, $IM \perp XY$, $IK \perp YK$ we get $\angle MIA_1 = \angle A_1XY < \angle KYX = \angle MIK$. Finally, we get

$$\begin{aligned} \angle A_1IK < \angle A_1IL &= (\angle A_1IM + \angle MIK) + (\angle KIY + \angle YIL) \\ &< 2\angle MIK + 2\angle KIY = 2\angle MIY < 60°. \end{aligned}$$

Hence, $\angle A_1IB_1 < 60°$, and therefore $\angle ACB = 180° - \angle A_1IB_1 > 120°$, as desired.

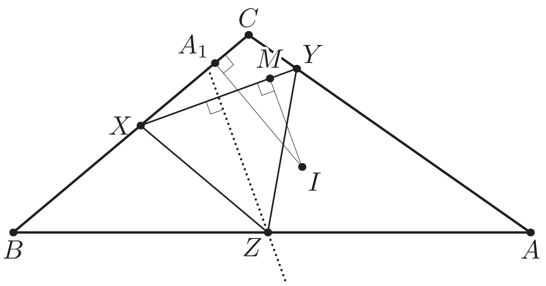


Fig. 5

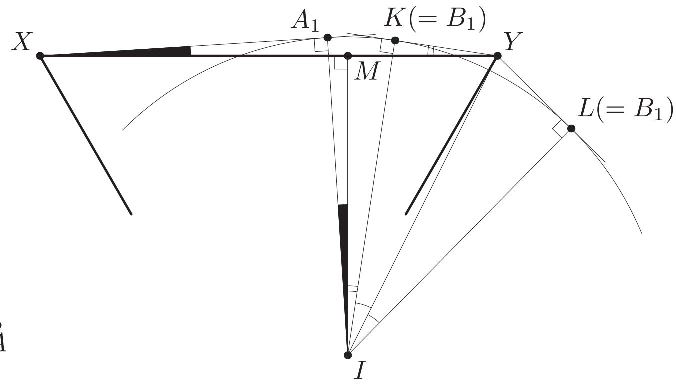


Fig. 6

Comment 1. The estimate claimed in G6′ is sharp. Actually, if $\angle BAC > 120°$, one can consider an equilateral triangle XYZ with $Z = A$, $Y \in AC$, $X \in BC$ (such triangle exists since $\angle ACB < 60°$). It intersects with the angle bisector of $\angle BAC$ only at point A, hence it does not contain I.

Comment 2. As in the previous solution, there is a generalization for an arbitrary triangle XYZ, but here we need some additional condition. The statement reads as follows.

Claim. Let the vertices X, Y, Z of a triangle XYZ lie respectively on the sides BC, CA, AB of a triangle ABC. Suppose that the incenter of triangle ABC lies outside triangle XYZ, α is the least angle of $\triangle XYZ$, and all sides of triangle XYZ are greater than $2r \cot \alpha$, where r is the inradius of $\triangle ABC$. Then one of the angles of triangle ABC is greater than 2α.

The additional condition is needed to verify that $XM > YM$ since it cannot be shown in the original way. Actually, we have $\angle MYI > \alpha$, $IM < r$, hence $YM < r \cot \alpha$. Now, if we have $XY = XM + YM > 2r \cot \alpha$, then surely $XM > YM$.

On the other hand, this additional condition follows easily from the conditions of the original problem. Actually, if $I \in \triangle AYZ$, then the diameter of ω parallel to YZ is contained in $\triangle AYZ$ and is thus shorter than YZ. Hence $YZ > 2r > 2r \cot 60°$.

G7. Three circular arcs γ_1, γ_2, and γ_3 connect the points A and C. These arcs lie in the same half-plane defined by line AC in such a way that arc γ_2 lies between the arcs γ_1 and γ_3. Point B lies on the segment AC. Let h_1, h_2, and h_3 be three rays starting at B, lying in the same half-plane, h_2 being between h_1 and h_3. For $i, j = 1, 2, 3$, denote by V_{ij} the point of intersection of h_i and γ_j (see the Figure below).

Denote by $\wideparen{V_{ij}V_{kj}}\wideparen{V_{k\ell}V_{i\ell}}$ the curved quadrilateral, whose sides are the segments $V_{ij}V_{i\ell}$, $V_{kj}V_{k\ell}$ and arcs $V_{ij}V_{kj}$ and $V_{i\ell}V_{k\ell}$. We say that this quadrilateral is *circumscribed* if there exists a circle touching these two segments and two arcs.

Prove that if the curved quadrilaterals $\wideparen{V_{11}V_{21}}\wideparen{V_{22}V_{12}}$, $\wideparen{V_{12}V_{22}}\wideparen{V_{23}V_{13}}$, $\wideparen{V_{21}V_{31}}\wideparen{V_{32}V_{22}}$ are circumscribed, then the curved quadrilateral $\wideparen{V_{22}V_{32}}\wideparen{V_{33}V_{23}}$ is circumscribed, too.

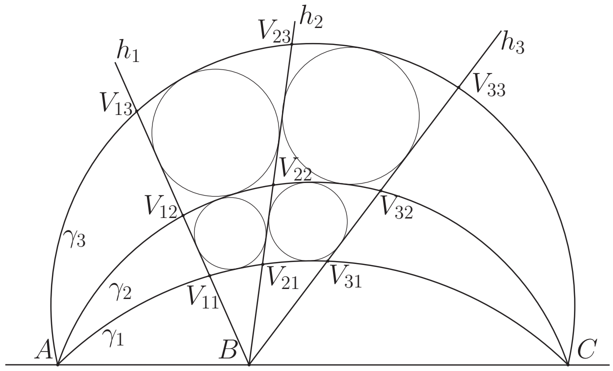


Fig. 1

(Hungary)

Solution. Denote by O_i and R_i the center and the radius of γ_i, respectively. Denote also by H the half-plane defined by AC which contains the whole configuration. For every point P in the half-plane H, denote by $d(P)$ the distance between P and line AC. Furthermore, for any $r > 0$, denote by $\Omega(P, r)$ the circle with center P and radius r.

Lemma 1. For every $1 \le i < j \le 3$, consider those circles $\Omega(P, r)$ in the half-plane H which are tangent to h_i and h_j.

(a) The locus of the centers of these circles is the angle bisector β_{ij} between h_i and h_j.

(b) There is a constant u_{ij} such that $r = u_{ij} \cdot d(P)$ for all such circles.

Proof. Part (a) is obvious. To prove part (b), notice that the circles which are tangent to h_i and h_j are homothetic with the common homothety center B (see Fig. 2). Then part (b) also becomes trivial. $\square$

Lemma 2. For every $1 \le i < j \le 3$, consider those circles $\Omega(P, r)$ in the half-plane H which are externally tangent to γ_i and internally tangent to γ_j.

(a) The locus of the centers of these circles is an ellipse arc ε_{ij} with end-points A and C.

(b) There is a constant v_{ij} such that $r = v_{ij} \cdot d(P)$ for all such circles.

Proof. (a) Notice that the circle $\Omega(P, r)$ is externally tangent to γ_i and internally tangent to γ_j if and only if $O_iP = R_i + r$ and $O_j = R_j - r$. Therefore, for each such circle we have

$$O_iP + O_jP = O_iA + O_jA = O_iC + O_jC = R_i + R_j.$$

Such points lie on an ellipse with foci O_i and O_j; the diameter of this ellipse is $R_i + R_j$, and it passes through the points A and C. Let ε_{ij} be that arc AC of the ellipse which runs inside the half plane H (see Fig. 3.)

This ellipse arc lies between the arcs γ_i and γ_j. Therefore, if some point P lies on ε_{ij}, then $O_iP > R_i$ and $O_jP < R_j$. Now, we choose $r = O_iP - R_i = R_j - O_jP > 0$; then the

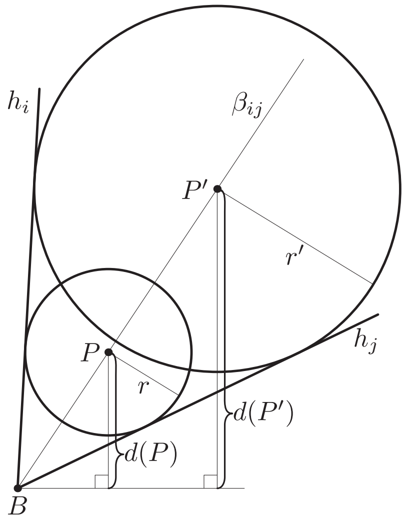


Fig. 2

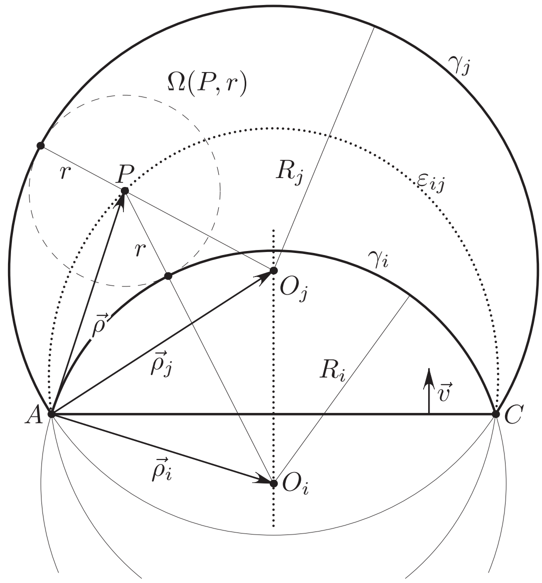


Fig. 3

circle $\Omega(P, r)$ touches γ_i externally and touches γ_j internally, so P belongs to the locus under investigation.

(b) Let $\vec{\rho} = \overrightarrow{AP}$, $\vec{\rho}_i = \overrightarrow{AO_i}$, and $\vec{\rho}_j = \overrightarrow{AO_j}$; let $d_{ij} = O_iO_j$, and let $\vec{v}$ be a unit vector orthogonal to AC and directed toward H. Then we have $|\vec{\rho}_i| = R_i$, $|\vec{\rho}_j| = R_j$, $|\overrightarrow{O_iP}| = |\vec{\rho} - \vec{\rho}_i| = R_i + r$, $|\overrightarrow{O_jP}| = |\vec{\rho} - \vec{\rho}_j| = R_j - r$, hence

$$(\vec{\rho} - \vec{\rho}_i)^2 - (\vec{\rho} - \vec{\rho}_j)^2 = (R_i + r)^2 - (R_j - r)^2,$$
$$(\vec{\rho}_i^{\,2} - \vec{\rho}_j^{\,2}) + 2\vec{\rho} \cdot (\vec{\rho}_j - \vec{\rho}_i) = (R_i^2 - R_j^2) + 2r(R_i + R_j),$$
$$d_{ij} \cdot d(P) = d_{ij}\vec{v} \cdot \vec{\rho} = (\vec{\rho}_j - \vec{\rho}_i) \cdot \vec{\rho} = r(R_i + R_j).$$

Therefore,

$$r = \frac{d_{ij}}{R_i + R_j} \cdot d(P),$$

and the value $v_{ij} = \dfrac{d_{ij}}{R_i + R_j}$ does not depend on P. □

Lemma 3. The curved quadrilateral $\mathcal{Q}_{ij} = \widehat{V_{i,j}V_{i+1,j}}\widehat{V_{i+1,j+1}V_{i,j+1}}$ is circumscribed if and only if $u_{i,i+1} = v_{j,j+1}$.

Proof. First suppose that the curved quadrilateral $\mathcal{Q}_{ij}$ is circumscribed and $\Omega(P, r)$ is its inscribed circle. By Lemma 1 and Lemma 2 we have $r = u_{i,i+1} \cdot d(P)$ and $r = v_{j,j+1} \cdot d(P)$ as well. Hence, $u_{i,i+1} = v_{j,j+1}$.

To prove the opposite direction, suppose $u_{i,i+1} = v_{j,j+1}$. Let P be the intersection of the angle bisector $\beta_{i,i+1}$ and the ellipse arc $\varepsilon_{j,j+1}$. Choose $r = u_{i,i+1} \cdot d(P) = v_{j,j+1} \cdot d(P)$. Then the circle $\Omega(P, r)$ is tangent to the half lines h_i and h_{i+1} by Lemma 1, and it is tangent to the arcs γ_j and γ_{j+1} by Lemma 2. Hence, the curved quadrilateral $\mathcal{Q}_{ij}$ is circumscribed. □

By Lemma 3, the statement of the problem can be reformulated to an obvious fact: If the equalities $u_{12} = v_{12}$, $u_{12} = v_{23}$, and $u_{23} = v_{12}$ hold, then $u_{23} = v_{23}$ holds as well.

Comment 1. Lemma 2(b) (together with the easy Lemma 1(b)) is the key tool in this solution. If one finds this fact, then the solution can be finished in many ways. That is, one can find a circle touching three of h_2, h_3, γ_2, and γ_3, and then prove that it is tangent to the fourth one in either synthetic or analytical way. Both approaches can be successful.

Here we present some discussion about this key Lemma.

1. In the solution above we chose an analytic proof for Lemma 2(b) because we expect that most students will use coordinates or vectors to examine the locus of the centers, and these approaches are less case-sensitive.

Here we outline a synthetic proof. We consider only the case when P does not lie in the line O_iO_j. The other case can be obtained as a limit case, or computed in a direct way.

Let S be the internal homothety center between the circles of γ_i and γ_j, lying on O_iO_j; this point does not depend on P. Let U and V be the points of tangency of circle $\sigma = \Omega(P, r)$ with γ_i and γ_j, respectively (then $r = PU = PV$); in other words, points U and V are the intersection points of rays O_iP, O_jP with arcs γ_i, γ_j respectively (see Fig. 4).

Due to the theorem on three homothety centers (or just to the Menelaus theorem applied to triangle O_iO_jP), the points U, V and S are collinear. Let T be the intersection point of line AC and the common tangent to σ and γ_i at U; then T is the radical center of σ, γ_i and γ_j, hence TV is the common tangent to σ and γ_j.

Let Q be the projection of P onto the line AC. By the right angles, the points U, V and Q lie on the circle with diameter PT. From this fact and the equality $PU = PV$ we get $\angle UQP = \angle UVP = \angle VUP = \angle SUO_i$. Since $O_iS \parallel PQ$, we have $\angle SO_iU = \angle QPU$. Hence, the triangles SO_iU and UPQ are similar and thus $\dfrac{r}{d(P)} = \dfrac{PU}{PQ} = \dfrac{O_iS}{O_iU} = \dfrac{O_iS}{R_i}$; the last expression is constant since S is a constant point. □

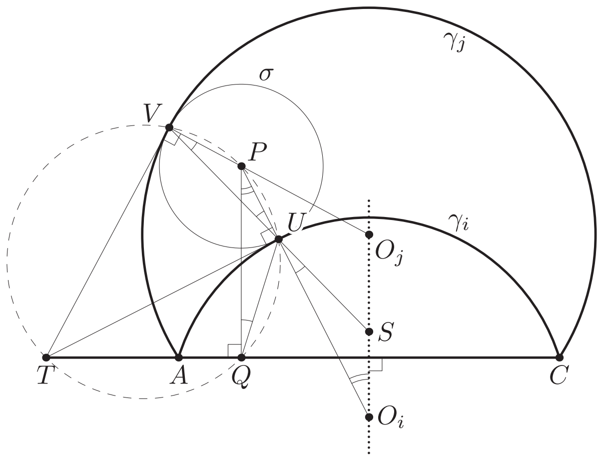


Fig. 4

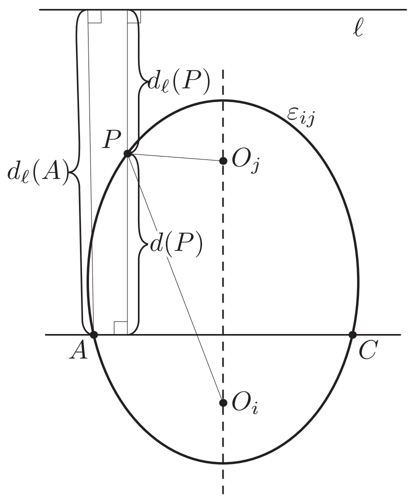


Fig. 5

2. Using some known facts about conics, the same statement can be proved in a very short way. Denote by ℓ the directrix of ellipse of ε_{ij} related to the focus O_j; since ε_{ij} is symmetrical about O_iO_j, we have $\ell \parallel AC$. Recall that for each point $P \in \varepsilon_{ij}$, we have $PO_j = \epsilon \cdot d_\ell(P)$, where $d_\ell(P)$ is the distance from P to ℓ, and ϵ is the eccentricity of ε_{ij} (see Fig. 5).

Now we have

$$r = R_j - (R_j - r) = AO_j - PO_j = \epsilon\big(d_\ell(A) - d_\ell(P)\big) = \epsilon\big(d(P) - d(A)\big) = \epsilon \cdot d(P),$$

and ϵ does not depend on P. □

Comment 2. One can find a spatial interpretations of the problem and the solution.

For every point (x, y) and radius $r > 0$, represent the circle $\Omega\big((x, y), r\big)$ by the point (x, y, r) in space. This point is the apex of the cone with base circle $\Omega\big((x, y), r\big)$ and height r. According to Lemma 1, the circles which are tangent to h_i and h_j correspond to the points of a half line β'_{ij}, starting at B.

Now we translate Lemma 2. Take some $1 \le i < j \le 3$, and consider those circles which are internally tangent to γ_j. It is easy to see that the locus of the points which represent these circles is a subset of a cone, containing γ_j. Similarly, the circles which are externally tangent to γ_i correspond to the points on the extension of another cone, which has its apex on the opposite side of the base plane Π. (See Fig. 6; for this illustration, the z-coordinates were multiplied by 2.)

The two cones are symmetric to each other (they have the same aperture, and their axes are parallel). As is well-known, it follows that the common points of the two cones are co-planar. So the intersection of the two cones is a a conic section — which is an ellipse, according to Lemma 2(a). The points which represent the circles touching γ_i and γ_j is an ellipse arc ε'_{ij} with end-points A and C.

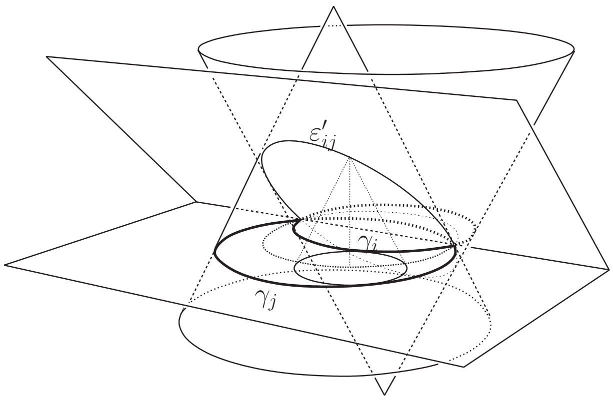


Fig. 6

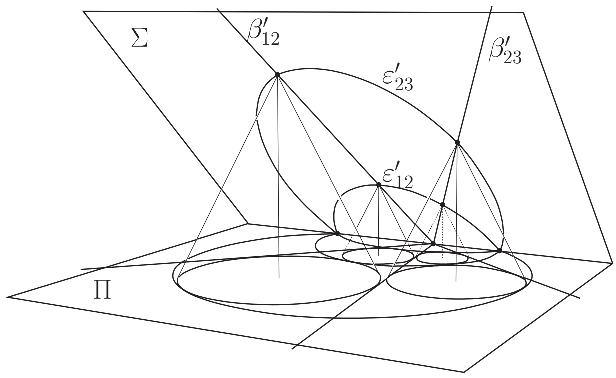


Fig. 7

Thus, the curved quadrilateral $\mathcal{Q}_{ij}$ is circumscribed if and only if $\beta'_{i,i+1}$ and $\varepsilon'_{j,j+1}$ intersect, i.e. if they are coplanar. If three of the four curved quadrilaterals are circumscribed, it means that ε'_{12}, ε'_{23}, β'_{12} and β'_{23} lie in the same plane Σ, and the fourth intersection comes to existence, too (see Fig. 7).

A connection between mathematics and real life:
the Palace of Creativity "Shabyt" ("Inspiration") in Astana

Number Theory

N1. Find the least positive integer n for which there exists a set $\{s_1, s_2, \ldots, s_n\}$ consisting of n distinct positive integers such that

$$\left(1-\frac{1}{s_1}\right)\left(1-\frac{1}{s_2}\right)\ldots\left(1-\frac{1}{s_n}\right)=\frac{51}{2010}.$$

N1′. Same as Problem N1, but the constant $\frac{51}{2010}$ is replaced by $\frac{42}{2010}$.

(Canada)

Answer for Problem N1. $n = 39$.

Solution for Problem N1. Suppose that for some n there exist the desired numbers; we may assume that $s_1 < s_2 < \cdots < s_n$. Surely $s_1 > 1$ since otherwise $1 - \frac{1}{s_1} = 0$. So we have $2 \le s_1 \le s_2 - 1 \le \cdots \le s_n - (n-1)$, hence $s_i \ge i + 1$ for each $i = 1, \ldots, n$. Therefore

$$\begin{aligned}\frac{51}{2010} &= \left(1-\frac{1}{s_1}\right)\left(1-\frac{1}{s_2}\right)\ldots\left(1-\frac{1}{s_n}\right)\\ &\ge \left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\ldots\left(1-\frac{1}{n+1}\right)=\frac{1}{2}\cdot\frac{2}{3}\cdots\frac{n}{n+1}=\frac{1}{n+1},\end{aligned}$$

which implies

$$n+1 \ge \frac{2010}{51} = \frac{670}{17} > 39,$$

so $n \ge 39$.

Now we are left to show that $n = 39$ fits. Consider the set $\{2, 3, \ldots, 33, 35, 36, \ldots, 40, 67\}$ which contains exactly 39 numbers. We have

$$\frac{1}{2}\cdot\frac{2}{3}\cdots\frac{32}{33}\cdot\frac{34}{35}\cdots\frac{39}{40}\cdot\frac{66}{67}=\frac{1}{33}\cdot\frac{34}{40}\cdot\frac{66}{67}=\frac{17}{670}=\frac{51}{2010}, \tag{1}$$

hence for $n = 39$ there exists a desired example.

Comment. One can show that the example (1) is unique.

Answer for Problem N1′. $n = 48$.

Solution for Problem N1′. Suppose that for some n there exist the desired numbers. In the same way we obtain that $s_i \ge i + 1$. Moreover, since the denominator of the fraction $\frac{42}{2010} = \frac{7}{335}$ is divisible by 67, some of s_i's should be divisible by 67, so $s_n \ge s_i \ge 67$. This means that

$$\frac{42}{2010} \ge \frac{1}{2}\cdot\frac{2}{3}\cdots\frac{n-1}{n}\cdot\left(1-\frac{1}{67}\right)=\frac{66}{67n},$$

which implies

$$n \geq \frac{2010 \cdot 66}{42 \cdot 67} = \frac{330}{7} > 47,$$

so $n \geq 48$.

Now we are left to show that $n = 48$ fits. Consider the set $\{2, 3, \ldots, 33, 36, 37, \ldots, 50, 67\}$ which contains exactly 48 numbers. We have

$$\frac{1}{2} \cdot \frac{2}{3} \cdots \frac{32}{33} \cdot \frac{35}{36} \cdots \frac{49}{50} \cdot \frac{66}{67} = \frac{1}{33} \cdot \frac{35}{50} \cdot \frac{66}{67} = \frac{7}{335} = \frac{42}{2010},$$

hence for $n = 48$ there exists a desired example.

Comment 1. In this version of the problem, the estimate needs one more step, hence it is a bit harder. On the other hand, the example in this version is not unique. Another example is

$$\frac{1}{2} \cdot \frac{2}{3} \cdots \frac{46}{47} \cdot \frac{66}{67} \cdot \frac{329}{330} = \frac{1}{67} \cdot \frac{66}{330} \cdot \frac{329}{47} = \frac{7}{67 \cdot 5} = \frac{42}{2010}.$$

Comment 2. N1′ was the Proposer's formulation of the problem. We propose N1 according to the number of current IMO.

N2. Find all pairs (m,n) of nonnegative integers for which

$$m^2+2\cdot 3^n=m\bigl(2^{n+1}-1\bigr). \tag{1}$$

(Australia)

Answer. $(6,3)$, $(9,3)$, $(9,5)$, $(54,5)$.

Solution. For fixed values of n, the equation (1) is a simple quadratic equation in m. For $n\le 5$ the solutions are listed in the following table.

case	equation	discriminant	integer roots
$n=0$	$m^2-m+2=0$	-7	none
$n=1$	$m^2-3m+6=0$	-15	none
$n=2$	$m^2-7m+18=0$	-23	none
$n=3$	$m^2-15m+54=0$	9	$m=6$ and $m=9$
$n=4$	$m^2-31m+162=0$	313	none
$n=5$	$m^2-63m+486=0$	$2025=45^2$	$m=9$ and $m=54$

We prove that there is no solution for $n\ge 6$.

Suppose that (m,n) satisfies (1) and $n\ge 6$. Since $m\mid 2\cdot 3^n=m\bigl(2^{n+1}-1\bigr)-m^2$, we have $m=3^p$ with some $0\le p\le n$ or $m=2\cdot 3^q$ with some $0\le q\le n$.

In the first case, let $q=n-p$; then

$$2^{n+1}-1=m+\frac{2\cdot 3^n}{m}=3^p+2\cdot 3^q.$$

In the second case let $p=n-q$. Then

$$2^{n+1}-1=m+\frac{2\cdot 3^n}{m}=2\cdot 3^q+3^p.$$

Hence, in both cases we need to find the nonnegative integer solutions of

$$3^p+2\cdot 3^q=2^{n+1}-1,\quad p+q=n. \tag{2}$$

Next, we prove bounds for p,q. From (2) we get

$$3^p<2^{n+1}=8^{\frac{n+1}{3}}<9^{\frac{n+1}{3}}=3^{\frac{2(n+1)}{3}}$$

and

$$2\cdot 3^q<2^{n+1}=2\cdot 8^{\frac{n}{3}}<2\cdot 9^{\frac{n}{3}}=2\cdot 3^{\frac{2n}{3}}<2\cdot 3^{\frac{2(n+1)}{3}},$$

so $p,q<\frac{2(n+1)}{3}$. Combining these inequalities with $p+q=n$, we obtain

$$\frac{n-2}{3}<p,q<\frac{2(n+1)}{3}. \tag{3}$$

Now let $h=\min(p,q)$. By (3) we have $h>\frac{n-2}{3}$; in particular, we have $h>1$. On the left-hand side of (2), both terms are divisible by 3^h, therefore $9\mid 3^h\mid 2^{n+1}-1$. It is easy check that $\operatorname{ord}_9(2)=6$, so $9\mid 2^{n+1}-1$ if and only if $6\mid n+1$. Therefore, $n+1=6r$ for some positive integer r, and we can write

$$2^{n+1}-1=4^{3r}-1=(4^{2r}+4^r+1)(2^r-1)(2^r+1). \tag{4}$$

Notice that the factor $4^{2r} + 4^r + 1 = (4^r - 1)^2 + 3 \cdot 4^r$ is divisible by 3, but it is never divisible by 9. The other two factors in (4), $2^r - 1$ and $2^r + 1$ are coprime: both are odd and their difference is 2. Since the whole product is divisible by 3^h, we have either $3^{h-1} \mid 2^r - 1$ or $3^{h-1} \mid 2^r + 1$. In any case, we have $3^{h-1} \leq 2^r + 1$. Then

$$3^{h-1} \leq 2^r + 1 \leq 3^r = 3^{\frac{n+1}{6}},$$
$$\frac{n-2}{3} - 1 < h - 1 \leq \frac{n+1}{6},$$
$$n < 11.$$

But this is impossible since we assumed $n \geq 6$, and we proved $6 \mid n + 1$.

N3. Find the smallest number n such that there exist polynomials $f_1, f_2, \ldots, f_n$ with rational coefficients satisfying

$$x^2 + 7 = f_1(x)^2 + f_2(x)^2 + \cdots + f_n(x)^2.$$

(Poland)

Answer. The smallest n is 5.

Solution 1. The equality $x^2 + 7 = x^2 + 2^2 + 1^2 + 1^2 + 1^2$ shows that $n \le 5$. It remains to show that $x^2 + 7$ is not a sum of four (or less) squares of polynomials with rational coefficients.

Suppose by way of contradiction that $x^2 + 7 = f_1(x)^2 + f_2(x)^2 + f_3(x)^2 + f_4(x)^2$, where the coefficients of polynomials f_1, f_2, f_3 and f_4 are rational (some of these polynomials may be zero).

Clearly, the degrees of f_1, f_2, f_3 and f_4 are at most 1. Thus $f_i(x) = a_i x + b_i$ for $i = 1, 2, 3, 4$ and some rationals a_1, b_1, a_2, b_2, a_3, b_3, a_4, b_4. It follows that $x^2 + 7 = \sum_{i=1}^{4}(a_i x + b_i)^2$ and hence

$$\sum_{i=1}^{4} a_i^2 = 1, \quad \sum_{i=1}^{4} a_i b_i = 0, \quad \sum_{i=1}^{4} b_i^2 = 7. \tag{1}$$

Let $p_i = a_i + b_i$ and $q_i = a_i - b_i$ for $i = 1, 2, 3, 4$. Then

$$\sum_{i=1}^{4} p_i^2 = \sum_{i=1}^{4} a_i^2 + 2\sum_{i=1}^{4} a_i b_i + \sum_{i=1}^{4} b_i^2 = 8,$$

$$\sum_{i=1}^{4} q_i^2 = \sum_{i=1}^{4} a_i^2 - 2\sum_{i=1}^{4} a_i b_i + \sum_{i=1}^{4} b_i^2 = 8$$

$$\text{and} \quad \sum_{i=1}^{4} p_i q_i = \sum_{i=1}^{4} a_i^2 - \sum_{i=1}^{4} b_i^2 = -6,$$

which means that there exist a solution in integers x_1, y_1, x_2, y_2, x_3, y_3, x_4, y_4 and $m > 0$ of the system of equations

$$\text{(i)} \ \sum_{i=1}^{4} x_i^2 = 8m^2, \qquad \text{(ii)} \ \sum_{i=1}^{4} y_i^2 = 8m^2, \qquad \text{(iii)} \ \sum_{i=1}^{4} x_i y_i = -6m^2.$$

We will show that such a solution does not exist.

Assume the contrary and consider a solution with minimal m. Note that if an integer x is odd then $x^2 \equiv 1 \pmod 8$. Otherwise (i.e., if x is even) we have $x^2 \equiv 0 \pmod 8$ or $x^2 \equiv 4 \pmod 8$. Hence, by (i), we get that x_1, x_2, x_3 and x_4 are even. Similarly, by (ii), we get that y_1, y_2, y_3 and y_4 are even. Thus the LHS of (iii) is divisible by 4 and m is also even. It follows that $\left(\frac{x_1}{2}, \frac{y_1}{2}, \frac{x_2}{2}, \frac{y_2}{2}, \frac{x_3}{2}, \frac{y_3}{2}, \frac{x_4}{2}, \frac{y_4}{2}, \frac{m}{2}\right)$ is a solution of the system of equations (i), (ii) and (iii), which contradicts the minimality of m.

Solution 2. We prove that $n \le 4$ is impossible. Define the numbers a_i, b_i for $i = 1, 2, 3, 4$ as in the previous solution.

By Euler's identity we have

$$\begin{aligned}(a_1^2 + a_2^2 + a_3^2 + a_4^2)(b_1^2 + b_2^2 + b_3^2 + b_4^2) =& (a_1 b_1 + a_2 b_2 + a_3 b_3 + a_4 b_4)^2 + (a_1 b_2 - a_2 b_1 + a_3 b_4 - a_4 b_3)^2 \\ &+ (a_1 b_3 - a_3 b_1 + a_4 b_2 - a_2 b_4)^2 + (a_1 b_4 - a_4 b_1 + a_2 b_3 - a_3 b_2)^2.\end{aligned}$$

So, using the relations (1) from the Solution 1 we get that

$$7 = \left(\frac{m_1}{m}\right)^2 + \left(\frac{m_2}{m}\right)^2 + \left(\frac{m_3}{m}\right)^2, \tag{2}$$

where

$$\begin{aligned}\frac{m_1}{m} &= a_1b_2 - a_2b_1 + a_3b_4 - a_4b_3,\\ \frac{m_2}{m} &= a_1b_3 - a_3b_1 + a_4b_2 - a_2b_4,\\ \frac{m_3}{m} &= a_1b_4 - a_4b_1 + a_2b_3 - a_3b_2\end{aligned}$$

and $m_1, m_2, m_3 \in \mathbb{Z}, m \in \mathbb{N}$.

Let m be a minimum positive integer number for which (2) holds. Then

$$8m^2 = m_1^2 + m_2^2 + m_3^2 + m^2.$$

As in the previous solution, we get that m_1, m_2, m_3, m are all even numbers. Then $\left(\frac{m_1}{2}, \frac{m_2}{2}, \frac{m_3}{2}, \frac{m}{2}\right)$ is also a solution of (2) which contradicts the minimality of m. So, we have $n \geq 5$. The example with $n = 5$ is already shown in Solution 1.

N4. Let a, b be integers, and let $P(x) = ax^3 + bx$. For any positive integer n we say that the pair (a, b) is *n-good* if $n \mid P(m) - P(k)$ implies $n \mid m - k$ for all integers m, k. We say that (a, b) is *very good* if (a, b) is n-good for infinitely many positive integers n.

(a) Find a pair (a, b) which is 51-good, but not very good.

(b) Show that all 2010-good pairs are very good.

(Turkey)

Solution. (a) We show that the pair $(1, -51^2)$ is good but not very good. Let $P(x) = x^3 - 51^2 x$. Since $P(51) = P(0)$, the pair $(1, -51^2)$ is not n-good for any positive integer that does not divide 51. Therefore, $(1, -51^2)$ is not very good.

On the other hand, if $P(m) \equiv P(k) \pmod{51}$, then $m^3 \equiv k^3 \pmod{51}$. By Fermat's theorem, from this we obtain

$$m \equiv m^3 \equiv k^3 \equiv k \pmod{3} \quad \text{and} \quad m \equiv m^{33} \equiv k^{33} \equiv k \pmod{17}.$$

Hence we have $m \equiv k \pmod{51}$. Therefore $(1, -51^2)$ is 51-good.

(b) We will show that if a pair (a, b) is 2010-good then (a, b) is 67^i-good for all positive integer i.

Claim 1. If (a, b) is 2010-good then (a, b) is 67-good.

Proof. Assume that $P(m) = P(k) \pmod{67}$. Since 67 and 30 are coprime, there exist integers m' and k' such that $k' \equiv k \pmod{67}$, $k' \equiv 0 \pmod{30}$, and $m' \equiv m \pmod{67}$, $m' \equiv 0 \pmod{30}$. Then we have $P(m') \equiv P(0) \equiv P(k') \pmod{30}$ and $P(m') \equiv P(m) \equiv P(k) \equiv P(k') \pmod{67}$, hence $P(m') \equiv P(k') \pmod{2010}$. This implies $m' \equiv k' \pmod{2010}$ as (a, b) is 2010-good. It follows that $m \equiv m' \equiv k' \equiv k \pmod{67}$. Therefore, (a, b) is 67-good. $\square$

Claim 2. If (a, b) is 67-good then $67 \mid a$.

Proof. Suppose that $67 \nmid a$. Consider the sets $\{at^2 \pmod{67} : 0 \le t \le 33\}$ and $\{-3as^2 - b \mod 67 : 0 \le s \le 33\}$. Since $a \not\equiv 0 \pmod{67}$, each of these sets has 34 elements. Hence they have at least one element in common. If $at^2 \equiv -3as^2 - b \pmod{67}$ then for $m = t \pm s$, $k = \mp 2s$ we have

$$\begin{aligned} P(m) - P(k) = a(m^3 - k^3) + b(m - k) &= (m - k)\big(a(m^2 + mk + k^2) + b\big) \\ &= (t \pm 3s)(at^2 + 3as^2 + b) \equiv 0 \pmod{67}. \end{aligned}$$

Since (a, b) is 67-good, we must have $m \equiv k \pmod{67}$ in both cases, that is, $t \equiv 3s \pmod{67}$ and $t \equiv -3s \pmod{67}$. This means $t \equiv s \equiv 0 \pmod{67}$ and $b \equiv -3as^2 - at^2 \equiv 0 \pmod{67}$. But then $67 \mid P(7) - P(2) = 67 \cdot 5a + 5b$ and $67 \nmid 7 - 2$, contradicting that (a, b) is 67-good. $\square$

Claim 3. If (a, b) is 2010-good then (a, b) is 67^i-good all $i \ge 1$.

Proof. By Claim 2, we have $67 \mid a$. If $67 \mid b$, then $P(x) \equiv P(0) \pmod{67}$ for all x, contradicting that (a, b) is 67-good. Hence, $67 \nmid b$.

Suppose that $67^i \mid P(m) - P(k) = (m - k)\big(a(m^2 + mk + k^2) + b\big)$. Since $67 \mid a$ and $67 \nmid b$, the second factor $a(m^2 + mk + k^2) + b$ is coprime to 67 and hence $67^i \mid m - k$. Therefore, (a, b) is 67^i-good. $\square$

Comment 1. In the proof of Claim 2, the following reasoning can also be used. Since 3 is not a quadratic residue modulo 67, either $au^2 \equiv -b \pmod{67}$ or $3av^2 \equiv -b \pmod{67}$ has a solution. The settings $(m, k) = (u, 0)$ in the first case and $(m, k) = (v, -2v)$ in the second case lead to $b \equiv 0 \pmod{67}$.

Comment 2. The pair $(67, 30)$ is n-good if and only if $n = d \cdot 67^i$, where $d \mid 30$ and $i \ge 0$. It shows that in part (b), one should deal with the large powers of 67 to reach the solution. The key property of number 67 is that it has the form $3k + 1$, so there exists a nontrivial cubic root of unity modulo 67.

N5. Let $\mathbb{N}$ be the set of all positive integers. Find all functions $f\colon \mathbb{N} \to \mathbb{N}$ such that the number $\big(f(m)+n\big)\big(m+f(n)\big)$ is a square for all $m, n \in \mathbb{N}$.

(U.S.A.)

Answer. All functions of the form $f(n) = n + c$, where $c \in \mathbb{N} \cup \{0\}$.

Solution. First, it is clear that all functions of the form $f(n) = n + c$ with a constant nonnegative integer c satisfy the problem conditions since $\big(f(m)+n\big)\big(f(n)+m\big) = (n+m+c)^2$ is a square.

We are left to prove that there are no other functions. We start with the following

Lemma. Suppose that $p \mid f(k) - f(\ell)$ for some prime p and positive integers k, ℓ. Then $p \mid k - \ell$.

Proof. Suppose first that $p^2 \mid f(k) - f(\ell)$, so $f(\ell) = f(k) + p^2 a$ for some integer a. Take some positive integer $D > \max\{f(k), f(\ell)\}$ which is not divisible by p and set $n = pD - f(k)$. Then the positive numbers $n + f(k) = pD$ and $n + f(\ell) = pD + \big(f(\ell) - f(k)\big) = p(D + pa)$ are both divisible by p but not by p^2. Now, applying the problem conditions, we get that both the numbers $\big(f(k)+n\big)\big(f(n)+k\big)$ and $\big(f(\ell)+n\big)\big(f(n)+\ell\big)$ are squares divisible by p (and thus by p^2); this means that the multipliers $f(n) + k$ and $f(n) + \ell$ are also divisible by p, therefore $p \mid \big(f(n)+k\big) - \big(f(n)+\ell\big) = k - \ell$ as well.

On the other hand, if $f(k) - f(\ell)$ is divisible by p but not by p^2, then choose the same number D and set $n = p^3 D - f(k)$. Then the positive numbers $f(k) + n = p^3 D$ and $f(\ell) + n = p^3 D + \big(f(\ell) - f(k)\big)$ are respectively divisible by p^3 (but not by p^4) and by p (but not by p^2). Hence in analogous way we obtain that the numbers $f(n) + k$ and $f(n) + \ell$ are divisible by p, therefore $p \mid \big(f(n)+k\big) - \big(f(n)+\ell\big) = k - \ell$. $\square$

We turn to the problem. First, suppose that $f(k) = f(\ell)$ for some $k, \ell \in \mathbb{N}$. Then by Lemma we have that $k - \ell$ is divisible by every prime number, so $k - \ell = 0$, or $k = \ell$. Therefore, the function f is injective.

Next, consider the numbers $f(k)$ and $f(k+1)$. Since the number $(k+1) - k = 1$ has no prime divisors, by Lemma the same holds for $f(k+1) - f(k)$; thus $|f(k+1) - f(k)| = 1$.

Now, let $f(2) - f(1) = q$, $|q| = 1$. Then we prove by induction that $f(n) = f(1) + q(n-1)$. The base for $n = 1, 2$ holds by the definition of q. For the step, if $n > 1$ we have $f(n+1) = f(n) \pm q = f(1) + q(n-1) \pm q$. Since $f(n) \neq f(n-2) = f(1) + q(n-2)$, we get $f(n) = f(1) + qn$, as desired.

Finally, we have $f(n) = f(1) + q(n-1)$. Then q cannot be -1 since otherwise for $n \geq f(1) + 1$ we have $f(n) \leq 0$ which is impossible. Hence $q = 1$ and $f(n) = (f(1) - 1) + n$ for each $n \in \mathbb{N}$, and $f(1) - 1 \geq 0$, as desired.

N6. The rows and columns of a $2^n \times 2^n$ table are numbered from 0 to $2^n - 1$. The cells of the table have been colored with the following property being satisfied: for each $0 \le i, j \le 2^n - 1$, the jth cell in the ith row and the $(i+j)$th cell in the jth row have the same color. (The indices of the cells in a row are considered modulo 2^n.)

Prove that the maximal possible number of colors is 2^n.

(Iran)

Solution. Throughout the solution we denote the cells of the table by coordinate pairs; (i, j) refers to the jth cell in the ith row.

Consider the directed graph, whose vertices are the cells of the board, and the edges are the arrows $(i, j) \to (j, i+j)$ for all $0 \le i, j \le 2^n - 1$. From each vertex (i, j), exactly one edge passes (to $(j, i + j \mod 2^n)$); conversely, to each cell (j, k) exactly one edge is directed (from the cell $(k - j \mod 2^n, j)$). Hence, the graph splits into cycles.

Now, in any coloring considered, the vertices of each cycle should have the same color by the problem condition. On the other hand, if each cycle has its own color, the obtained coloring obviously satisfies the problem conditions. Thus, the maximal possible number of colors is the same as the number of cycles, and we have to prove that this number is 2^n.

Next, consider any cycle $(i_1, j_1), (i_2, j_2), \ldots$; we will describe it in other terms. Define a sequence $(a_0, a_1, \ldots)$ by the relations $a_0 = i_1$, $a_1 = j_1$, $a_{n+1} = a_n + a_{n-1}$ for all $n \ge 1$ (we say that such a sequence is *a Fibonacci-type sequence*). Then an obvious induction shows that $i_k \equiv a_{k-1} \pmod{2^n}$, $j_k \equiv a_k \pmod{2^n}$. Hence we need to investigate the behavior of Fibonacci-type sequences modulo 2^n.

Denote by $F_0, F_1, \ldots$ the Fibonacci numbers defined by $F_0 = 0$, $F_1 = 1$, and $F_{n+2} = F_{n+1} + F_n$ for $n \ge 0$. We also set $F_{-1} = 1$ according to the recurrence relation.

For every positive integer m, denote by $\nu(m)$ the exponent of 2 in the prime factorization of m, i.e. for which $2^{\nu(m)} \mid m$ but $2^{\nu(m)+1} \nmid m$.

Lemma 1. For every Fibonacci-type sequence $a_0, a_1, a_2, \ldots$, and every $k \ge 0$, we have $a_k = F_{k-1}a_0 + F_k a_1$.

Proof. Apply induction on k. The base cases $k = 0, 1$ are trivial. For the step, from the induction hypothesis we get

$$a_{k+1} = a_k + a_{k-1} = (F_{k-1}a_0 + F_k a_1) + (F_{k-2}a_0 + F_{k-1}a_1) = F_k a_0 + F_{k+1}a_1. \qquad \square$$

Lemma 2. For every $m \ge 3$,

(a) we have $\nu(F_{3\cdot 2^{m-2}}) = m$;

(b) $d = 3 \cdot 2^{m-2}$ is the least positive index for which $2^m \mid F_d$;

(c) $F_{3\cdot 2^{m-2}+1} \equiv 1 + 2^{m-1} \pmod{2^m}$.

Proof. Apply induction on m. In the base case $m = 3$ we have $\nu(F_{3\cdot 2^{m-2}}) = F_6 = 8$, so $\nu(F_{3\cdot 2^{m-2}}) = \nu(8) = 3$, the preceding Fibonacci-numbers are not divisible by 8, and indeed $F_{3\cdot 2^{m-2}+1} = F_7 = 13 \equiv 1 + 4 \pmod 8$.

Now suppose that $m > 3$ and let $k = 3 \cdot 2^{m-3}$. By applying Lemma 1 to the Fibonacci-type sequence $F_k, F_{k+1}, \ldots$ we get

$$F_{2k} = F_{k-1}F_k + F_k F_{k+1} = (F_{k+1} - F_k)F_k + F_{k+1}F_k = 2F_{k+1}F_k - F_k^2,$$
$$F_{2k+1} = F_k \cdot F_k + F_{k+1} \cdot F_{k+1} = F_k^2 + F_{k+1}^2.$$

By the induction hypothesis, $\nu(F_k) = m - 1$, and F_{k+1} is odd. Therefore we get $\nu(F_k^2) = 2(m-1) > (m-1) + 1 = \nu(2F_k F_{k+1})$, which implies $\nu(F_{2k}) = m$, establishing statement (a).

Moreover, since $F_{k+1} = 1 + 2^{m-2} + a2^{m-1}$ for some integer a, we get

$$F_{2k+1} = F_k^2 + F_{k+1}^2 \equiv 0 + (1 + 2^{m-2} + a2^{m-1})^2 \equiv 1 + 2^{m-1} \pmod{2^m},$$

as desired in statement (c).

We are left to prove that $2^m \nmid F_\ell$ for $\ell < 2k$. Assume the contrary. Since $2^{m-1} \mid F_\ell$, from the induction hypothesis it follows that $\ell > k$. But then we have $F_\ell = F_{k-1}F_{\ell-k} + F_kF_{\ell-k+1}$, where the second summand is divisible by 2^{m-1} but the first one is not (since F_{k-1} is odd and $\ell - k < k$). Hence the sum is not divisible even by 2^{m-1}. A contradiction. □

Now, for every pair of integers $(a, b) \neq (0, 0)$, let $\mu(a, b) = \min\{\nu(a), \nu(b)\}$. By an obvious induction, for every Fibonacci-type sequence $A = (a_0, a_1, \dots)$ we have $\mu(a_0, a_1) = \mu(a_1, a_2) = \dots$; denote this common value by $\mu(A)$. Also denote by $p_n(A)$ the period of this sequence modulo 2^n, that is, the least $p > 0$ such that $a_{k+p} \equiv a_k \pmod{2^n}$ for all $k \geq 0$.

Lemma 3. Let $A = (a_0, a_1, \dots)$ be a Fibonacci-type sequence such that $\mu(A) = k < n$. Then $p_n(A) = 3 \cdot 2^{n-1-k}$.

Proof. First, we note that the sequence $(a_0, a_1, \dots)$ has period p modulo 2^n if and only if the sequence $(a_0/2^k, a_1/2^k, \dots)$ has period p modulo 2^{n-k}. Hence, passing to this sequence we can assume that $k = 0$.

We prove the statement by induction on n. It is easy to see that for $n = 1, 2$ the claim is true; actually, each Fibonacci-type sequence A with $\mu(A) = 0$ behaves as $0, 1, 1, 0, 1, 1, \dots$ modulo 2, and as $0, 1, 1, 2, 3, 1, 0, 1, 1, 2, 3, 1, \dots$ modulo 4 (all pairs of residues from which at least one is odd appear as a pair of consecutive terms in this sequence).

Now suppose that $n \geq 3$ and consider an arbitrary Fibonacci-type sequence $A = (a_0, a_1, \dots)$ with $\mu(A) = 0$. Obviously we should have $p_{n-1}(A) \mid p_n(A)$, or, using the induction hypothesis, $s = 3 \cdot 2^{n-2} \mid p_n(A)$. Next, we may suppose that a_0 is even; hence a_1 is odd, and $a_0 = 2b_0$, $a_1 = 2b_1 + 1$ for some integers b_0, b_1.

Consider the Fibonacci-type sequence $B = (b_0, b_1, \dots)$ starting with (b_0, b_1). Since $a_0 = 2b_0 + F_0$, $a_1 = 2b_1 + F_1$, by an easy induction we get $a_k = 2b_k + F_k$ for all $k \geq 0$. By the induction hypothesis, we have $p_{n-1}(B) \mid s$, hence the sequence $(2b_0, 2b_1, \dots)$ is s-periodic modulo 2^n. On the other hand, by Lemma 2 we have $F_{s+1} \equiv 1 + 2^{n-1} \pmod{2^n}$, $F_{2s} \equiv 0 \pmod{2^n}$, $F_{2s+1} \equiv 1 \pmod{2^n}$, hence

$$\begin{aligned} a_{s+1} = 2b_{s+1} + F_{s+1} \equiv 2b_1 + 1 + 2^{n-1} \not\equiv 2b_1 + 1 = a_1 \pmod{2^n}, \\ a_{2s} = 2b_{2s} + F_{2s} \equiv 2b_0 + 0 = a_0 \pmod{2^n}, \\ a_{2s+1} = 2b_{2s+1} + F_{2s+1} \equiv 2b_1 + 1 = a_1 \pmod{2^n}. \end{aligned}$$

The first line means that A is not s-periodic, while the other two provide that $a_{2s} \equiv a_0$, $a_{2s+1} \equiv a_1$ and hence $a_{2s+t} \equiv a_t$ for all $t \geq 0$. Hence $s \mid p_n(A) \mid 2s$ and $p_n(A) \neq s$, which means that $p_n(A) = 2s$, as desired. □

Finally, Lemma 3 provides a straightforward method of counting the number of cycles. Actually, take any number $0 \leq k \leq n-1$ and consider all the cells (i, j) with $\mu(i, j) = k$. The total number of such cells is $2^{2(n-k)} - 2^{2(n-k-1)} = 3 \cdot 2^{2n-2k-2}$. On the other hand, they are split into cycles, and by Lemma 3 the length of each cycle is $3 \cdot 2^{n-1-k}$. Hence the number of cycles consisting of these cells is exactly $\dfrac{3 \cdot 2^{2n-2k-2}}{3 \cdot 2^{n-1-k}} = 2^{n-k-1}$. Finally, there is only one cell $(0, 0)$ which is not mentioned in the previous computation, and it forms a separate cycle. So the total number of cycles is

$$1 + \sum_{k=0}^{n-1} 2^{n-1-k} = 1 + (1 + 2 + 4 + \dots + 2^{n-1}) = 2^n.$$

Comment. We outline a different proof for the essential part of Lemma 3. That is, we assume that $k = 0$ and show that in this case the period of (a_i) modulo 2^n coincides with the period of the Fibonacci numbers modulo 2^n; then the proof can be finished by the arguments from Lemma 2..

Note that p is a (not necessarily minimal) period of the sequence (a_i) modulo 2^n if and only if we have $a_0 \equiv a_p \pmod{2^n}$, $a_1 \equiv a_{p+1} \pmod{2^n}$, that is,

$$\begin{aligned} a_0 \equiv a_p &\equiv F_{p-1}a_0 + F_p a_1 = F_p(a_1 - a_0) + F_{p+1}a_0 \pmod{2^n}, \\ a_1 \equiv a_{p+1} &= F_p a_0 + F_{p+1}a_1 \pmod{2^n}. \end{aligned} \tag{1}$$

Now, If p is a period of (F_i) then we have $F_p \equiv F_0 = 0 \pmod{2^n}$ and $F_{p+1} \equiv F_1 = 1 \pmod{2^n}$, which by (1) implies that p is a period of (a_i) as well.

Conversely, suppose that p is a period of (a_i). Combining the relations of (1) we get

$$\begin{aligned} 0 = a_1 \cdot a_0 - a_0 \cdot a_1 &\equiv a_1\big(F_p(a_1 - a_0) + F_{p+1}a_0\big) - a_0(F_p a_0 + F_{p+1}a_1) \\ &= F_p(a_1^2 - a_1 a_0 - a_0^2) \pmod{2^n}, \\ a_1^2 - a_1 a_0 - a_0^2 = (a_1 - a_0)a_1 - a_0 \cdot a_0 &\equiv (a_1 - a_0)(F_p a_0 + F_{p+1}a_1) - a_0\big(F_p(a_1 - a_0) + F_{p+1}a_0\big) \\ &= F_{p+1}(a_1^2 - a_1 a_0 - a_0^2) \pmod{2^n}. \end{aligned}$$

Since at least one of the numbers a_0, a_1 is odd, the number $a_1^2 - a_1 a_0 - a_0^2$ is odd as well. Therefore the previous relations are equivalent with $F_p \equiv 0 \pmod{2^n}$ and $F_{p+1} \equiv 1 \pmod{2^n}$, which means exactly that p is a period of $(F_0, F_1, \dots)$ modulo 2^n.

So, the sets of periods of (a_i) and (F_i) coincide, and hence the minimal periods coincide as well.

52nd International Mathematical Olympiad

12 – 24 July 2011

Amsterdam

The Netherlands

Problem Shortlist with Solutions

IMO2011 Amsterdam

International Mathematical Olympiad Amsterdam 2011

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Problem shortlist with solutions

IMPORTANT

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The committee gratefully acknowledges the receipt of 142 problem proposals by the following 46 countries:

Armenia, Australia, Austria, Belarus, Belgium, Bosnia and Herzegovina, Brazil, Bulgaria, Canada, Colombia, Cyprus, Denmark, Estonia, Finland, France, Germany, Greece, Hong Kong, Hungary, India, Islamic Republic of Iran, Ireland, Israel, Japan, Kazakhstan, Republic of Korea, Luxembourg, Malaysia, Mexico, Mongolia, Montenegro, Pakistan, Poland, Romania, Russian Federation, Saudi Arabia, Serbia, Slovakia, Slovenia, Sweden, Taiwan, Thailand, Turkey, Ukraine, United Kingdom, United States of America

Algebra

A1

A1

For any set $A = \{a_1, a_2, a_3, a_4\}$ of four distinct positive integers with sum $s_A = a_1 + a_2 + a_3 + a_4$, let p_A denote the number of pairs (i, j) with $1 \leq i < j \leq 4$ for which $a_i + a_j$ divides s_A. Among all sets of four distinct positive integers, determine those sets A for which p_A is maximal.

A2

A2

Determine all sequences $(x_1, x_2, \ldots, x_{2011})$ of positive integers such that for every positive integer n there is an integer a with

$$x_1^n + 2x_2^n + \cdots + 2011x_{2011}^n = a^{n+1} + 1.$$

A3

A3

Determine all pairs (f, g) of functions from the set of real numbers to itself that satisfy

$$g(f(x + y)) = f(x) + (2x + y)g(y)$$

for all real numbers x and y.

A4

A4

Determine all pairs (f, g) of functions from the set of positive integers to itself that satisfy

$$f^{g(n)+1}(n) + g^{f(n)}(n) = f(n + 1) - g(n + 1) + 1$$

for every positive integer n. Here, $f^k(n)$ means $\underbrace{f(f(\ldots f}_{k}(n)\ldots))$.

A5

A5

Prove that for every positive integer n, the set $\{2, 3, 4, \ldots, 3n + 1\}$ can be partitioned into n triples in such a way that the numbers from each triple are the lengths of the sides of some obtuse triangle.

A6

A6

Let f be a function from the set of real numbers to itself that satisfies

$$f(x+y) \leq yf(x) + f(f(x))$$

for all real numbers x and y. Prove that $f(x) = 0$ for all $x \leq 0$.

A7

A7

Let a, b, and c be positive real numbers satisfying $\min(a+b, b+c, c+a) > \sqrt{2}$ and $a^2+b^2+c^2 = 3$. Prove that

$$\frac{a}{(b+c-a)^2} + \frac{b}{(c+a-b)^2} + \frac{c}{(a+b-c)^2} \geq \frac{3}{(abc)^2}.$$

Combinatorics

C1

C1

Let $n > 0$ be an integer. We are given a balance and n weights of weight $2^0, 2^1, \ldots, 2^{n-1}$. In a sequence of n moves we place all weights on the balance. In the first move we choose a weight and put it on the left pan. In each of the following moves we choose one of the remaining weights and we add it either to the left or to the right pan. Compute the number of ways in which we can perform these n moves in such a way that the right pan is never heavier than the left pan.

C2

C2

Suppose that 1000 students are standing in a circle. Prove that there exists an integer k with $100 \leq k \leq 300$ such that in this circle there exists a contiguous group of $2k$ students, for which the first half contains the same number of girls as the second half.

C3

C3

Let $\mathcal{S}$ be a finite set of at least two points in the plane. Assume that no three points of $\mathcal{S}$ are collinear. By a *windmill* we mean a process as follows. Start with a line ℓ going through a point $P \in \mathcal{S}$. Rotate ℓ clockwise around the *pivot* P until the line contains another point Q of $\mathcal{S}$. The point Q now takes over as the new pivot. This process continues indefinitely, with the pivot always being a point from $\mathcal{S}$.

Show that for a suitable $P \in \mathcal{S}$ and a suitable starting line ℓ containing P, the resulting windmill will visit each point of $\mathcal{S}$ as a pivot infinitely often.

C4

C4

Determine the greatest positive integer k that satisfies the following property: The set of positive integers can be partitioned into k subsets $A_1, A_2, \ldots, A_k$ such that for all integers $n \geq 15$ and all $i \in \{1, 2, \ldots, k\}$ there exist two distinct elements of A_i whose sum is n.

C5

C5

Let m be a positive integer and consider a checkerboard consisting of m by m unit squares. At the midpoints of some of these unit squares there is an ant. At time 0, each ant starts moving with speed 1 parallel to some edge of the checkerboard. When two ants moving in opposite directions meet, they both turn 90° clockwise and continue moving with speed 1. When more than two ants meet, or when two ants moving in perpendicular directions meet, the ants continue moving in the same direction as before they met. When an ant reaches one of the edges of the checkerboard, it falls off and will not re-appear.

Considering all possible starting positions, determine the latest possible moment at which the last ant falls off the checkerboard or prove that such a moment does not necessarily exist.

C6

C6

Let n be a positive integer and let $W = \ldots x_{-1}x_0x_1x_2\ldots$ be an infinite periodic word consisting of the letters a and b. Suppose that the minimal period N of W is greater than 2^n.

A finite nonempty word U is said to *appear* in W if there exist indices $k \le \ell$ such that $U = x_kx_{k+1}\ldots x_\ell$. A finite word U is called *ubiquitous* if the four words Ua, Ub, aU, and bU all appear in W. Prove that there are at least n ubiquitous finite nonempty words.

C7

C7

On a square table of 2011 by 2011 cells we place a finite number of napkins that each cover a square of 52 by 52 cells. In each cell we write the number of napkins covering it, and we record the maximal number k of cells that all contain the same nonzero number. Considering all possible napkin configurations, what is the largest value of k?

Geometry

G1

G1

Let ABC be an acute triangle. Let ω be a circle whose center L lies on the side BC. Suppose that ω is tangent to AB at B' and to AC at C'. Suppose also that the circumcenter O of the triangle ABC lies on the shorter arc $B'C'$ of ω. Prove that the circumcircle of ABC and ω meet at two points.

G2

G2

Let $A_1A_2A_3A_4$ be a non-cyclic quadrilateral. Let O_1 and r_1 be the circumcenter and the circumradius of the triangle $A_2A_3A_4$. Define O_2, O_3, O_4 and r_2, r_3, r_4 in a similar way. Prove that

$$\frac{1}{O_1A_1^2 - r_1^2} + \frac{1}{O_2A_2^2 - r_2^2} + \frac{1}{O_3A_3^2 - r_3^2} + \frac{1}{O_4A_4^2 - r_4^2} = 0.$$

G3

G3

Let $ABCD$ be a convex quadrilateral whose sides AD and BC are not parallel. Suppose that the circles with diameters AB and CD meet at points E and F inside the quadrilateral. Let ω_E be the circle through the feet of the perpendiculars from E to the lines AB, BC, and CD. Let ω_F be the circle through the feet of the perpendiculars from F to the lines CD, DA, and AB. Prove that the midpoint of the segment EF lies on the line through the two intersection points of ω_E and ω_F.

G4

G4

Let ABC be an acute triangle with circumcircle Ω. Let B_0 be the midpoint of AC and let C_0 be the midpoint of AB. Let D be the foot of the altitude from A, and let G be the centroid of the triangle ABC. Let ω be a circle through B_0 and C_0 that is tangent to the circle Ω at a point $X \neq A$. Prove that the points D, G, and X are collinear.

G5

G5

Let ABC be a triangle with incenter I and circumcircle ω. Let D and E be the second intersection points of ω with the lines AI and BI, respectively. The chord DE meets AC at a point F, and BC at a point G. Let P be the intersection point of the line through F parallel to AD and the line through G parallel to BE. Suppose that the tangents to ω at A and at B meet at a point K. Prove that the three lines AE, BD, and KP are either parallel or concurrent.

G6

G6

Let ABC be a triangle with $AB = AC$, and let D be the midpoint of AC. The angle bisector of $\angle BAC$ intersects the circle through D, B, and C in a point E inside the triangle ABC. The line BD intersects the circle through A, E, and B in two points B and F. The lines AF and BE meet at a point I, and the lines CI and BD meet at a point K. Show that I is the incenter of triangle KAB.

G7

G7

Let $ABCDEF$ be a convex hexagon all of whose sides are tangent to a circle ω with center O. Suppose that the circumcircle of triangle ACE is concentric with ω. Let J be the foot of the perpendicular from B to CD. Suppose that the perpendicular from B to DF intersects the line EO at a point K. Let L be the foot of the perpendicular from K to DE. Prove that $DJ = DL$.

G8

G8

Let ABC be an acute triangle with circumcircle ω. Let t be a tangent line to ω. Let t_a, t_b, and t_c be the lines obtained by reflecting t in the lines BC, CA, and AB, respectively. Show that the circumcircle of the triangle determined by the lines t_a, t_b, and t_c is tangent to the circle ω.

Number Theory

N1

N1

For any integer $d > 0$, let $f(d)$ be the smallest positive integer that has exactly d positive divisors (so for example we have $f(1) = 1$, $f(5) = 16$, and $f(6) = 12$). Prove that for every integer $k \geq 0$ the number $f(2^k)$ divides $f(2^{k+1})$.

N2

N2

Consider a polynomial $P(x) = (x + d_1)(x + d_2) \cdot \ldots \cdot (x + d_9)$, where $d_1, d_2, \ldots, d_9$ are nine distinct integers. Prove that there exists an integer N such that for all integers $x \geq N$ the number $P(x)$ is divisible by a prime number greater than 20.

N3

N3

Let $n \geq 1$ be an odd integer. Determine all functions f from the set of integers to itself such that for all integers x and y the difference $f(x) - f(y)$ divides $x^n - y^n$.

N4

N4

For each positive integer k, let $t(k)$ be the largest odd divisor of k. Determine all positive integers a for which there exists a positive integer n such that all the differences

$$t(n+a) - t(n), \quad t(n+a+1) - t(n+1), \quad \ldots, \quad t(n+2a-1) - t(n+a-1)$$

are divisible by 4.

N5

N5

Let f be a function from the set of integers to the set of positive integers. Suppose that for any two integers m and n, the difference $f(m) - f(n)$ is divisible by $f(m - n)$. Prove that for all integers m, n with $f(m) \leq f(n)$ the number $f(n)$ is divisible by $f(m)$.

N6

N6

Let $P(x)$ and $Q(x)$ be two polynomials with integer coefficients such that no nonconstant polynomial with rational coefficients divides both $P(x)$ and $Q(x)$. Suppose that for every positive integer n the integers $P(n)$ and $Q(n)$ are positive, and $2^{Q(n)} - 1$ divides $3^{P(n)} - 1$. Prove that $Q(x)$ is a constant polynomial.

N7

N7

Let p be an odd prime number. For every integer a, define the number

$$S_a = \frac{a}{1} + \frac{a^2}{2} + \cdots + \frac{a^{p-1}}{p-1}.$$

Let m and n be integers such that

$$S_3 + S_4 - 3S_2 = \frac{m}{n}.$$

Prove that p divides m.

N8

N8

Let k be a positive integer and set $n = 2^k + 1$. Prove that n is a prime number if and only if the following holds: there is a permutation $a_1, \ldots, a_{n-1}$ of the numbers $1, 2, \ldots, n-1$ and a sequence of integers $g_1, g_2, \ldots, g_{n-1}$ such that n divides $g_i^{a_i} - a_{i+1}$ for every $i \in \{1, 2, \ldots, n-1\}$, where we set $a_n = a_1$.

A1

For any set $A = \{a_1, a_2, a_3, a_4\}$ of four distinct positive integers with sum $s_A = a_1+a_2+a_3+a_4$, let p_A denote the number of pairs (i,j) with $1 \le i < j \le 4$ for which $a_i + a_j$ divides s_A. Among all sets of four distinct positive integers, determine those sets A for which p_A is maximal.

Answer. The sets A for which p_A is maximal are the sets the form $\{d, 5d, 7d, 11d\}$ and $\{d, 11d, 19d, 29d\}$, where d is any positive integer. For all these sets p_A is 4.

Solution. Firstly, we will prove that the maximum value of p_A is at most 4. Without loss of generality, we may assume that $a_1 < a_2 < a_3 < a_4$. We observe that for each pair of indices (i,j) with $1 \le i < j \le 4$, the sum $a_i + a_j$ divides s_A if and only if $a_i + a_j$ divides $s_A - (a_i + a_j) = a_k + a_l$, where k and l are the other two indices. Since there are 6 distinct pairs, we have to prove that at least two of them do not satisfy the previous condition. We claim that two such pairs are (a_2, a_4) and (a_3, a_4). Indeed, note that $a_2 + a_4 > a_1 + a_3$ and $a_3 + a_4 > a_1 + a_2$. Hence $a_2 + a_4$ and $a_3 + a_4$ do not divide s_A. This proves $p_A \le 4$.

Now suppose $p_A = 4$. By the previous argument we have

$$\begin{array}{lcl} a_1 + a_4 \mid a_2 + a_3 & \text{and} & a_2 + a_3 \mid a_1 + a_4, \\ a_1 + a_2 \mid a_3 + a_4 & \text{and} & a_3 + a_4 \nmid a_1 + a_2, \\ a_1 + a_3 \mid a_2 + a_4 & \text{and} & a_2 + a_4 \nmid a_1 + a_3. \end{array}$$

Hence, there exist positive integers m and n with $m > n \ge 2$ such that

$$\begin{cases} a_1 + a_4 = a_2 + a_3 \\ m(a_1 + a_2) = a_3 + a_4 \\ n(a_1 + a_3) = a_2 + a_4. \end{cases}$$

Adding up the first equation and the third one, we get $n(a_1 + a_3) = 2a_2 + a_3 - a_1$. If $n \ge 3$, then $n(a_1 + a_3) > 3a_3 > 2a_2 + a_3 > 2a_2 + a_3 - a_1$. This is a contradiction. Therefore $n = 2$. If we multiply by 2 the sum of the first equation and the third one, we obtain

$$6a_1 + 2a_3 = 4a_2,$$

while the sum of the first one and the second one is

$$(m+1)a_1 + (m-1)a_2 = 2a_3.$$

Adding up the last two equations we get

$$(m+7)a_1 = (5-m)a_2.$$

It follows that $5 - m \geq 1$, because the left-hand side of the last equation and a_2 are positive. Since we have $m > n = 2$, the integer m can be equal only to either 3 or 4. Substituting $(3, 2)$ and $(4, 2)$ for (m, n) and solving the previous system of equations, we find the families of solutions $\{d, 5d, 7d, 11d\}$ and $\{d, 11d, 19d, 29d\}$, where d is any positive integer.

A2

Determine all sequences $(x_1, x_2, \dots, x_{2011})$ of positive integers such that for every positive integer n there is an integer a with

$$x_1^n + 2x_2^n + \cdots + 2011x_{2011}^n = a^{n+1} + 1.$$

Answer. The only sequence that satisfies the condition is

$$(x_1, \dots, x_{2011}) = (1, k, \dots, k) \quad \text{with } k = 2 + 3 + \cdots + 2011 = 2023065.$$

Solution. Throughout this solution, the set of positive integers will be denoted by $\mathbb{Z}_+$.

Put $k = 2 + 3 + \cdots + 2011 = 2023065$. We have

$$1^n + 2k^n + \cdots 2011k^n = 1 + k \cdot k^n = k^{n+1} + 1$$

for all n, so $(1, k, \dots, k)$ is a valid sequence. We shall prove that it is the only one.

Let a valid sequence $(x_1, \dots, x_{2011})$ be given. For each $n \in \mathbb{Z}_+$ we have some $y_n \in \mathbb{Z}_+$ with

$$x_1^n + 2x_2^n + \cdots + 2011x_{2011}^n = y_n^{n+1} + 1.$$

Note that $x_1^n + 2x_2^n + \cdots + 2011x_{2011}^n < (x_1 + 2x_2 + \cdots + 2011x_{2011})^{n+1}$, which implies that the sequence (y_n) is bounded. In particular, there is some $y \in \mathbb{Z}_+$ with $y_n = y$ for infinitely many n.

Let m be the maximum of all the x_i. Grouping terms with equal x_i together, the sum $x_1^n + 2x_2^n + \cdots + 2011x_{2011}^n$ can be written as

$$x_1^n + 2x_2^n + \cdots + x_{2011}^n = a_m m^n + a_{m-1}(m-1)^n + \cdots + a_1$$

with $a_i \geq 0$ for all i and $a_1 + \cdots + a_m = 1 + 2 + \cdots + 2011$. So there exist arbitrarily large values of n, for which

$$a_m m^n + \cdots + a_1 - 1 - y \cdot y^n = 0. \tag{1}$$

The following lemma will help us to determine the a_i and y:

Lemma. Let integers $b_1, \dots, b_N$ be given and assume that there are arbitrarily large positive integers n with $b_1 + b_2 2^n + \cdots + b_N N^n = 0$. Then $b_i = 0$ for all i.

Proof. Suppose that not all b_i are zero. We may assume without loss of generality that $b_N \neq 0$.

Dividing through by N^n gives

$$|b_N| = \left| b_{N-1}\left(\frac{N-1}{N}\right)^n + \cdots + b_1\left(\frac{1}{N}\right)^n \right| \leq (|b_{N-1}| + \cdots + |b_1|)\left(\frac{N-1}{N}\right)^n.$$

The expression $\left(\frac{N-1}{N}\right)^n$ can be made arbitrarily small for n large enough, contradicting the assumption that b_N be non-zero. $\square$

We obviously have $y > 1$. Applying the lemma to (1) we see that $a_m = y = m$, $a_1 = 1$, and all the other a_i are zero. This implies $(x_1, \ldots, x_{2011}) = (1, m, \ldots, m)$. But we also have $1 + m = a_1 + \cdots + a_m = 1 + \cdots + 2011 = 1 + k$ so $m = k$, which is what we wanted to show.

A3

Determine all pairs (f, g) of functions from the set of real numbers to itself that satisfy

$$g(f(x+y)) = f(x) + (2x+y)g(y)$$

for all real numbers x and y.

Answer. Either both f and g vanish identically, or there exists a real number C such that $f(x) = x^2 + C$ and $g(x) = x$ for all real numbers x.

Solution. Clearly all these pairs of functions satisfy the functional equation in question, so it suffices to verify that there cannot be any further ones. Substituting $-2x$ for y in the given functional equation we obtain

$$g(f(-x)) = f(x). \tag{1}$$

Using this equation for $-x-y$ in place of x we obtain

$$f(-x-y) = g(f(x+y)) = f(x) + (2x+y)g(y). \tag{2}$$

Now for any two real numbers a and b, setting $x = -b$ and $y = a+b$ we get

$$f(-a) = f(-b) + (a-b)g(a+b).$$

If c denotes another arbitrary real number we have similarly

$$f(-b) = f(-c) + (b-c)g(b+c)$$

as well as

$$f(-c) = f(-a) + (c-a)g(c+a).$$

Adding all these equations up, we obtain

$$\big((a+c)-(b+c)\big)g(a+b) + \big((a+b)-(a+c)\big)g(b+c) + \big((b+c)-(a+b)\big)g(a+c) = 0.$$

Now given any three real numbers x, y, and z one may determine three reals a, b, and c such that $x = b+c$, $y = c+a$, and $z = a+b$, so that we get

$$(y-x)g(z) + (z-y)g(x) + (x-z)g(y) = 0.$$

This implies that the three points $(x, g(x))$, $(y, g(y))$, and $(z, g(z))$ from the graph of g are collinear. Hence that graph is a line, i.e., g is either a constant or a linear function.

Let us write $g(x) = Ax + B$, where A and B are two real numbers. Substituting $(0, -y)$ for (x, y) in (2) and denoting $C = f(0)$, we have $f(y) = Ay^2 - By + C$. Now, comparing the coefficients of x^2 in (1) we see that $A^2 = A$, so $A = 0$ or $A = 1$.

If $A = 0$, then (1) becomes $B = -Bx + C$ and thus $B = C = 0$, which provides the first of the two solutions mentioned above.

Now suppose $A = 1$. Then (1) becomes $x^2 - Bx + C + B = x^2 - Bx + C$, so $B = 0$. Thus, $g(x) = x$ and $f(x) = x^2 + C$, which is the second solution from above.

Comment. Another way to show that $g(x)$ is either a constant or a linear function is the following. If we interchange x and y in the given functional equation and subtract this new equation from the given one, we obtain

$$f(x) - f(y) = (2y + x)g(x) - (2x + y)g(y).$$

Substituting $(x, 0)$, $(1, x)$, and $(0, 1)$ for (x, y), we get

$$\begin{aligned} f(x) - f(0) &= xg(x) - 2xg(0), \\ f(1) - f(x) &= (2x + 1)g(1) - (x + 2)g(x), \\ f(0) - f(1) &= 2g(0) - g(1). \end{aligned}$$

Taking the sum of these three equations and dividing by 2, we obtain

$$g(x) = x\big(g(1) - g(0)\big) + g(0).$$

This proves that $g(x)$ is either a constant of a linear function.

A4

Determine all pairs (f,g) of functions from the set of positive integers to itself that satisfy

$$f^{g(n)+1}(n)+g^{f(n)}(n)=f(n+1)-g(n+1)+1$$

for every positive integer n. Here, $f^k(n)$ means $\underbrace{f(f(\ldots f}_{k}(n)\ldots))$.

Answer. The only pair (f,g) of functions that satisfies the equation is given by $f(n)=n$ and $g(n)=1$ for all n.

Solution. The given relation implies

$$f\left(f^{g(n)}(n)\right)<f(n+1) \quad \text{for all } n, \tag{1}$$

which will turn out to be sufficient to determine f.

Let $y_1<y_2<\ldots$ be all the values attained by f (this sequence might be either finite or infinite). We will prove that for every positive n the function f attains at least n values, and we have $(\mathrm{i})_n$: $f(x)=y_n$ if and only if $x=n$, and $(\mathrm{ii})_n$: $y_n=n$. The proof will follow the scheme

$$(\mathrm{i})_1,\ (\mathrm{ii})_1,\ (\mathrm{i})_2,\ (\mathrm{ii})_2,\ \ldots,\ (\mathrm{i})_n,\ (\mathrm{ii})_n,\ \ldots \tag{2}$$

To start, consider any x such that $f(x)=y_1$. If $x>1$, then (1) reads $f\big(f^{g(x-1)}(x-1)\big)<y_1$, contradicting the minimality of y_1. So we have that $f(x)=y_1$ is equivalent to $x=1$, establishing $(\mathrm{i})_1$.

Next, assume that for some n statement $(\mathrm{i})_n$ is established, as well as all the previous statements in (2). Note that these statements imply that for all $k\geq 1$ and $a<n$ we have $f^k(x)=a$ if and only if $x=a$.

Now, each value y_i with $1\leq i\leq n$ is attained at the unique integer i, so y_{n+1} exists. Choose an arbitrary x such that $f(x)=y_{n+1}$; we necessarily have $x>n$. Substituting $x-1$ into (1) we have $f\big(f^{g(x-1)}(x-1)\big)<y_{n+1}$, which implies

$$f^{g(x-1)}(x-1)\in\{1,\ldots,n\} \tag{3}$$

Set $b=f^{g(x-1)}(x-1)$. If $b<n$ then we would have $x-1=b$ which contradicts $x>n$. So $b=n$, and hence $y_n=n$, which proves $(\mathrm{ii})_n$. Next, from $(\mathrm{i})_n$ we now get $f(k)=n \iff k=n$, so removing all the iterations of f in (3) we obtain $x-1=b=n$, which proves $(\mathrm{i})_{n+1}$.

So, all the statements in (2) are valid and hence $f(n)=n$ for all n. The given relation between f and g now reads $n+g^n(n)=n+1-g(n+1)+1$ or $g^n(n)+g(n+1)=2$, from which it immediately follows that we have $g(n) = 1$ for all n.

Comment. Several variations of the above solution are possible. For instance, one may first prove by induction that the smallest n values of f are exactly $f(1) < \cdots < f(n)$ and proceed as follows. We certainly have $f(n) \geq n$ for all n. If there is an n with $f(n) > n$, then $f(x) > x$ for all $x \geq n$. From this we conclude $f^{g(n)+1}(n) > f^{g(n)}(n) > \cdots > f(n)$. But we also have $f^{g(n)+1} < f(n+1)$. Having squeezed in a function value between $f(n)$ and $f(n+1)$, we arrive at a contradiction.

In any case, the inequality (1) plays an essential rôle.

A5

Prove that for every positive integer n, the set $\{2, 3, 4, \ldots, 3n+1\}$ can be partitioned into n triples in such a way that the numbers from each triple are the lengths of the sides of some obtuse triangle.

Solution. Throughout the solution, we denote by $[a, b]$ the set $\{a, a+1, \ldots, b\}$. We say that $\{a, b, c\}$ is an *obtuse triple* if a, b, c are the sides of some obtuse triangle.

We prove by induction on n that there exists a partition of $[2, 3n+1]$ into n obtuse triples A_i $(2 \leq i \leq n+1)$ having the form $A_i = \{i, a_i, b_i\}$. For the base case $n = 1$, one can simply set $A_2 = \{2, 3, 4\}$. For the induction step, we need the following simple lemma.

Lemma. Suppose that the numbers $a < b < c$ form an obtuse triple, and let x be any positive number. Then the triple $\{a, b+x, c+x\}$ is also obtuse.

Proof. The numbers $a < b+x < c+x$ are the sides of a triangle because $(c+x) - (b+x) = c - b < a$. This triangle is obtuse since $(c+x)^2 - (b+x)^2 = (c-b)(c+b+2x) > (c-b)(c+b) > a^2$. $\square$

Now we turn to the induction step. Let $n > 1$ and put $t = \lfloor n/2 \rfloor < n$. By the induction hypothesis, there exists a partition of the set $[2, 3t+1]$ into t obtuse triples $A'_i = \{i, a'_i, b'_i\}$ $(i \in [2, t+1])$. For the same values of i, define $A_i = \{i, a'_i + (n-t), b'_i + (n-t)\}$. The constructed triples are obviously disjoint, and they are obtuse by the lemma. Moreover, we have

$$\bigcup_{i=2}^{t+1} A_i = [2, t+1] \cup [n+2, n+2t+1].$$

Next, for each $i \in [t+2, n+1]$, define $A_i = \{i, n+t+i, 2n+i\}$. All these sets are disjoint, and

$$\bigcup_{i=t+2}^{n+1} A_i = [t+2, n+1] \cup [n+2t+2, 2n+t+1] \cup [2n+t+2, 3n+1],$$

so

$$\bigcup_{i=2}^{n+1} A_i = [2, 3n+1].$$

Thus, we are left to prove that the triple A_i is obtuse for each $i \in [t+2, n+1]$.

Since $(2n+i) - (n+t+i) = n - t < t + 2 \leq i$, the elements of A_i are the sides of a triangle. Next, we have

$$(2n+i)^2 - (n+t+i)^2 = (n-t)(3n+t+2i) \geq \frac{n}{2} \cdot (3n+3(t+1)+1) > \frac{n}{2} \cdot \frac{9n}{2} \geq (n+1)^2 \geq i^2,$$

so this triangle is obtuse. The proof is completed.

A6

Let f be a function from the set of real numbers to itself that satisfies

$$f(x+y) \leq yf(x) + f(f(x)) \tag{1}$$

for all real numbers x and y. Prove that $f(x) = 0$ for all $x \leq 0$.

Solution 1. Substituting $y = t - x$, we rewrite (1) as

$$f(t) \leq tf(x) - xf(x) + f(f(x)). \tag{2}$$

Consider now some real numbers a, b and use (2) with $t = f(a)$, $x = b$ as well as with $t = f(b)$, $x = a$. We get

$$f(f(a)) - f(f(b)) \leq f(a)f(b) - bf(b),$$
$$f(f(b)) - f(f(a)) \leq f(a)f(b) - af(a).$$

Adding these two inequalities yields

$$2f(a)f(b) \geq af(a) + bf(b).$$

Now, substitute $b = 2f(a)$ to obtain $2f(a)f(b) \geq af(a) + 2f(a)f(b)$, or $af(a) \leq 0$. So, we get

$$f(a) \geq 0 \quad \text{for all } a < 0. \tag{3}$$

Now suppose $f(x) > 0$ for some real number x. From (2) we immediately get that for every $t < \dfrac{xf(x) - f(f(x))}{f(x)}$ we have $f(t) < 0$. This contradicts (3); therefore

$$f(x) \leq 0 \quad \text{for all real } x, \tag{4}$$

and by (3) again we get $f(x) = 0$ for all $x < 0$.

We are left to find $f(0)$. Setting $t = x < 0$ in (2) we get

$$0 \leq 0 - 0 + f(0),$$

so $f(0) \geq 0$. Combining this with (4) we obtain $f(0) = 0$.

Solution 2. We will also use the condition of the problem in form (2). For clarity we divide the argument into four steps.

Step 1. We begin by proving that f attains nonpositive values only. Assume that there exist some real number z with $f(z) > 0$. Substituting $x = z$ into (2) and setting $A = f(z)$, $B = -zf(z) - f(f(z))$ we get $f(t) \leq At + B$ for all real t. Hence, if for any positive real number t we substitute $x = -t$, $y = t$ into (1), we get

$$\begin{aligned} f(0) &\leq tf(-t) + f(f(-t)) \leq t(-At + B) + Af(-t) + B \\ &\leq -t(At - B) + A(-At + B) + B = -At^2 - (A^2 - B)t + (A+1)B. \end{aligned}$$

But surely this cannot be true if we take t to be large enough. This contradiction proves that we have indeed $f(x) \leq 0$ for all real numbers x. Note that for this reason (1) entails

$$f(x+y) \leq yf(x) \tag{5}$$

for all real numbers x and y.

Step 2. We proceed by proving that f has at least one zero. If $f(0) = 0$, we are done. Otherwise, in view of Step 1 we get $f(0) < 0$. Observe that (5) tells us now $f(y) \leq yf(0)$ for all real numbers y. Thus we can specify a positive real number a that is so large that $f(a)^2 > -f(0)$. Put $b = f(a)$ and substitute $x = b$ and $y = -b$ into (5); we learn $-b^2 < f(0) \leq -bf(b)$, i.e. $b < f(b)$. Now we apply (2) to $x = b$ and $t = f(b)$, which yields

$$f(f(b)) \leq \big(f(b) - b\big)f(b) + f(f(b)),$$

i.e. $f(b) \geq 0$. So in view of Step 1, b is a zero of f.

Step 3. Next we show that if $f(a) = 0$ and $b < a$, then $f(b) = 0$ as well. To see this, we just substitute $x = b$ and $y = a - b$ into (5), thus getting $f(b) \geq 0$, which suffices by Step 1.

Step 4. By Step 3, the solution of the problem is reduced to showing $f(0) = 0$. Pick any zero r of f and substitute $x = r$ and $y = -1$ into (1). Because of $f(r) = f(r-1) = 0$ this gives $f(0) \geq 0$ and hence $f(0) = 0$ by Step 1 again.

Comment 1. Both of these solutions also show $f(x) \leq 0$ for all real numbers x. As one can see from Solution 1, this task gets much easier if one already knows that f takes nonnegative values for sufficiently small arguments. Another way of arriving at this statement, suggested by the proposer, is as follows:

Put $a = f(0)$ and substitute $x = 0$ into (1). This gives $f(y) \leq ay + f(a)$ for all real numbers y. Thus if for any real number x we plug $y = a - x$ into (1), we obtain

$$f(a) \leq (a-x)f(x) + f(f(x)) \leq (a-x)f(x) + af(x) + f(a)$$

and hence $0 \leq (2a - x)f(x)$. In particular, if $x < 2a$, then $f(x) \geq 0$.

Having reached this point, one may proceed almost exactly as in the first solution to deduce $f(x) \leq 0$ for all x. Afterwards the problem can be solved in a few lines as shown in steps 3 and 4 of the second solution.

Comment 2. The original problem also contained the question whether a nonzero function satisfying the problem condition exists. Here we present a family of such functions.

Notice first that if $g : (0, \infty) \longrightarrow [0, \infty)$ denotes any function such that

$$g(x + y) \geq yg(x) \tag{6}$$

for all positive real numbers x and y, then the function f given by

$$f(x) = \begin{cases} -g(x) & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases} \tag{7}$$

automatically satisfies (1). Indeed, we have $f(x) \leq 0$ and hence also $f(f(x)) = 0$ for all real numbers x. So (1) reduces to (5); moreover, this inequality is nontrivial only if x and y are positive. In this last case it is provided by (6).

Now it is not hard to come up with a nonzero function g obeying (6). E.g. $g(z) = Ce^z$ (where C is a positive constant) fits since the inequality $e^y > y$ holds for all (positive) real numbers y. One may also consider the function $g(z) = e^z - 1$; in this case, we even have that f is continuous.

A7

Let a, b, and c be positive real numbers satisfying $\min(a+b, b+c, c+a) > \sqrt{2}$ and $a^2+b^2+c^2 = 3$. Prove that

$$\frac{a}{(b+c-a)^2} + \frac{b}{(c+a-b)^2} + \frac{c}{(a+b-c)^2} \geq \frac{3}{(abc)^2}. \tag{1}$$

Throughout both solutions, we denote the sums of the form $f(a,b,c) + f(b,c,a) + f(c,a,b)$ by $\sum f(a,b,c)$.

Solution 1. The condition $b+c > \sqrt{2}$ implies $b^2+c^2 > 1$, so $a^2 = 3 - (b^2+c^2) < 2$, i.e. $a < \sqrt{2} < b+c$. Hence we have $b+c-a > 0$, and also $c+a-b > 0$ and $a+b-c > 0$ for similar reasons.

We will use the variant of Hölder's inequality

$$\frac{x_1^{p+1}}{y_1^p} + \frac{x_1^{p+1}}{y_1^p} + \ldots + \frac{x_n^{p+1}}{y_n^p} \geq \frac{(x_1+x_2+\ldots+x_n)^{p+1}}{(y_1+y_2+\ldots+y_n)^p},$$

which holds for all positive real numbers $p, x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n$. Applying it to the left-hand side of (1) with $p = 2$ and $n = 3$, we get

$$\sum \frac{a}{(b+c-a)^2} = \sum \frac{(a^2)^3}{a^5(b+c-a)^2} \geq \frac{(a^2+b^2+c^2)^3}{\left(\sum a^{5/2}(b+c-a)\right)^2} = \frac{27}{\left(\sum a^{5/2}(b+c-a)\right)^2}. \tag{2}$$

To estimate the denominator of the right-hand part, we use an instance of Schur's inequality, namely

$$\sum a^{3/2}(a-b)(a-c) \geq 0,$$

which can be rewritten as

$$\sum a^{5/2}(b+c-a) \leq abc(\sqrt{a}+\sqrt{b}+\sqrt{c}).$$

Moreover, by the inequality between the arithmetic mean and the fourth power mean we also have

$$\left(\frac{\sqrt{a}+\sqrt{b}+\sqrt{c}}{3}\right)^4 \leq \frac{a^2+b^2+c^2}{3} = 1,$$

i.e., $\sqrt{a}+\sqrt{b}+\sqrt{c} \leq 3$. Hence, (2) yields

$$\sum \frac{a}{(b+c-a)^2} \geq \frac{27}{\left(abc(\sqrt{a}+\sqrt{b}+\sqrt{c})\right)^2} \geq \frac{3}{a^2b^2c^2},$$

thus solving the problem.

Comment. In this solution, one may also start from the following version of HÖLDER's inequality

$$\left(\sum_{i=1}^{n} a_i^3\right)\left(\sum_{i=1}^{n} b_i^3\right)\left(\sum_{i=1}^{n} c_i^3\right) \geq \left(\sum_{i=1}^{n} a_i b_i c_i\right)^3$$

applied as

$$\sum \frac{a}{(b+c-a)^2} \cdot \sum a^3(b+c-a) \cdot \sum a^2(b+c-a) \geq 27.$$

After doing that, one only needs the slightly better known instances

$$\sum a^3(b+c-a) \leq (a+b+c)abc \qquad \text{and} \qquad \sum a^2(b+c-a) \leq 3abc$$

of SCHUR's Inequality.

Solution 2. As in Solution 1, we mention that all the numbers $b+c-a$, $a+c-b$, $a+b-c$ are positive. We will use only this restriction and the condition

$$a^5 + b^5 + c^5 \geq 3, \tag{3}$$

which is weaker than the given one. Due to the symmetry we may assume that $a \geq b \geq c$.

In view of (3), it suffices to prove the inequality

$$\sum \frac{a^3 b^2 c^2}{(b+c-a)^2} \geq \sum a^5,$$

or, moving all the terms into the left-hand part,

$$\sum \frac{a^3}{(b+c-a)^2}\left((bc)^2 - (a(b+c-a))^2\right) \geq 0. \tag{4}$$

Note that the signs of the expressions $(yz)^2 - (x(y+z-x))^2$ and $yz - x(y+z-x) = (x-y)(x-z)$ are the same for every positive x, y, z satisfying the triangle inequality. So the terms in (4) corresponding to a and c are nonnegative, and hence it is sufficient to prove that the sum of the terms corresponding to a and b is nonnegative. Equivalently, we need the relation

$$\frac{a^3}{(b+c-a)^2}(a-b)(a-c)(bc+a(b+c-a)) \geq \frac{b^3}{(a+c-b)^2}(a-b)(b-c)(ac+b(a+c-b)).$$

Obviously, we have

$$a^3 \geq b^3 \geq 0, \quad 0 < b+c-a \leq a+c-b, \quad \text{and} \quad a-c \geq b-c \geq 0,$$

hence it suffices to prove that

$$\frac{ab+ac+bc-a^2}{b+c-a} \geq \frac{ab+ac+bc-b^2}{c+a-b}.$$

Since all the denominators are positive, it is equivalent to

$$(c+a-b)(ab+ac+bc-a^2)-(ab+ac+bc-b^2)(b+c-a)\geq 0,$$

or

$$(a-b)(2ab-a^2-b^2+ac+bc)\geq 0.$$

Since $a\geq b$, the last inequality follows from

$$c(a+b)>(a-b)^2$$

which holds since $c>a-b\geq 0$ and $a+b>a-b\geq 0$.

C1

Let $n > 0$ be an integer. We are given a balance and n weights of weight $2^0, 2^1, \ldots, 2^{n-1}$. In a sequence of n moves we place all weights on the balance. In the first move we choose a weight and put it on the left pan. In each of the following moves we choose one of the remaining weights and we add it either to the left or to the right pan. Compute the number of ways in which we can perform these n moves in such a way that the right pan is never heavier than the left pan.

Answer. The number $f(n)$ of ways of placing the n weights is equal to the product of all odd positive integers less than or equal to $2n-1$, i.e. $f(n) = (2n-1)!! = 1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-1)$.

Solution 1. Assume $n \geq 2$. We claim

$$f(n) = (2n-1)f(n-1). \tag{1}$$

Firstly, note that after the first move the left pan is always at least 1 heavier than the right one. Hence, any valid way of placing the n weights on the scale gives rise, by not considering weight 1, to a valid way of placing the weights $2, 2^2, \ldots, 2^{n-1}$.

If we divide the weight of each weight by 2, the answer does not change. So these $n-1$ weights can be placed on the scale in $f(n-1)$ valid ways. Now we look at weight 1. If it is put on the scale in the first move, then it has to be placed on the left side, otherwise it can be placed either on the left or on the right side, because after the first move the difference between the weights on the left pan and the weights on the right pan is at least 2. Hence, there are exactly $2n-1$ different ways of inserting weight 1 in each of the $f(n-1)$ valid sequences for the $n-1$ weights in order to get a valid sequence for the n weights. This proves the claim.

Since $f(1) = 1$, by induction we obtain for all positive integers n

$$f(n) = (2n-1)!! = 1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-1).$$

Comment 1. The word "compute" in the statement of the problem is probably too vague. An alternative but more artificial question might ask for the smallest n for which the number of valid ways is divisible by 2011. In this case the answer would be 1006.

Comment 2. It is useful to remark that the answer is the same for any set of weights where each weight is heavier than the sum of the lighter ones. Indeed, in such cases the given condition is equivalent to asking that during the process the heaviest weight on the balance is always on the left pan.

Comment 3. Instead of considering the lightest weight, one may also consider the last weight put on the balance. If this weight is 2^{n-1} then it should be put on the left pan. Otherwise it may be put on any pan; the inequality would not be violated since at this moment the heaviest weight is already put onto the left pan. In view of the previous comment, in each of these $2n-1$ cases the number of ways to place the previous weights is exactly $f(n-1)$, which yields (1).

Solution 2. We present a different way of obtaining (1). Set $f(0) = 1$. Firstly, we find a recurrent formula for $f(n)$.

Assume $n \geq 1$. Suppose that weight 2^{n-1} is placed on the balance in the i-th move with $1 \leq i \leq n$. This weight has to be put on the left pan. For the previous moves we have $\binom{n-1}{i-1}$ choices of the weights and from Comment 2 there are $f(i-1)$ valid ways of placing them on the balance. For later moves there is no restriction on the way in which the weights are to be put on the pans. Therefore, all $(n-i)!2^{n-i}$ ways are possible. This gives

$$f(n) = \sum_{i=1}^{n} \binom{n-1}{i-1} f(i-1)(n-i)!2^{n-i} = \sum_{i=1}^{n} \frac{(n-1)!f(i-1)2^{n-i}}{(i-1)!}. \tag{2}$$

Now we are ready to prove (1). Using $n-1$ instead of n in (2) we get

$$f(n-1) = \sum_{i=1}^{n-1} \frac{(n-2)!f(i-1)2^{n-1-i}}{(i-1)!}.$$

Hence, again from (2) we get

$$\begin{aligned} f(n) &= 2(n-1)\sum_{i=1}^{n-1} \frac{(n-2)!f(i-1)2^{n-1-i}}{(i-1)!} + f(n-1) \\ &= (2n-2)f(n-1) + f(n-1) = (2n-1)f(n-1), \end{aligned}$$

QED.

Comment. There exist different ways of obtaining the formula (2). Here we show one of them.

Suppose that in the first move we use weight 2^{n-i+1}. Then the lighter $n-i$ weights may be put on the balance at any moment and on either pan. This gives $2^{n-i} \cdot (n-1)!/(i-1)!$ choices for the moves (moments and choices of pan) with the lighter weights. The remaining $i-1$ moves give a valid sequence for the $i-1$ heavier weights and this is the only requirement for these moves, so there are $f(i-1)$ such sequences. Summing over all $i = 1, 2, \ldots, n$ we again come to (2).

C2

Suppose that 1000 students are standing in a circle. Prove that there exists an integer k with $100 \le k \le 300$ such that in this circle there exists a contiguous group of $2k$ students, for which the first half contains the same number of girls as the second half.

Solution. Number the students consecutively from 1 to 1000. Let $a_i = 1$ if the ith student is a girl, and $a_i = 0$ otherwise. We expand this notion for all integers i by setting $a_{i+1000} = a_{i-1000} = a_i$. Next, let

$$S_k(i) = a_i + a_{i+1} + \cdots + a_{i+k-1}.$$

Now the statement of the problem can be reformulated as follows:

There exist an integer k with $100 \le k \le 300$ and an index i such that $S_k(i) = S_k(i+k)$.

Assume now that this statement is false. Choose an index i such that $S_{100}(i)$ attains the maximal possible value. In particular, we have $S_{100}(i-100) - S_{100}(i) < 0$ and $S_{100}(i) - S_{100}(i+100) > 0$, for if we had an equality, then the statement would hold. This means that the function $S(j) - S(j+100)$ changes sign somewhere on the segment $[i-100, i]$, so there exists some index $j \in [i-100, i-1]$ such that

$$S_{100}(j) \le S_{100}(j+100) - 1, \quad \text{but} \quad S_{100}(j+1) \ge S_{100}(j+101) + 1. \tag{1}$$

Subtracting the first inequality from the second one, we get $a_{j+100} - a_j \ge a_{j+200} - a_{j+100} + 2$, so

$$a_j = 0, \quad a_{j+100} = 1, \quad a_{j+200} = 0.$$

Substituting this into the inequalities of (1), we also obtain $S_{99}(j+1) \le S_{99}(j+101) \le S_{99}(j+1)$, which implies

$$S_{99}(j+1) = S_{99}(j+101). \tag{2}$$

Now let k and ℓ be the least positive integers such that $a_{j-k} = 1$ and $a_{j+200+\ell} = 1$. By symmetry, we may assume that $k \ge \ell$. If $k \ge 200$ then we have $a_j = a_{j-1} = \cdots = a_{j-199} = 0$, so $S_{100}(j-199) = S_{100}(j-99) = 0$, which contradicts the initial assumption. Hence $\ell \le k \le 199$.

Finally, we have

$$S_{100+\ell}(j-\ell+1) = (a_{j-\ell+1} + \cdots + a_j) + S_{99}(j+1) + a_{j+100} = S_{99}(j+1) + 1,$$
$$S_{100+\ell}(j+101) = S_{99}(j+101) + (a_{j+200} + \cdots + a_{j+200+\ell-1}) + a_{j+200+\ell} = S_{99}(j+101) + 1.$$

Comparing with (2) we get $S_{100+\ell}(j-\ell+1) = S_{100+\ell}(j+101)$ and $100 + \ell \le 299$, which again contradicts our assumption.

Comment. It may be seen from the solution that the number 300 from the problem statement can be replaced by 299. Here we consider some improvements of this result. Namely, we investigate which interval can be put instead of $[100, 300]$ in order to keep the problem statement valid.

First of all, the two examples

$$\underbrace{1,1,\dots,1}_{167},\underbrace{0,0,\dots,0}_{167},\underbrace{1,1,\dots,1}_{167},\underbrace{0,0,\dots,0}_{167},\underbrace{1,1,\dots,1}_{167},\underbrace{0,0,\dots,0}_{165}$$

and

$$\underbrace{1,1,\dots,1}_{249},\underbrace{0,0,\dots,0}_{251},\underbrace{1,1,\dots,1}_{249},\underbrace{0,0,\dots,0}_{251}$$

show that the interval can be changed neither to $[84, 248]$ nor to $[126, 374]$.

On the other hand, we claim that this interval can be changed to $[125, 250]$. Note that this statement is invariant under replacing all 1's by 0's and vice versa. Assume, to the contrary, that there is no admissible $k \in [125, 250]$. The arguments from the solution easily yield the following lemma.

Lemma. Under our assumption, suppose that for some indices $i < j$ we have $S_{125}(i) \le S_{125}(i + 125)$ but $S_{125}(j) \ge S_{125}(j+125)$. Then there exists some $t \in [i, j-1]$ such that $a_t = a_{t-1} = \dots = a_{t-125} = 0$ and $a_{t+250} = a_{t+251} = \dots = a_{t+375} = 0$. $\square$

Let us call a segment $[i, j]$ of indices a *crowd*, if (a) $a_i = a_{i+1} = \dots = a_j$, but $a_{i-1} \ne a_i \ne a_{j+1}$, and (b) $j - i \ge 125$. Now, using the lemma, one can get in the same way as in the solution that there exists some crowd. Take all the crowds in the circle, and enumerate them in cyclic order as $A_1, \dots, A_d$. We also assume always that $A_{s+d} = A_{s-d} = A_s$.

Consider one of the crowds, say A_1. We have $A_1 = [i, i + t]$ with $125 \le t \le 248$ (if $t \ge 249$, then $a_i = a_{i+1} = \dots = a_{i+249}$ and therefore $S_{125}(i) = S_{125}(i + 125)$, which contradicts our assumption). We may assume that $a_i = 1$. Then we have $S_{125}(i + t - 249) \le 125 = S_{125}(i + t - 124)$ and $S_{125}(i) = 125 \ge S_{125}(i + 125)$, so by the lemma there exists some index $j \in [i + t - 249, i - 1]$ such that the segments $[j - 125, j]$ and $[j + 250, j + 375]$ are contained in some crowds.

Let us fix such j and denote the segment $[j + 1, j + 249]$ by B_1. Clearly, $A_1 \subseteq B_1$. Moreover, B_1 cannot contain any crowd other than A_1 since $|B_1| = 249 < 2 \cdot 126$. Hence it is clear that $j \in A_d$ and $j + 250 \in A_2$. In particular, this means that the genders of A_d and A_2 are different from that of A_1.

Performing this procedure for every crowd A_s, we find segments $B_s = [j_s + 1, j_s + 249]$ such that $|B_s| = 249$, $A_s \subseteq B_s$, and $j_s \in A_{s-1}$, $j_s + 250 \in A_{s+1}$. So, B_s covers the whole segment between A_{s-1} and A_{s+1}, hence the sets $B_1, \dots, B_d$ cover some 1000 consecutive indices. This implies $249d \ge 1000$, and $d \ge 5$. Moreover, the gender of A_i is alternating, so d is even; therefore $d \ge 6$.

Consider now three segments $A_1 = [i_1, i'_1]$, $B_2 = [j_2 + 1, j_2 + 249]$, $A_3 = [i_3, i'_3]$. By construction, we have $[j_2 - 125, j_2] \subseteq A_1$ and $[j_2 + 250, j_2 + 375] \subseteq A_3$, whence $i_1 \le j_2 - 125$, $i'_3 \ge j_2 + 375$. Therefore $i'_3 - i_1 \ge 500$. Analogously, if $A_4 = [i_4, i'_4]$, $A_6 = [i_6, i'_6]$ then $i'_6 - i_4 \ge 500$. But from $d \ge 6$ we get $i_1 < i'_3 < i_4 < i'_6 < i_1 + 1000$, so $1000 > (i'_3 - i_1) + (i'_6 - i_4) \ge 500 + 500$. This final contradiction shows that our claim holds.

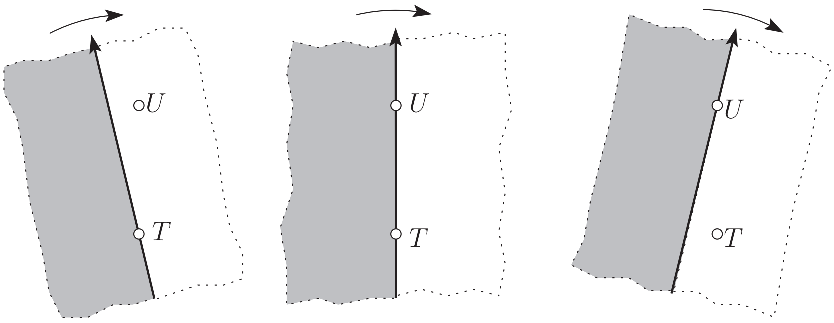
One may even show that the interval in the statement of the problem may be replaced by $[125, 249]$ (both these numbers cannot be improved due to the examples above). But a proof of this fact is a bit messy, and we do not present it here.

C3

Let $\mathcal{S}$ be a finite set of at least two points in the plane. Assume that no three points of $\mathcal{S}$ are collinear. By a *windmill* we mean a process as follows. Start with a line ℓ going through a point $P \in \mathcal{S}$. Rotate ℓ clockwise around the *pivot* P until the line contains another point Q of $\mathcal{S}$. The point Q now takes over as the new pivot. This process continues indefinitely, with the pivot always being a point from $\mathcal{S}$.

Show that for a suitable $P \in \mathcal{S}$ and a suitable starting line ℓ containing P, the resulting windmill will visit each point of $\mathcal{S}$ as a pivot infinitely often.

Solution. Give the rotating line an orientation and distinguish its sides as the *oranje side* and the *blue side*. Notice that whenever the pivot changes from some point T to another point U, after the change, T is on the same side as U was before. Therefore, the number of elements of $\mathcal{S}$ on the oranje side and the number of those on the blue side remain the same throughout the whole process (except for those moments when the line contains two points).



First consider the case that $|\mathcal{S}| = 2n+1$ is odd. We claim that through any point $T \in \mathcal{S}$, there is a line that has n points on each side. To see this, choose an oriented line through T containing no other point of $\mathcal{S}$ and suppose that it has $n+r$ points on its oranje side. If $r = 0$ then we have established the claim, so we may assume that $r \neq 0$. As the line rotates through 180° around T, the number of points of $\mathcal{S}$ on its oranje side changes by 1 whenever the line passes through a point; after 180°, the number of points on the oranje side is $n - r$. Therefore there is an intermediate stage at which the oranje side, and thus also the blue side, contains n points.

Now select the point P arbitrarily, and choose a line through P that has n points of $\mathcal{S}$ on each side to be the initial state of the windmill. We will show that during a rotation over 180°, the line of the windmill visits each point of $\mathcal{S}$ as a pivot. To see this, select any point T of $\mathcal{S}$ and select a line ℓ through T that separates $\mathcal{S}$ into equal halves. The point T is the unique point of $\mathcal{S}$ through which a line in this direction can separate the points of $\mathcal{S}$ into equal halves (parallel translation would disturb the balance). Therefore, when the windmill line is parallel to ℓ, it must be ℓ itself, and so pass through T.

Next suppose that $|\mathcal{S}| = 2n$. Similarly to the odd case, for every $T \in \mathcal{S}$ there is an oriented line through T with $n-1$ points on its oranje side and n points on its blue side. Select such an oriented line through an arbitrary P to be the initial state of the windmill.

We will now show that during a rotation over 360°, the line of the windmill visits each point of $\mathcal{S}$ as a pivot. To see this, select any point T of $\mathcal{S}$ and an oriented line ℓ through T that separates $\mathcal{S}$ into two subsets with $n-1$ points on its oranje and n points on its blue side. Again, parallel translation would change the numbers of points on the two sides, so when the windmill line is parallel to ℓ with the same orientation, the windmill line must pass through T.

Comment. One may shorten this solution in the following way.

Suppose that $|\mathcal{S}| = 2n+1$. Consider any line ℓ that separates $\mathcal{S}$ into equal halves; this line is unique given its direction and contains some point $T \in \mathcal{S}$. Consider the windmill starting from this line. When the line has made a rotation of 180°, it returns to the same location but the oranje side becomes blue and vice versa. So, for each point there should have been a moment when it appeared as pivot, as this is the only way for a point to pass from on side to the other.

Now suppose that $|\mathcal{S}| = 2n$. Consider a line having $n-1$ and n points on the two sides; it contains some point T. Consider the windmill starting from this line. After having made a rotation of 180°, the windmill line contains some different point R, and each point different from T and R has changed the color of its side. So, the windmill should have passed through all the points.

C4

Determine the greatest positive integer k that satisfies the following property: The set of positive integers can be partitioned into k subsets $A_1, A_2, \ldots, A_k$ such that for all integers $n \geq 15$ and all $i \in \{1, 2, \ldots, k\}$ there exist two distinct elements of A_i whose sum is n.

Answer. The greatest such number k is 3.

Solution 1. There are various examples showing that $k = 3$ does indeed have the property under consideration. E.g. one can take

$$A_1 = \{1, 2, 3\} \cup \{3m \mid m \geq 4\},$$
$$A_2 = \{4, 5, 6\} \cup \{3m - 1 \mid m \geq 4\},$$
$$A_3 = \{7, 8, 9\} \cup \{3m - 2 \mid m \geq 4\}.$$

To check that this partition fits, we notice first that the sums of two distinct elements of A_i obviously represent all numbers $n \geq 1 + 12 = 13$ for $i = 1$, all numbers $n \geq 4 + 11 = 15$ for $i = 2$, and all numbers $n \geq 7 + 10 = 17$ for $i = 3$. So, we are left to find representations of the numbers 15 and 16 as sums of two distinct elements of A_3. These are $15 = 7+8$ and $16 = 7+9$.

Let us now suppose that for some $k \geq 4$ there exist sets $A_1, A_2, \ldots, A_k$ satisfying the given property. Obviously, the sets $A_1, A_2, A_3, A_4 \cup \cdots \cup A_k$ also satisfy the same property, so one may assume $k = 4$.

Put $B_i = A_i \cap \{1, 2, \ldots, 23\}$ for $i = 1, 2, 3, 4$. Now for any index i each of the ten numbers $15, 16, \ldots, 24$ can be written as sum of two distinct elements of B_i. Therefore this set needs to contain at least five elements. As we also have $|B_1| + |B_2| + |B_3| + |B_4| = 23$, there has to be some index j for which $|B_j| = 5$. Let $B_j = \{x_1, x_2, x_3, x_4, x_5\}$. Finally, now the sums of two distinct elements of A_j representing the numbers $15, 16, \ldots, 24$ should be exactly all the pairwise sums of the elements of B_j. Calculating the sum of these numbers in two different ways, we reach

$$4(x_1 + x_2 + x_3 + x_4 + x_5) = 15 + 16 + \ldots + 24 = 195.$$

Thus the number 195 should be divisible by 4, which is false. This contradiction completes our solution.

Comment. There are several variation of the proof that k should not exceed 3. E.g., one may consider the sets $C_i = A_i \cap \{1, 2, \ldots, 19\}$ for $i = 1, 2, 3, 4$. As in the previous solution one can show that for some index j one has $|C_j| = 4$, and the six pairwise sums of the elements of C_j should represent all numbers $15, 16, \ldots, 20$. Let $C_j = \{y_1, y_2, y_3, y_4\}$ with $y_1 < y_2 < y_3 < y_4$. It is not hard to deduce

$C_j = \{7, 8, 9, 11\}$, so in particular we have $1 \notin C_j$. Hence it is impossible to represent 21 as sum of two distinct elements of A_j, which completes our argument.

Solution 2. Again we only prove that $k \leq 3$. Assume that $A_1, A_2, \ldots, A_k$ is a partition satisfying the given property. We construct a graph $\mathcal{G}$ on the set $V = \{1, 2, \ldots, 18\}$ of *vertices* as follows. For each $i \in \{1, 2, \ldots, k\}$ and each $d \in \{15, 16, 17, 19\}$ we choose one pair of distinct elements $a, b \in A_i$ with $a + b = d$, and we draw an *edge* in the i^{th} color connecting a with b. By hypothesis, $\mathcal{G}$ has exactly 4 edges of each color.

Claim. The graph $\mathcal{G}$ contains at most one circuit.

Proof. Note that all the connected components of $\mathcal{G}$ are monochromatic and hence contain at most four edges. Thus also all circuits of $\mathcal{G}$ are monochromatic and have length at most four. Moreover, each component contains at most one circuit since otherwise it should contain at least five edges.

Suppose that there is a 4-cycle in $\mathcal{G}$, say with vertices a, b, c, and d in order. Then $\{a + b, b + c, c + d, d + a\} = \{15, 16, 17, 19\}$. Taking sums we get $2(a + b + c + d) = 15 + 16 + 17 + 19$ which is impossible for parity reasons. Thus all circuits of $\mathcal{G}$ are triangles.

Now if the vertices a, b, and c form such a triangle, then by a similar reasoning the set $\{a + b, b + c, c + a\}$ coincides with either $\{15, 16, 17\}$, or $\{15, 16, 19\}$, or $\{16, 17, 19\}$, or $\{15, 17, 19\}$. The last of these alternatives can be excluded for parity reasons again, whilst in the first three cases the set $\{a, b, c\}$ appears to be either $\{7, 8, 9\}$, or $\{6, 9, 10\}$, or $\{7, 9, 10\}$, respectively. Thus, a component containing a circuit should contain 9 as a vertex. Therefore there is at most one such component and hence at most one circuit. $\square$

By now we know that $\mathcal{G}$ is a graph with $4k$ edges, at least k components and at most one circuit. Consequently, $\mathcal{G}$ must have at least $4k + k - 1$ vertices. Thus $5k - 1 \leq 18$, and $k \leq 3$.

C5

Let m be a positive integer and consider a checkerboard consisting of m by m unit squares. At the midpoints of some of these unit squares there is an ant. At time 0, each ant starts moving with speed 1 parallel to some edge of the checkerboard. When two ants moving in opposite directions meet, they both turn 90° clockwise and continue moving with speed 1. When more than two ants meet, or when two ants moving in perpendicular directions meet, the ants continue moving in the same direction as before they met. When an ant reaches one of the edges of the checkerboard, it falls off and will not re-appear.

Considering all possible starting positions, determine the latest possible moment at which the last ant falls off the checkerboard or prove that such a moment does not necessarily exist.

Antswer. The latest possible moment for the last ant to fall off is $\frac{3m}{2} - 1$.

Solution. For $m = 1$ the answer is clearly correct, so assume $m > 1$. In the sequel, the word *collision* will be used to denote meeting of exactly two ants, moving in opposite directions.

If at the beginning we place an ant on the southwest corner square facing east and an ant on the southeast corner square facing west, then they will meet in the middle of the bottom row at time $\frac{m-1}{2}$. After the collision, the ant that moves to the north will stay on the board for another $m - \frac{1}{2}$ time units and thus we have established an example in which the last ant falls off at time $\frac{m-1}{2} + m - \frac{1}{2} = \frac{3m}{2} - 1$. So, we are left to prove that this is the latest possible moment.

Consider any collision of two ants a and a'. Let us change the rule for this collision, and enforce these two ants to turn anticlockwise. Then the succeeding behavior of all the ants does not change; the only difference is that a and a' swap their positions. These arguments may be applied to any collision separately, so we may assume that at any collision, either both ants rotate clockwise or both of them rotate anticlockwise by our own choice.

For instance, we may assume that there are only two types of ants, depending on their initial direction: *NE-ants*, which move only north or east, and *SW-ants*, moving only south and west. Then we immediately obtain that all ants will have fallen off the board after $2m - 1$ time units. However, we can get a better bound by considering the last moment at which a given ant collides with another ant.

Choose a coordinate system such that the corners of the checkerboard are $(0, 0)$, $(m, 0)$, (m, m) and $(0, m)$. At time t, there will be no NE-ants in the region $\{(x, y) : x + y < t + 1\}$ and no SW-ants in the region $\{(x, y) : x + y > 2m - t - 1\}$. So if two ants collide at (x, y) at time t, we have

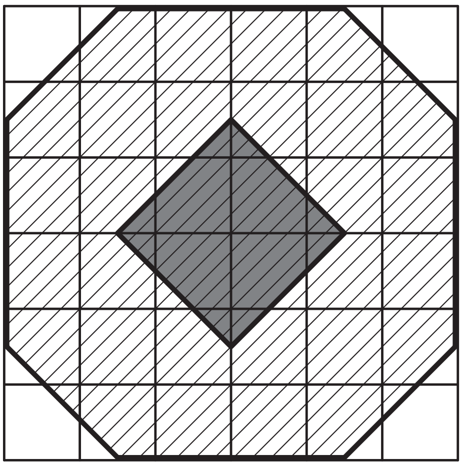
$$t + 1 \leq x + y \leq 2m - t - 1. \tag{1}$$

Analogously, we may change the rules so that each ant would move either alternatingly north and west, or alternatingly south and east. By doing so, we find that apart from (1) we also have $|x-y| \leq m-t-1$ for each collision at point (x, y) and time t.

To visualize this, put

$$B(t)=\left\{(x, y) \in[0, m]^{2}: t+1 \leq x+y \leq 2m-t-1 \text{ and } |x-y| \leq m-t-1\right\}.$$

An ant can thus only collide with another ant at time t if it happens to be in the region $B(t)$. The following figure displays $B(t)$ for $t=\frac{1}{2}$ and $t=\frac{7}{2}$ in the case $m=6$:



Now suppose that an NE-ant has its last collision at time t and that it does so at the point (x, y) (if the ant does not collide at all, it will fall off the board within $m-\frac{1}{2}<\frac{3m}{2}-1$ time units, so this case can be ignored). Then we have $(x, y) \in B(t)$ and thus $x+y \geq t+1$ and $x-y \geq -(m-t-1)$. So we get

$$x \geq \frac{(t+1)-(m-t-1)}{2}=t+1-\frac{m}{2}.$$

By symmetry we also have $y \geq t+1-\frac{m}{2}$, and hence $\min\{x, y\} \geq t+1-\frac{m}{2}$. After this collision, the ant will move directly to an edge, which will take at most $m-\min\{x, y\}$ units of time. In sum, the total amount of time the ant stays on the board is at most

$$t+(m-\min\{x, y\}) \leq t+m-\left(t+1-\frac{m}{2}\right)=\frac{3m}{2}-1.$$

By symmetry, the same bound holds for SW-ants as well.

C6

Let n be a positive integer and let $W = \ldots x_{-1}x_0x_1x_2\ldots$ be an infinite periodic word consisting of the letters a and b. Suppose that the minimal period N of W is greater than 2^n.

A finite nonempty word U is said to *appear* in W if there exist indices $k \leq \ell$ such that $U = x_kx_{k+1}\ldots x_\ell$. A finite word U is called *ubiquitous* if the four words Ua, Ub, aU, and bU all appear in W. Prove that there are at least n ubiquitous finite nonempty words.

Solution. Throughout the solution, all the words are nonempty. For any word R of length m, we call the number of indices $i \in \{1, 2, \ldots, N\}$ for which R coincides with the subword $x_{i+1}x_{i+2}\ldots x_{i+m}$ of W the *multiplicity* of R and denote it by $\mu(R)$. Thus a word R appears in W if and only if $\mu(R) > 0$. Since each occurrence of a word in W is both succeeded by either the letter a or the letter b and similarly preceded by one of those two letters, we have

$$\mu(R) = \mu(Ra) + \mu(Rb) = \mu(aR) + \mu(bR) \tag{1}$$

for all words R.

We claim that the condition that N is in fact the minimal period of W guarantees that each word of length N has multiplicity 1 or 0 depending on whether it appears or not. Indeed, if the words $x_{i+1}x_{i+2}\ldots x_{i+N}$ and $x_{j+1}\ldots x_{j+N}$ are equal for some $1 \leq i < j \leq N$, then we have $x_{i+a} = x_{j+a}$ for every integer a, and hence $j - i$ is also a period.

Moreover, since $N > 2^n$, at least one of the two words a and b has a multiplicity that is strictly larger than 2^{n-1}.

For each $k = 0, 1, \ldots, n-1$, let U_k be a subword of W whose multiplicity is strictly larger than 2^k and whose length is maximal subject to this property. Note that such a word exists in view of the two observations made in the two previous paragraphs.

Fix some index $k \in \{0, 1, \ldots, n-1\}$. Since the word U_kb is longer than U_k, its multiplicity can be at most 2^k, so in particular $\mu(U_kb) < \mu(U_k)$. Therefore, the word U_ka has to appear by (1). For a similar reason, the words U_kb, aU_k, and bU_k have to appear as well. Hence, the word U_k is ubiquitous. Moreover, if the multiplicity of U_k were strictly greater than 2^{k+1}, then by (1) at least one of the two words U_ka and U_kb would have multiplicity greater than 2^k and would thus violate the maximality condition imposed on U_k.

So we have $\mu(U_0) \leq 2 < \mu(U_1) \leq 4 < \ldots \leq 2^{n-1} < \mu(U_{n-1})$, which implies in particular that the words $U_0, U_1, \ldots, U_{n-1}$ have to be distinct. As they have been proved to be ubiquitous as well, the problem is solved.

Comment 1. There is an easy construction for obtaining ubiquitous words from appearing words whose multiplicity is at least two. Starting with any such word U we may simply extend one of its occurrences in W forwards and backwards as long as its multiplicity remains fixed, thus arriving at a word that one might call the *ubiquitous prolongation* $p(U)$ of U.

There are several variants of the argument in the second half of the solution using the concept of prolongation. For instance, one may just take all ubiquitous words $U_1, U_2, \ldots, U_\ell$ ordered by increasing multiplicity and then prove for $i \in \{1, 2, \ldots, \ell\}$ that $\mu(U_i) \le 2^i$. Indeed, assume that i is a minimal counterexample to this statement; then by the arguments similar to those presented above, the ubiquitous prolongation of one of the words $U_i a$, $U_i b$, aU_i or bU_i violates the definition of U_i.

Now the multiplicity of one of the two letters a and b is strictly greater than 2^{n-1}, so passing to ubiquitous prolongations once more we obtain $2^{n-1} < \mu(U_\ell) \le 2^\ell$, which entails $\ell \ge n$, as needed.

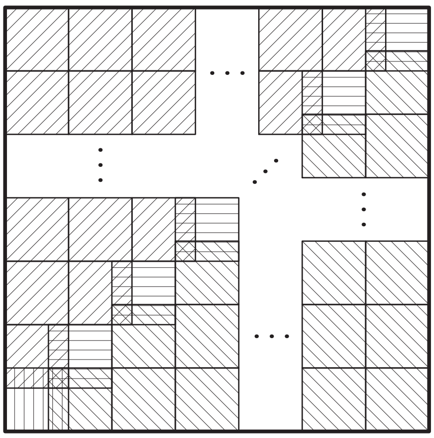
Comment 2. The bound n for the number of ubiquitous subwords in the problem statement is not optimal, but it is close to an optimal one in the following sense. There is a universal constant $C > 0$ such that for each positive integer n there exists an infinite periodic word W whose minimal period is greater than 2^n but for which there exist fewer than Cn ubiquitous words.

C7

On a square table of 2011 by 2011 cells we place a finite number of napkins that each cover a square of 52 by 52 cells. In each cell we write the number of napkins covering it, and we record the maximal number k of cells that all contain the same nonzero number. Considering all possible napkin configurations, what is the largest value of k?

Answer. $2011^2 - \big((52^2 - 35^2) \cdot 39 - 17^2\big) = 4044121 - 57392 = 3986729.$

Solution 1. Let $m = 39$, then $2011 = 52m - 17$. We begin with an example showing that there can exist 3986729 cells carrying the same positive number.



To describe it, we number the columns from the left to the right and the rows from the bottom to the top by $1, 2, \ldots, 2011$. We will denote each napkin by the coordinates of its lower-left cell. There are four kinds of napkins: first, we take all napkins $(52i + 36, 52j + 1)$ with $0 \le j \le i \le m - 2$; second, we use all napkins $(52i + 1, 52j + 36)$ with $0 \le i \le j \le m - 2$; third, we use all napkins $(52i + 36, 52i + 36)$ with $0 \le i \le m - 2$; and finally the napkin $(1, 1)$. Different groups of napkins are shown by different types of hatchings in the picture.

Now except for those squares that carry two or more different hatchings, all squares have the number 1 written into them. The number of these exceptional cells is easily computed to be $(52^2 - 35^2)m - 17^2 = 57392$.

We are left to prove that 3986729 is an upper bound for the number of cells containing the same number. Consider any configuration of napkins and any positive integer M. Suppose there are g cells with a number different from M. Then it suffices to show $g \ge 57392$. Throughout the solution, a *line* will mean either a row or a column.

Consider any line ℓ. Let $a_1, \ldots, a_{52m-17}$ be the numbers written into its consecutive cells. For $i = 1, 2, \ldots, 52$, let $s_i = \sum_{t \equiv i \pmod{52}} a_t$. Note that $s_1, \ldots, s_{35}$ have m terms each, while $s_{36}, \ldots, s_{52}$ have $m-1$ terms each. Every napkin intersecting ℓ contributes exactly 1 to each s_i;

hence the number s of all those napkins satisfies $s_1 = \cdots = s_{52} = s$. Call the line ℓ *rich* if $s > (m-1)M$ and *poor* otherwise.

Suppose now that ℓ is rich. Then in each of the sums $s_{36}, \ldots, s_{52}$ there exists a term greater than M; consider all these terms and call the corresponding cells the *rich bad cells* for this line. So, each rich line contains at least 17 cells that are bad for this line.

If, on the other hand, ℓ is poor, then certainly $s < mM$ so in each of the sums $s_1, \ldots, s_{35}$ there exists a term less than M; consider all these terms and call the corresponding cells the *poor bad cells* for this line. So, each poor line contains at least 35 cells that are bad for this line.

Let us call all indices congruent to $1, 2, \ldots,$ or 35 modulo 52 *small*, and all other indices, i.e. those congruent to $36, 37, \ldots,$ or 52 modulo 52, *big*. Recall that we have numbered the columns from the left to the right and the rows from the bottom to the top using the numbers $1, 2, \ldots, 52m - 17$; we say that a line is *big* or *small* depending on whether its index is big or small. By definition, all rich bad cells for the rows belong to the big columns, while the poor ones belong to the small columns, and vice versa.

In each line, we put a *strawberry* on each cell that is bad for this line. In addition, for each **small rich** line we put an *extra* strawberry on each of its (rich) bad cells. A cell gets the strawberries from its row and its column independently.

Notice now that a cell with a strawberry on it contains a number different from M. If this cell gets a strawberry by the extra rule, then it contains a number greater than M. Moreover, it is either in a small row and in a big column, or vice versa. Suppose that it is in a small row, then it is not bad for its column. So it has not more than two strawberries in this case. On the other hand, if the extra rule is not applied to some cell, then it also has not more than two strawberries. So, the total number N of strawberries is at most $2g$.

We shall now estimate N in a different way. For each of the $2 \cdot 35m$ small lines, we have introduced at least 34 strawberries if it is rich and at least 35 strawberries if it is poor, so at least 34 strawberries in any case. Similarly, for each of the $2 \cdot 17(m-1)$ big lines, we put at least $\min(17, 35) = 17$ strawberries. Summing over all lines we obtain

$$2g \geq N \geq 2(35m \cdot 34 + 17(m-1) \cdot 17) = 2(1479m - 289) = 2 \cdot 57392,$$

as desired.

Comment. The same reasoning applies also if we replace 52 by R and 2011 by $Rm - H$, where m, R, and H are integers with $m, R \geq 1$ and $0 \leq H \leq \frac{1}{3}R$. More detailed information is provided after the next solution.

Solution 2. We present a different proof of the estimate which is the hard part of the problem. Let $S = 35$, $H = 17$, $m = 39$; so the table size is $2011 = Sm + H(m-1)$, and the napkin size is $52 = S + H$. Fix any positive integer M and call a cell *vicious* if it contains a number distinct from M. We will prove that there are at least $H^2(m-1)+2SHm$ vicious cells.

Firstly, we introduce some terminology. As in the previous solution, we number rows and columns and we use the same notions of *small* and *big* indices and lines; so, an index is small if it is congruent to one of the numbers $1, 2, \ldots, S$ modulo $(S+H)$. The numbers $1, 2, \ldots, S+H$ will be known as *residues*. For two residues i and j, we say that a cell is of *type* (i, j) if the index of its row is congruent to i and the index of its column to j modulo $(S+H)$. The number of vicious cells of this type is denoted by v_{ij}.

Let s, s' be two variables ranging over small residues and let h, h' be two variables ranging over big residues. A cell is said to be of *class* A, B, C, or D if its type is of shape (s, s'), (s, h), (h, s), or (h, h'), respectively. The numbers of vicious cells belonging to these classes are denoted in this order by a, b, c, and d. Observe that each cell belongs to exactly one class.

Claim 1. We have

$$m \leq \frac{a}{S^2} + \frac{b+c}{2SH}. \tag{1}$$

Proof. Consider an arbitrary small row r. Denote the numbers of vicious cells on r belonging to the classes A and B by α and β, respectively. As in the previous solution, we obtain that $\alpha \geq S$ or $\beta \geq H$. So in each case we have $\frac{\alpha}{S} + \frac{\beta}{H} \geq 1$.

Performing this argument separately for each small row and adding up all the obtained inequalities, we get $\frac{a}{S} + \frac{b}{H} \geq mS$. Interchanging rows and columns we similarly get $\frac{a}{S} + \frac{c}{H} \geq mS$. Summing these inequalities and dividing by $2S$ we get what we have claimed. ☐

Claim 2. Fix two small residue s, s' and two big residues h, h'. Then $2m-1 \leq v_{ss'} + v_{sh'} + v_{hh'}$.

Proof. Each napkin covers exactly one cell of type (s, s'). Removing all napkins covering a vicious cell of this type, we get another collection of napkins, which covers each cell of type (s, s') either 0 or M times depending on whether the cell is vicious or not. Hence $(m^2 - v_{ss'})M$ napkins are left and throughout the proof of Claim 2 we will consider only these remaining napkins. Now, using a red pen, write in each cell the number of napkins covering it. Notice that a cell containing a red number greater than M is surely vicious.

We call two cells *neighbors* if they can be simultaneously covered by some napkin. So, each cell of type (h, h') has not more than four neighbors of type (s, s'), while each cell of type (s, h') has not more than two neighbors of each of the types (s, s') and (h, h'). Therefore, each red number at a cell of type (h, h') does not exceed $4M$, while each red number at a cell of type (s, h') does not exceed $2M$.

Let x, y, and z be the numbers of cells of type (h, h') whose red number belongs to $(M, 2M]$, $(2M, 3M]$, and $(3M, 4M]$, respectively. All these cells are vicious, hence $x+y+z \leq v_{hh'}$. The red numbers appearing in cells of type (h, h') clearly sum up to $(m^2 - v_{ss'})M$. Bounding each of these numbers by a multiple of M we get

$$(m^2 - v_{ss'})M \leq \big((m-1)^2 - (x+y+z)\big)\, M + 2xM + 3yM + 4zM,$$

i.e.

$$2m-1 \le v_{ss'} + x + 2y + 3z \le v_{ss'} + v_{hh'} + y + 2z.$$

So, to prove the claim it suffices to prove that $y + 2z \le v_{sh'}$.

For a cell δ of type (h, h') and a cell β of type (s, h') we say that δ *forces* β if there are more than M napkins covering both of them. Since each red number in a cell of type (s, h') does not exceed $2M$, it cannot be forced by more than one cell.

On the other hand, if a red number in a (h, h')-cell belongs to $(2M, 3M]$, then it forces at least one of its neighbors of type (s, h') (since the sum of red numbers in their cells is greater than $2M$). Analogously, an (h, h')-cell with the red number in $(3M, 4M]$ forces both its neighbors of type (s, h'), since their red numbers do not exceed $2M$. Therefore there are at least $y + 2z$ forced cells and clearly all of them are vicious, as desired. □

Claim 3. We have

$$2m-1 \le \frac{a}{S^2} + \frac{b+c}{2SH} + \frac{d}{H^2}. \tag{2}$$

Proof. Averaging the previous result over all S^2H^2 possibilities for the quadruple (s, s', h, h'), we get $2m-1 \le \frac{a}{S^2} + \frac{b}{SH} + \frac{d}{H^2}$. Due to the symmetry between rows and columns, the same estimate holds with b replaced by c. Averaging these two inequalities we arrive at our claim.
□

Now let us multiply (2) by H^2, multiply (1) by $(2SH - H^2)$ and add them; we get

$$H^2(2m-1)+(2SH-H^2)m \le a\cdot\frac{H^2+2SH-H^2}{S^2}+(b+c)\frac{H^2+2SH-H^2}{2SH}+d = a\cdot\frac{2H}{S}+b+c+d.$$

The left-hand side is exactly $H^2(m-1) + 2SHm$, while the right-hand side does not exceed $a + b + c + d$ since $2H \le S$. Hence we come to the desired inequality.

Comment 1. Claim 2 is the key difference between the two solutions, because it allows to get rid of the notions of rich and poor cells. However, one may prove it by the "strawberry method" as well. It suffices to put a strawberry on each cell which is bad for an s-row, and a strawberry on each cell which is bad for an h'-column. Then each cell would contain not more than one strawberry.

Comment 2. Both solutions above work if the residue of the table size T modulo the napkin size R is at least $\frac{2}{3}R$, or equivalently if $T = Sm + H(m-1)$ and $R = S + H$ for some positive integers S, H, m such that $S \ge 2H$. Here we discuss all other possible combinations.

Case 1. If $2H \ge S \ge H/2$, then the sharp bound for the number of vicious cells is $mS^2 + (m-1)H^2$; it can be obtained by the same methods as in any of the solutions. To obtain an example showing that the bound is sharp, one may simply remove the napkins of the third kind from the example in Solution 1 (with an obvious change in the numbers).

Case 2. If $2S \le H$, the situation is more difficult. If $(S + H)^2 > 2H^2$, then the answer and the example are the same as in the previous case; otherwise the answer is $(2m-1)S^2 + 2SH(m-1)$, and the example is provided simply by $(m-1)^2$ nonintersecting napkins.

Now we sketch the proof of both estimates for Case 2. We introduce a more appropriate notation based on that from Solution 2. Denote by a_- and a_+ the number of cells of class A that contain the number which is strictly less than M and strictly greater than M, respectively. The numbers $b_\pm$, $c_\pm$, and $d_\pm$ are defined in a similar way. One may notice that the proofs of Claim 1 and Claims 2, 3 lead in fact to the inequalities

$$m-1 \leq \frac{b_- + c_-}{2SH} + \frac{d_+}{H^2} \quad \text{and} \quad 2m-1 \leq \frac{a}{S^2} + \frac{b_+ + c_+}{2SH} + \frac{d_+}{H^2}$$

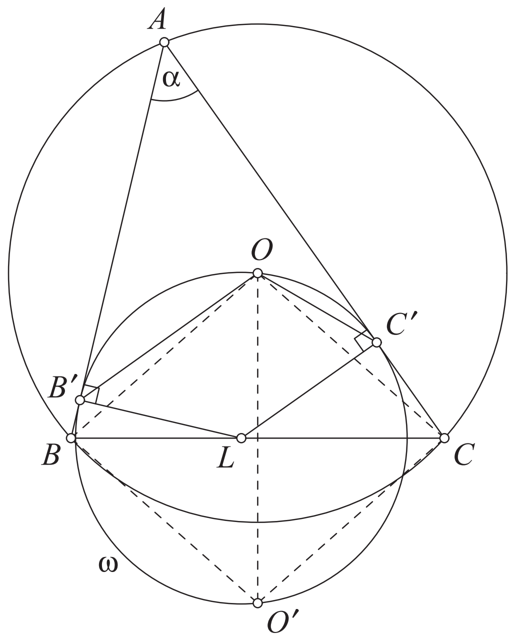
(to obtain the first one, one needs to look at the big lines instead of the small ones). Combining these inequalities, one may obtain the desired estimates.

These estimates can also be proved in some different ways, e.g. without distinguishing rich and poor cells.

G1

Let ABC be an acute triangle. Let ω be a circle whose center L lies on the side BC. Suppose that ω is tangent to AB at B' and to AC at C'. Suppose also that the circumcenter O of the triangle ABC lies on the shorter arc $B'C'$ of ω. Prove that the circumcircle of ABC and ω meet at two points.

Solution. The point B', being the perpendicular foot of L, is an interior point of side AB. Analogously, C' lies in the interior of AC. The point O is located inside the triangle $AB'C'$, hence $\angle COB < \angle C'OB'$.



Let $\alpha = \angle CAB$. The angles $\angle CAB$ and $\angle C'OB'$ are inscribed into the two circles with centers O and L, respectively, so $\angle COB = 2\angle CAB = 2\alpha$ and $2\angle C'OB' = 360^\circ - \angle C'LB'$. From the kite $AB'LC'$ we have $\angle C'LB' = 180^\circ - \angle C'AB' = 180^\circ - \alpha$. Combining these, we get

$$2\alpha = \angle COB < \angle C'OB' = \frac{360^\circ - \angle C'LB'}{2} = \frac{360^\circ - (180^\circ - \alpha)}{2} = 90^\circ + \frac{\alpha}{2},$$

so

$$\alpha < 60^\circ.$$

Let O' be the reflection of O in the line BC. In the quadrilateral $ABO'C$ we have

$$\angle CO'B + \angle CAB = \angle COB + \angle CAB = 2\alpha + \alpha < 180^\circ,$$

so the point O' is outside the circle ABC. Hence, O and O' are two points of ω such that one of them lies inside the circumcircle, while the other one is located outside. Therefore, the two circles intersect.

Comment. There are different ways of reducing the statement of the problem to the case $\alpha < 60^\circ$. E.g., since the point O lies in the interior of the isosceles triangle $AB'C'$, we have $OA < AB'$. So, if $AB' \leq 2LB'$ then $OA < 2LO$, which means that ω intersects the circumcircle of ABC. Hence the only interesting case is $AB' > 2LB'$, and this condition implies $\angle CAB = 2\angle B'AL < 2 \cdot 30^\circ = 60^\circ$.

G2

Let $A_1A_2A_3A_4$ be a non-cyclic quadrilateral. Let O_1 and r_1 be the circumcenter and the circumradius of the triangle $A_2A_3A_4$. Define O_2, O_3, O_4 and r_2, r_3, r_4 in a similar way. Prove that

$$\frac{1}{O_1A_1^2-r_1^2}+\frac{1}{O_2A_2^2-r_2^2}+\frac{1}{O_3A_3^2-r_3^2}+\frac{1}{O_4A_4^2-r_4^2}=0.$$

Solution 1. Let M be the point of intersection of the diagonals A_1A_3 and A_2A_4. On each diagonal choose a direction and let x, y, z, and w be the signed distances from M to the points A_1, A_2, A_3, and A_4, respectively.

Let ω_1 be the circumcircle of the triangle $A_2A_3A_4$ and let B_1 be the second intersection point of ω_1 and A_1A_3 (thus, $B_1=A_3$ if and only if A_1A_3 is tangent to ω_1). Since the expression $O_1A_1^2-r_1^2$ is the power of the point A_1 with respect to ω_1, we get

$$O_1A_1^2-r_1^2=A_1B_1\cdot A_1A_3.$$

On the other hand, from the equality $MB_1\cdot MA_3=MA_2\cdot MA_4$ we obtain $MB_1=yw/z$. Hence, we have

$$O_1A_1^2-r_1^2=\left(\frac{yw}{z}-x\right)(z-x)=\frac{z-x}{z}(yw-xz).$$

Substituting the analogous expressions into the sought sum we get

$$\sum_{i=1}^{4}\frac{1}{O_iA_i^2-r_i^2}=\frac{1}{yw-xz}\left(\frac{z}{z-x}-\frac{w}{w-y}+\frac{x}{x-z}-\frac{y}{y-w}\right)=0,$$

as desired.

Comment. One might reformulate the problem by assuming that the quadrilateral $A_1A_2A_3A_4$ is convex. This should not really change the difficulty, but proofs that distinguish several cases may become shorter.

Solution 2. Introduce a Cartesian coordinate system in the plane. Every circle has an equation of the form $p(x,y)=x^2+y^2+l(x,y)=0$, where $l(x,y)$ is a polynomial of degree at most 1. For any point $A=(x_A,y_A)$ we have $p(x_A,y_A)=d^2-r^2$, where d is the distance from A to the center of the circle and r is the radius of the circle.

For each i in $\{1,2,3,4\}$ let $p_i(x,y)=x^2+y^2+l_i(x,y)=0$ be the equation of the circle with center O_i and radius r_i and let d_i be the distance from A_i to O_i. Consider the equation

$$\sum_{i=1}^{4}\frac{p_i(x,y)}{d_i^2-r_i^2}=1. \tag{1}$$

Since the coordinates of the points A_1, A_2, A_3, and A_4 satisfy (1) but these four points do not lie on a circle or on an line, equation (1) defines neither a circle, nor a line. Hence, the equation is an identity and the coefficient of the quadratic term $x^2 + y^2$ also has to be zero, i.e.

$$\sum_{i=1}^{4} \frac{1}{d_i^2 - r_i^2} = 0.$$

Comment. Using the determinant form of the equation of the circle through three given points, the same solution can be formulated as follows.

For $i = 1, 2, 3, 4$ let (u_i, v_i) be the coordinates of A_i and define

$$\Delta = \begin{vmatrix} u_1^2+v_1^2 & u_1 & v_1 & 1 \\ u_2^2+v_2^2 & u_2 & v_2 & 1 \\ u_3^2+v_3^2 & u_3 & v_3 & 1 \\ u_4^2+v_4^2 & u_4 & v_4 & 1 \end{vmatrix} \qquad \text{and} \qquad \Delta_i = \begin{vmatrix} u_{i+1} & v_{i+1} & 1 \\ u_{i+2} & v_{i+2} & 1 \\ u_{i+3} & v_{i+3} & 1 \end{vmatrix},$$

where $i+1$, $i+2$, and $i+3$ have to be read modulo 4 as integers in the set $\{1, 2, 3, 4\}$.

Expanding $\begin{vmatrix} u_1 & v_1 & 1 & 1 \\ u_2 & v_2 & 1 & 1 \\ u_3 & v_3 & 1 & 1 \\ u_4 & v_4 & 1 & 1 \end{vmatrix} = 0$ along the third column, we get $\Delta_1 - \Delta_2 + \Delta_3 - \Delta_4 = 0$.

The circle through A_{i+1}, A_{i+2}, and A_{i+3} is given by the equation

$$\frac{1}{\Delta_i} \begin{vmatrix} x^2+y^2 & x & y & 1 \\ u_{i+1}^2+v_{i+1}^2 & u_{i+1} & v_{i+1} & 1 \\ u_{i+2}^2+v_{i+2}^2 & u_{i+2} & v_{i+2} & 1 \\ u_{i+3}^2+v_{i+3}^2 & u_{i+3} & v_{i+3} & 1 \end{vmatrix} = 0. \tag{2}$$

On the left-hand side, the coefficient of $x^2 + y^2$ is equal to 1. Substituting (u_i, v_i) for (x, y) in (2) we obtain the power of point A_i with respect to the circle through A_{i+1}, A_{i+2}, and A_{i+3}:

$$d_i^2 - r_i^2 = \frac{1}{\Delta_i} \begin{vmatrix} u_i^2+v_i^2 & u_i & v_i & 1 \\ u_{i+1}^2+v_{i+1}^2 & u_{i+1} & v_{i+1} & 1 \\ u_{i+2}^2+v_{i+2}^2 & u_{i+2} & v_{i+2} & 1 \\ u_{i+3}^2+v_{i+3}^2 & u_{i+3} & v_{i+3} & 1 \end{vmatrix} = (-1)^{i+1} \frac{\Delta}{\Delta_i}.$$

Thus, we have

$$\sum_{i=1}^{4} \frac{1}{d_i^2 - r_i^2} = \frac{\Delta_1 - \Delta_2 + \Delta_3 - \Delta_4}{\Delta} = 0.$$

G3

Let $ABCD$ be a convex quadrilateral whose sides AD and BC are not parallel. Suppose that the circles with diameters AB and CD meet at points E and F inside the quadrilateral. Let ω_E be the circle through the feet of the perpendiculars from E to the lines AB, BC, and CD. Let ω_F be the circle through the feet of the perpendiculars from F to the lines CD, DA, and AB. Prove that the midpoint of the segment EF lies on the line through the two intersection points of ω_E and ω_F.

Solution. Denote by P, Q, R, and S the projections of E on the lines DA, AB, BC, and CD respectively. The points P and Q lie on the circle with diameter AE, so $\angle QPE = \angle QAE$; analogously, $\angle QRE = \angle QBE$. So $\angle QPE + \angle QRE = \angle QAE + \angle QBE = 90^\circ$. By similar reasons, we have $\angle SPE + \angle SRE = 90^\circ$, hence we get $\angle QPS + \angle QRS = 90^\circ + 90^\circ = 180^\circ$, and the quadrilateral $PQRS$ is inscribed in ω_E. Analogously, all four projections of F onto the sides of $ABCD$ lie on ω_F.

Denote by K the meeting point of the lines AD and BC. Due to the arguments above, there is no loss of generality in assuming that A lies on segment DK. Suppose that $\angle CKD \geq 90^\circ$; then the circle with diameter CD covers the whole quadrilateral $ABCD$, so the points E, F cannot lie inside this quadrilateral. Hence our assumption is wrong. Therefore, the lines EP and BC intersect at some point P', while the lines ER and AD intersect at some point R'.

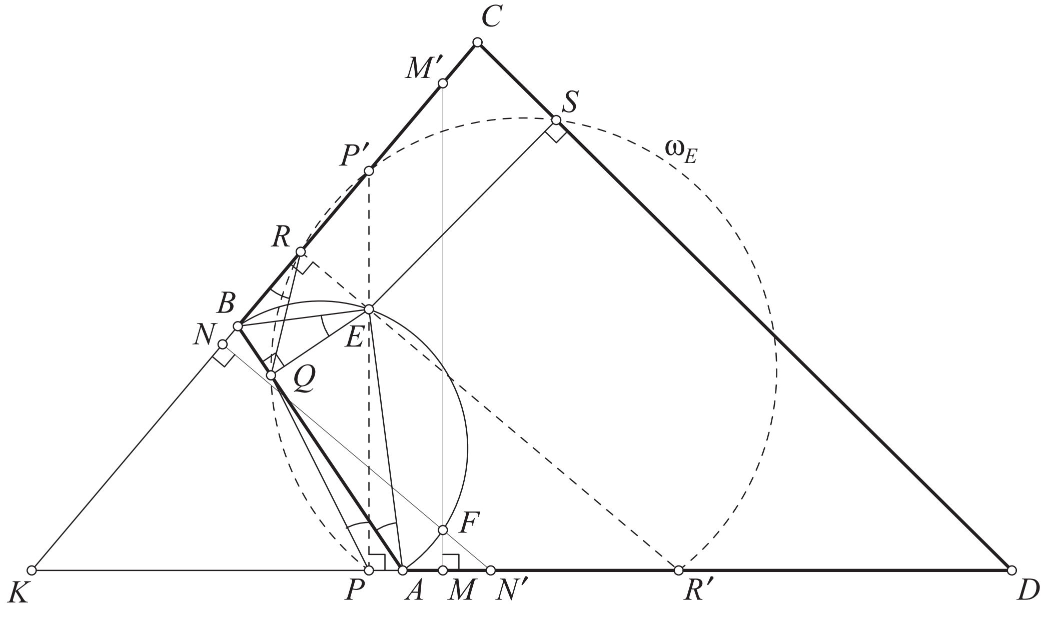


Figure 1

We claim that the points P' and R' also belong to ω_E. Since the points R, E, Q, B are concyclic, $\angle QRK = \angle QEB = 90^\circ - \angle QBE = \angle QAE = \angle QPE$. So $\angle QRK = \angle QPP'$, which means that the point P' lies on ω_E. Analogously, R' also lies on ω_E.

In the same manner, denote by M and N the projections of F on the lines AD and BC

respectively, and let $M' = FM \cap BC$, $N' = FN \cap AD$. By the same arguments, we obtain that the points M' and N' belong to ω_F.

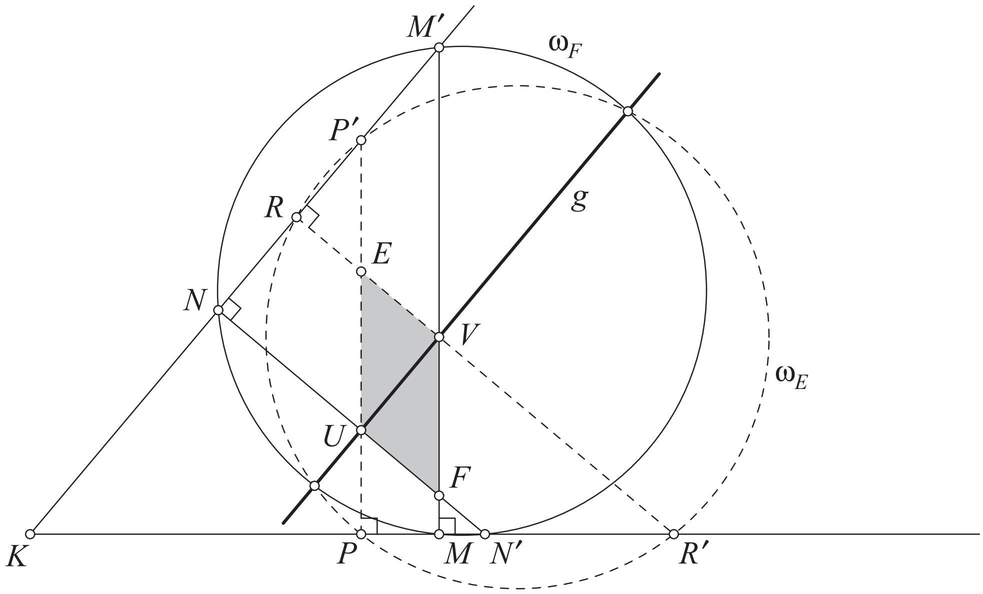


Figure 2

Now we concentrate on Figure 2, where all unnecessary details are removed. Let $U = NN' \cap PP'$, $V = MM' \cap RR'$. Due to the right angles at N and P, the points N, N', P, P' are concyclic, so $UN \cdot UN' = UP \cdot UP'$ which means that U belongs to the radical axis g of the circles ω_E and ω_F. Analogously, V also belongs to g.

Finally, since $EUFV$ is a parallelogram, the radical axis UV of ω_E and ω_F bisects EF.

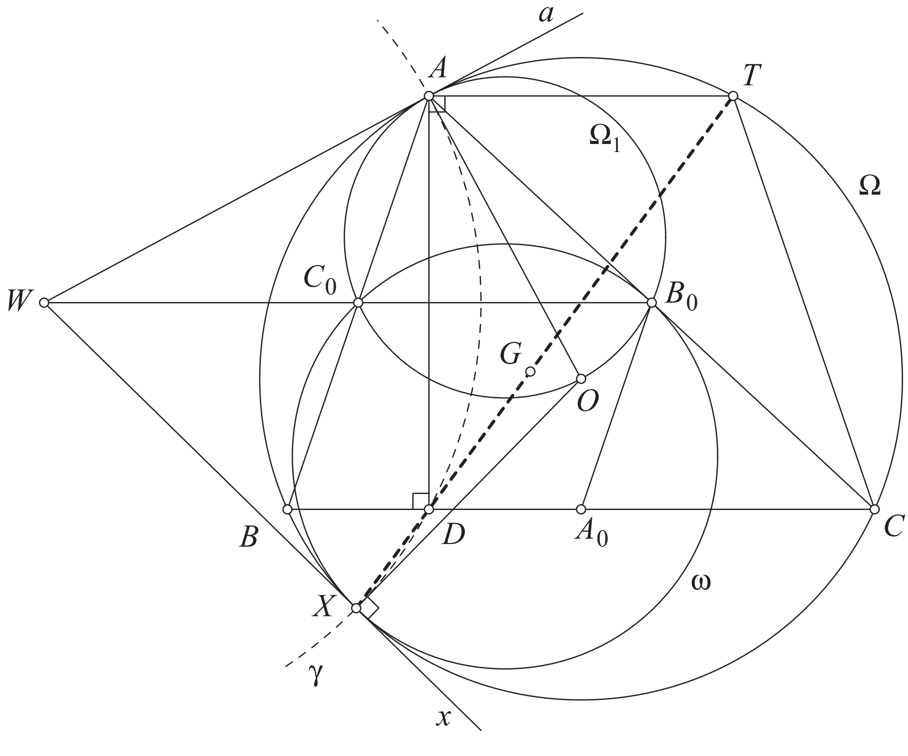
G4

Let ABC be an acute triangle with circumcircle Ω. Let B_0 be the midpoint of AC and let C_0 be the midpoint of AB. Let D be the foot of the altitude from A, and let G be the centroid of the triangle ABC. Let ω be a circle through B_0 and C_0 that is tangent to the circle Ω at a point $X \neq A$. Prove that the points D, G, and X are collinear.

Solution 1. If $AB = AC$, then the statement is trivial. So without loss of generality we may assume $AB < AC$. Denote the tangents to Ω at points A and X by a and x, respectively.

Let Ω_1 be the circumcircle of triangle AB_0C_0. The circles Ω and Ω_1 are homothetic with center A, so they are tangent at A, and a is their radical axis. Now, the lines a, x, and B_0C_0 are the three radical axes of the circles Ω, Ω_1, and ω. Since $a \nparallel B_0C_0$, these three lines are concurrent at some point W.

The points A and D are symmetric with respect to the line B_0C_0; hence $WX = WA = WD$. This means that W is the center of the circumcircle γ of triangle ADX. Moreover, we have $\angle WAO = \angle WXO = 90°$, where O denotes the center of Ω. Hence $\angle AWX + \angle AOX = 180°$.



Denote by T the second intersection point of Ω and the line DX. Note that O belongs to Ω_1. Using the circles γ and Ω, we find $\angle DAT = \angle ADX - \angle ATD = \frac{1}{2}(360° - \angle AWX) - \frac{1}{2}\angle AOX = 180° - \frac{1}{2}(\angle AWX + \angle AOX) = 90°$. So, $AD \perp AT$, and hence $AT \parallel BC$. Thus, $ATCB$ is an isosceles trapezoid inscribed in Ω.

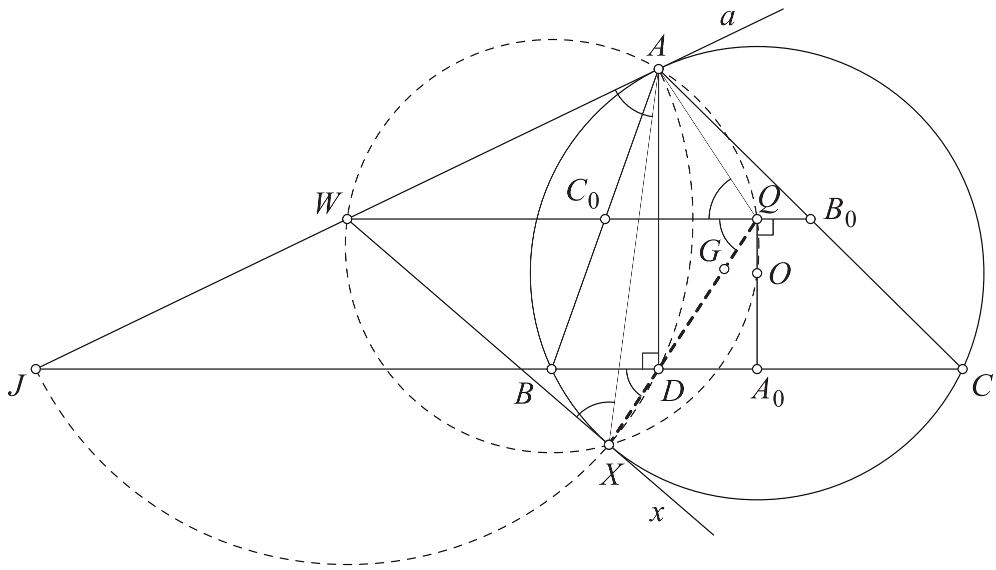
Denote by A_0 the midpoint of BC, and consider the image of $ATCB$ under the homothety h with center G and factor $-\frac{1}{2}$. We have $h(A) = A_0$, $h(B) = B_0$, and $h(C) = C_0$. From the symmetry about B_0C_0, we have $\angle TCB = \angle CBA = \angle B_0C_0A = \angle DC_0B_0$. Using $AT \parallel DA_0$, we conclude $h(T) = D$. Hence the points D, G, and T are collinear, and X lies on the same line.

Solution 2. We define the points A_0, O, and W as in the previous solution and we concentrate on the case $AB < AC$. Let Q be the perpendicular projection of A_0 on B_0C_0.

Since $\angle WAO = \angle WQO = \angle OXW = 90°$, the five points A, W, X, O, and Q lie on a common circle. Furthermore, the reflections with respect to B_0C_0 and OW map A to D and X, respectively. For these reasons, we have

$$\angle WQD = \angle AQW = \angle AXW = \angle WAX = \angle WQX.$$

Thus the three points Q, D, and X lie on a common line, say ℓ.



To complete the argument, we note that the homothety centered at G sending the triangle ABC to the triangle $A_0B_0C_0$ maps the altitude AD to the altitude A_0Q. Therefore it maps D to Q, so the points D, G, and Q are collinear. Hence G lies on ℓ as well.

Comment. There are various other ways to prove the collinearity of Q, D, and X obtained in the middle part of Solution 2. Introduce for instance the point J where the lines AW and BC intersect. Then the four points A, D, X, and J lie at the same distance from W, so the quadrilateral $ADXJ$ is cyclic. In combination with the fact that $AWXQ$ is cyclic as well, this implies

$$\angle JDX = \angle JAX = \angle WAX = \angle WQX.$$

Since $BC \parallel WQ$, it follows that Q, D, and X are indeed collinear.

G5

Let ABC be a triangle with incenter I and circumcircle ω. Let D and E be the second intersection points of ω with the lines AI and BI, respectively. The chord DE meets AC at a point F, and BC at a point G. Let P be the intersection point of the line through F parallel to AD and the line through G parallel to BE. Suppose that the tangents to ω at A and at B meet at a point K. Prove that the three lines AE, BD, and KP are either parallel or concurrent.

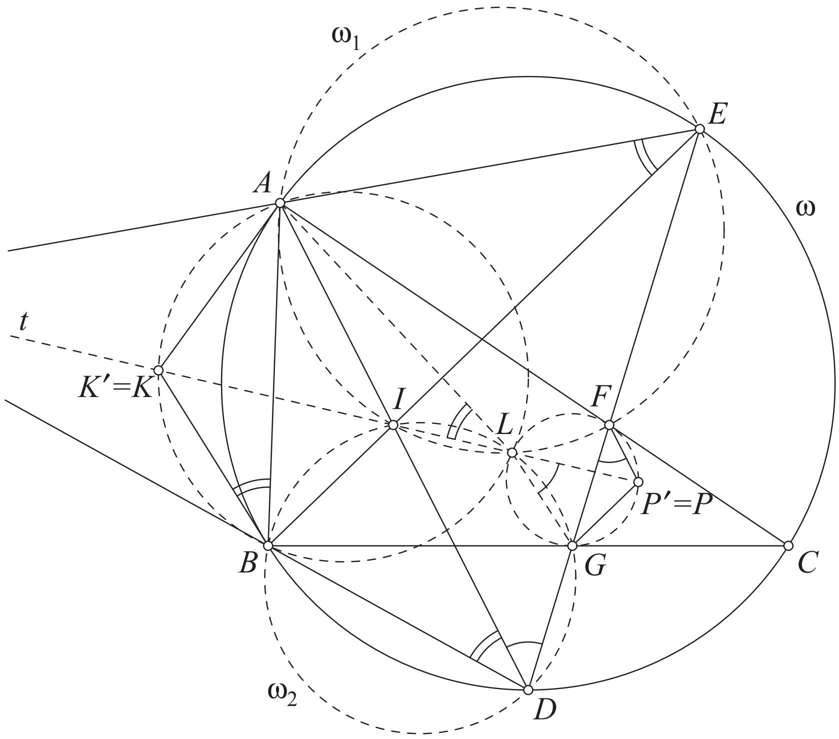
Solution 1. Since

$$\angle IAF = \angle DAC = \angle BAD = \angle BED = \angle IEF$$

the quadrilateral $AIFE$ is cyclic. Denote its circumcircle by ω_1. Similarly, the quadrilateral $BDGI$ is cyclic; denote its circumcircle by ω_2.

The line AE is the radical axis of ω and ω_1, and the line BD is the radical axis of ω and ω_2. Let t be the radical axis of ω_1 and ω_2. These three lines meet at the radical center of the three circles, or they are parallel to each other. We will show that t is in fact the line PK.

Let L be the second intersection point of ω_1 and ω_2, so $t = IL$. (If the two circles are tangent to each other then $L = I$ and t is the common tangent.)



Let the line t meet the circumcircles of the triangles ABL and FGL at $K' \neq L$ and $P' \neq L$, respectively. Using oriented angles we have

$$\angle(AB, BK') = \angle(AL, LK') = \angle(AL, LI) = \angle(AE, EI) = \angle(AE, EB) = \angle(AB, BK),$$

so $BK' \parallel BK$. Similarly we have $AK' \parallel AK$, and therefore $K' = K$. Next, we have

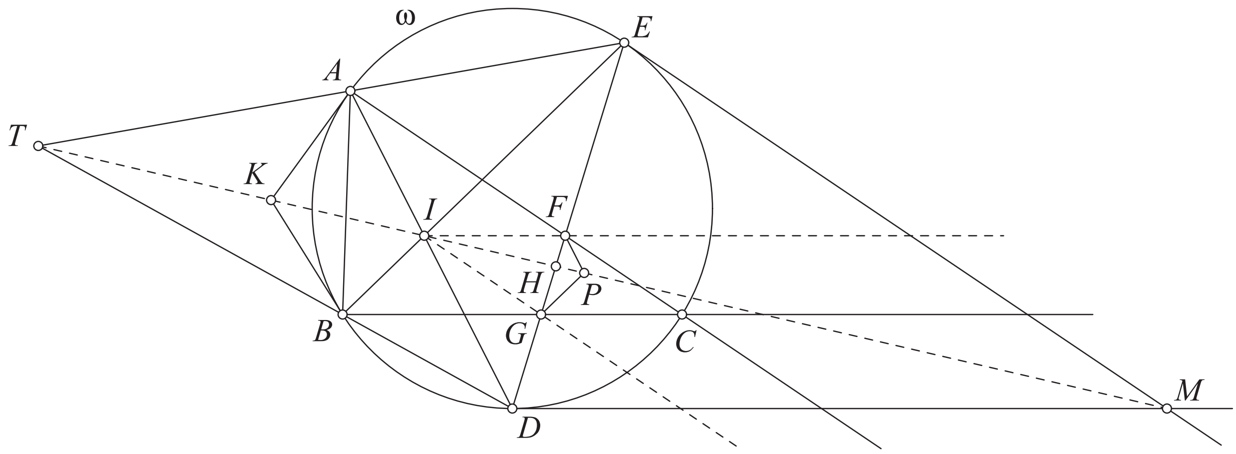
$$\angle(P'F, FG) = \angle(P'L, LG) = \angle(IL, LG) = \angle(ID, DG) = \angle(AD, DE) = \angle(PF, FG),$$

hence $P'F \parallel PF$ and similarly $P'G \parallel PG$. Therefore $P' = P$. This means that t passes through K and P, which finishes the proof.

Solution 2. Let M be the intersection point of the tangents to ω at D and E, and let the lines AE and BD meet at T; if AE and BD are parallel, then let T be their common ideal point. It is well-known that the points K and M lie on the line TI (as a consequence of Pascal's theorem, applied to the inscribed degenerate hexagons $AADBBE$ and $ADDBEE$).

The lines AD and BE are the angle bisectors of the angles $\angle CAB$ and $\angle ABC$, respectively, so D and E are the midpoints of the arcs BC and CA of the circle ω, respectively. Hence, DM is parallel to BC and EM is parallel to AC.

Apply Pascal's theorem to the degenerate hexagon $CADDEB$. By the theorem, the points $CA \cap DE = F$, $AD \cap EB = I$ and the common ideal point of lines DM and BC are collinear, therefore FI is parallel to BC and DM. Analogously, the line GI is parallel to AC and EM.



Now consider the homothety with scale factor $-\frac{FG}{ED}$ which takes E to G and D to F. Since the triangles EDM and GFI have parallel sides, the homothety takes M to I. Similarly, since the triangles DEI and FGP have parallel sides, the homothety takes I to P. Hence, the points M, I, P and the homothety center H must lie on the same line. Therefore, the point P also lies on the line $TKIM$.

Comment. One may prove that $IF \parallel BC$ and $IG \parallel AC$ in a more elementary way. Since $\angle ADE = \angle EDC$ and $\angle DEB = \angle CED$, the points I and C are symmetric about DE. Moreover, since the arcs AE and EC are equal and the arcs CD and DB are equal, we have $\angle CFG = \angle FGC$, so the triangle CFG is isosceles. Hence, the quadrilateral $IFCG$ is a rhombus.

G6

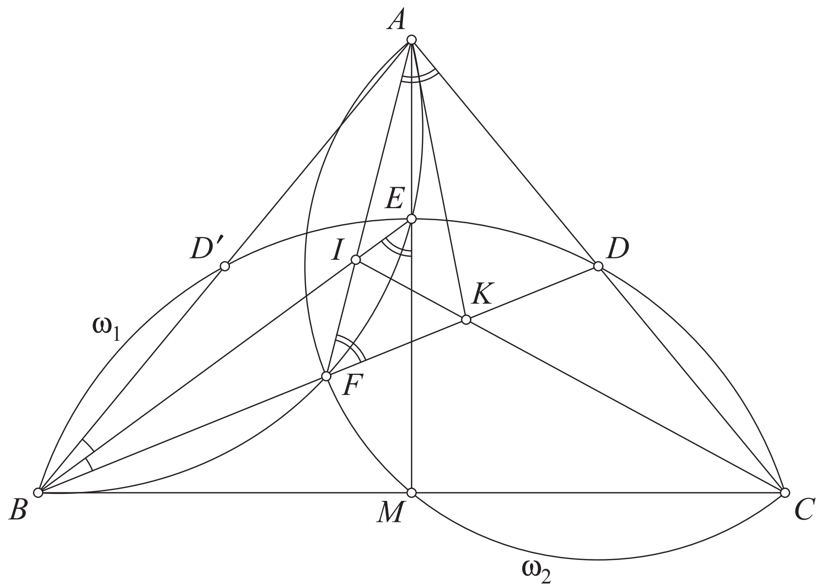
Let ABC be a triangle with $AB = AC$, and let D be the midpoint of AC. The angle bisector of $\angle BAC$ intersects the circle through D, B, and C in a point E inside the triangle ABC. The line BD intersects the circle through A, E, and B in two points B and F. The lines AF and BE meet at a point I, and the lines CI and BD meet at a point K. Show that I is the incenter of triangle KAB.

Solution 1. Let D' be the midpoint of the segment AB, and let M be the midpoint of BC. By symmetry at line AM, the point D' has to lie on the circle BCD. Since the arcs $D'E$ and ED of that circle are equal, we have $\angle ABI = \angle D'BE = \angle EBD = IBK$, so I lies on the angle bisector of $\angle ABK$. For this reason it suffices to prove in the sequel that the ray AI bisects the angle $\angle BAK$.

From

$$\angle DFA = 180^\circ - \angle BFA = 180^\circ - \angle BEA = \angle MEB = \frac{1}{2}\angle CEB = \frac{1}{2}\angle CDB$$

we derive $\angle DFA = \angle DAF$ so the triangle AFD is isosceles with $AD = DF$.



Applying Menelaus's theorem to the triangle ADF with respect to the line CIK, and applying the angle bisector theorem to the triangle ABF, we infer

$$1 = \frac{AC}{CD} \cdot \frac{DK}{KF} \cdot \frac{FI}{IA} = 2 \cdot \frac{DK}{KF} \cdot \frac{BF}{AB} = 2 \cdot \frac{DK}{KF} \cdot \frac{BF}{2 \cdot AD} = \frac{DK}{KF} \cdot \frac{BF}{AD}$$

and therefore

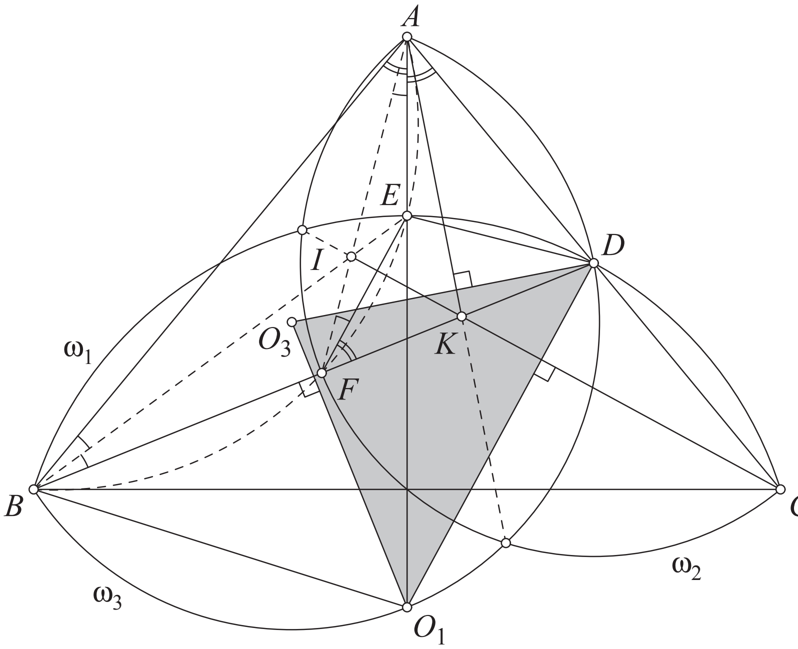
$$\frac{BD}{AD} = \frac{BF + FD}{AD} = \frac{BF}{AD} + 1 = \frac{KF}{DK} + 1 = \frac{DF}{DK} = \frac{AD}{DK}.$$

It follows that the triangles ADK and BDA are similar, hence $\angle DAK = \angle ABD$. Then

$$\angle IAB = \angle AFD - \angle ABD = \angle DAF - \angle DAK = \angle KAI$$

shows that the point K is indeed lying on the angle bisector of $\angle BAK$.

Solution 2. It can be shown in the same way as in the first solution that I lies on the angle bisector of $\angle ABK$. Here we restrict ourselves to proving that KI bisects $\angle AKB$.



Denote the circumcircle of triangle BCD and its center by ω_1 and by O_1, respectively. Since the quadrilateral $ABFE$ is cyclic, we have $\angle DFE = \angle BAE = \angle DAE$. By the same reason, we have $\angle EAF = \angle EBF = \angle ABE = \angle AFE$. Therefore $\angle DAF = \angle DFA$, and hence $DF = DA = DC$. So triangle AFC is inscribed in a circle ω_2 with center D.

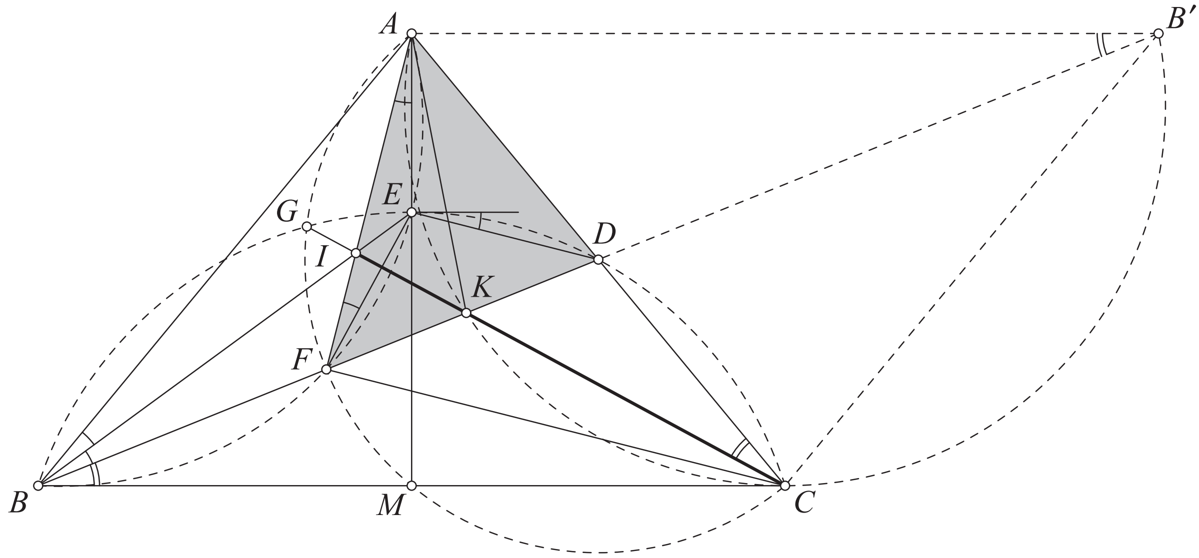
Denote the circumcircle of triangle ABD by ω_3, and let its center be O_3. Since the arcs BE and EC of circle ω_1 are equal, and the triangles ADE and FDE are congruent, we have $\angle AO_1B = 2\angle BDE = \angle BDA$, so O_1 lies on ω_3. Hence $\angle O_3O_1D = \angle O_3DO_1$.

The line BD is the radical axis of ω_1 and ω_3. Point C belongs to the radical axis of ω_1 and ω_2, and I also belongs to it since $AI \cdot IF = BI \cdot IE$. Hence $K = BD \cap CI$ is the radical center of ω_1, ω_2, and ω_3, and AK is the radical axis of ω_2 and ω_3. Now, the radical axes AK, BK and IK are perpendicular to the central lines O_3D, O_3O_1 and O_1D, respectively. By $\angle O_3O_1D = \angle O_3DO_1$, we get that KI is the angle bisector of $\angle AKB$.

Solution 3. Again, let M be the midpoint of BC. As in the previous solutions, we can deduce $\angle ABI = \angle IBK$. We show that the point I lies on the angle bisector of $\angle KAB$.

Let G be the intersection point of the circles AFC and BCD, different from C. The lines

CG, AF, and BE are the radical axes of the three circles $AGFC$, CDB, and $ABFE$, so $I = AF \cap BE$ is the radical center of the three circles and CG also passes through I.



The angle between line DE and the tangent to the circle BCD at E is equal to $\angle EBD = \angle EAF = \angle ABE = \angle AFE$. As the tangent at E is perpendicular to AM, the line DE is perpendicular to AF. The triangle AFE is isosceles, so DE is the perpendicular bisector of AF and thus $AD = DF$. Hence, the point D is the center of the circle AFC, and this circle passes through M as well since $\angle AMC = 90°$.

Let B' be the reflection of B in the point D, so $ABCB'$ is a parallelogram. Since $DC = DG$ we have $\angle GCD = \angle DBC = \angle KB'A$. Hence, the quadrilateral $AKCB'$ is cyclic and thus $\angle CAK = \angle CB'K = \angle ABD = 2\angle MAI$. Then

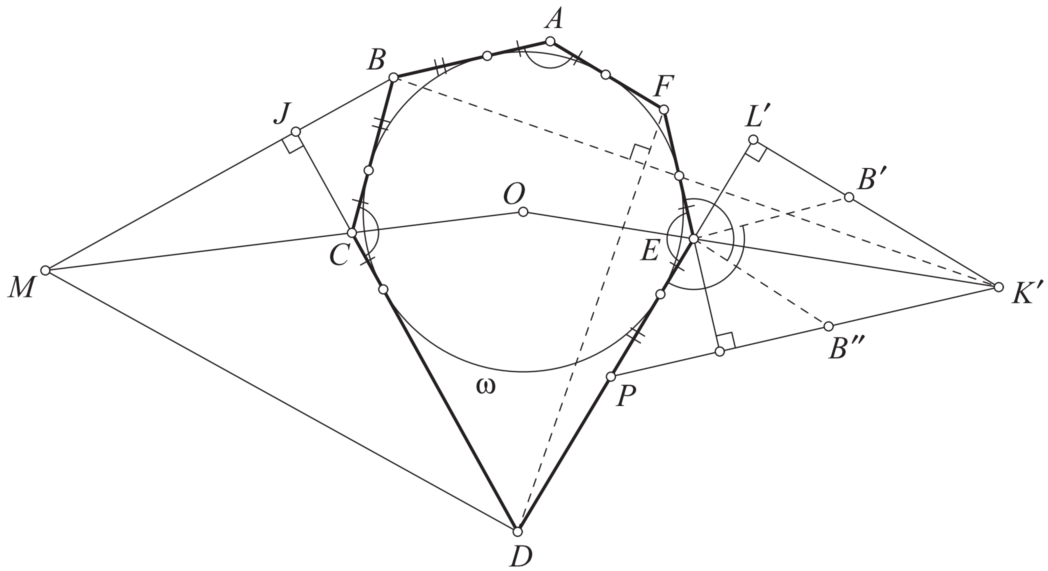
$$\angle IAB = \angle MAB - \angle MAI = \frac{1}{2}\angle CAB - \frac{1}{2}\angle CAK = \frac{1}{2}\angle KAB$$

and therefore AI is the angle bisector of $\angle KAB$.

G7

Let $ABCDEF$ be a convex hexagon all of whose sides are tangent to a circle ω with center O. Suppose that the circumcircle of triangle ACE is concentric with ω. Let J be the foot of the perpendicular from B to CD. Suppose that the perpendicular from B to DF intersects the line EO at a point K. Let L be the foot of the perpendicular from K to DE. Prove that $DJ = DL$.

Solution 1. Since ω and the circumcircle of triangle ACE are concentric, the tangents from A, C, and E to ω have equal lengths; that means that $AB = BC$, $CD = DE$, and $EF = FA$. Moreover, we have $\angle BCD = \angle DEF = \angle FAB$.



Consider the rotation around point D mapping C to E; let B' and L' be the images of the points B and J, respectively, under this rotation. Then one has $DJ = DL'$ and $B'L' \perp DE$; moreover, the triangles $B'ED$ and BCD are congruent. Since $\angle DEO < 90°$, the lines EO and $B'L'$ intersect at some point K'. We intend to prove that $K'B \perp DF$; this would imply $K = K'$, therefore $L = L'$, which proves the problem statement.

Analogously, consider the rotation around F mapping A to E; let B'' be the image of B under this rotation. Then the triangles FAB and FEB'' are congruent. We have $EB'' = AB = BC = EB'$ and $\angle FEB'' = \angle FAB = \angle BCD = \angle DEB'$, so the points B' and B'' are symmetrical with respect to the angle bisector EO of $\angle DEF$. So, from $K'B' \perp DE$ we get $K'B'' \perp EF$.

From these two relations we obtain

$$K'D^2 - K'E^2 = B'D^2 - B'E^2 \quad \text{and} \quad K'E^2 - K'F^2 = B''E^2 - B''F^2.$$

Adding these equalities and taking into account that $B'E = B''E$ we obtain

$$K'D^2 - K'F^2 = B'D^2 - B''F^2 = BD^2 - BF^2,$$

which means exactly that $K'B \perp DF$.

Comment. There are several variations of this solution. For instance, let us consider the intersection point M of the lines BJ and OC. Define the point K' as the reflection of M in the line DO. Then one has

$$DK'^2 - DB^2 = DM^2 - DB^2 = CM^2 - CB^2.$$

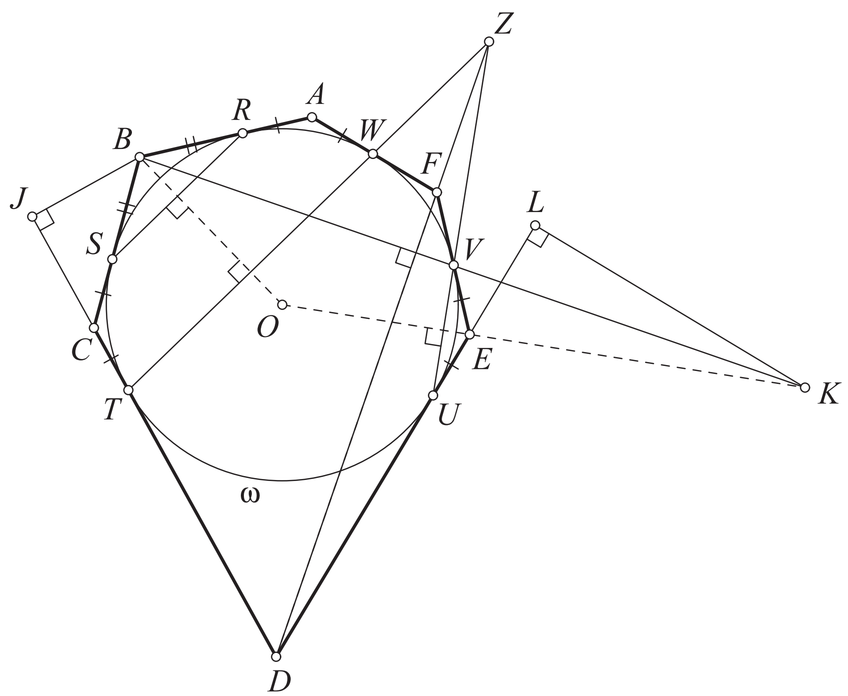
Next, consider the rotation around O which maps CM to EK'. Let P be the image of B under this rotation; so P lies on ED. Then $EF \perp K'P$, so

$$CM^2 - CB^2 = EK'^2 - EP^2 = FK'^2 - FP^2 = FK'^2 - FB^2,$$

since the triangles FEP and FAB are congruent.

Solution 2. Let us denote the points of tangency of AB, BC, CD, DE, EF, and FA to ω by R, S, T, U, V, and W, respectively. As in the previous solution, we mention that $AR = AW = CS = CT = EU = EV$.

The reflection in the line BO maps R to S, therefore A to C and thus also W to T. Hence, both lines RS and WT are perpendicular to OB, therefore they are parallel. On the other hand, the lines UV and WT are not parallel, since otherwise the hexagon $ABCDEF$ is symmetric with respect to the line BO and the lines defining the point K coincide, which contradicts the conditions of the problem. Therefore we can consider the intersection point Z of UV and WT.



Next, we recall a well-known fact that the points D, F, Z are collinear. Actually, D is the pole of the line UT, F is the pole of VW, and $Z = TW \cap UV$; so all these points belong to the polar line of $TU \cap VW$.

Now, we put O into the origin, and identify each point (say X) with the vector $\overrightarrow{OX}$. So, from now on all the products of points refer to the scalar products of the corresponding vectors.

Since $OK \perp UZ$ and $OB \perp TZ$, we have $K \cdot (Z - U) = 0 = B \cdot (Z - T)$. Next, the condition $BK \perp DZ$ can be written as $K \cdot (D - Z) = B \cdot (D - Z)$. Adding these two equalities we get

$$K \cdot (D - U) = B \cdot (D - T).$$

By symmetry, we have $D \cdot (D - U) = D \cdot (D - T)$. Subtracting this from the previous equation, we obtain $(K - D) \cdot (D - U) = (B - D) \cdot (D - T)$ and rewrite it in vector form as

$$\overrightarrow{DK} \cdot \overrightarrow{UD} = \overrightarrow{DB} \cdot \overrightarrow{TD}.$$

Finally, projecting the vectors $\overrightarrow{DK}$ and $\overrightarrow{DB}$ onto the lines UD and TD respectively, we can rewrite this equality in terms of segment lengths as $DL \cdot UD = DJ \cdot TD$, thus $DL = DJ$.

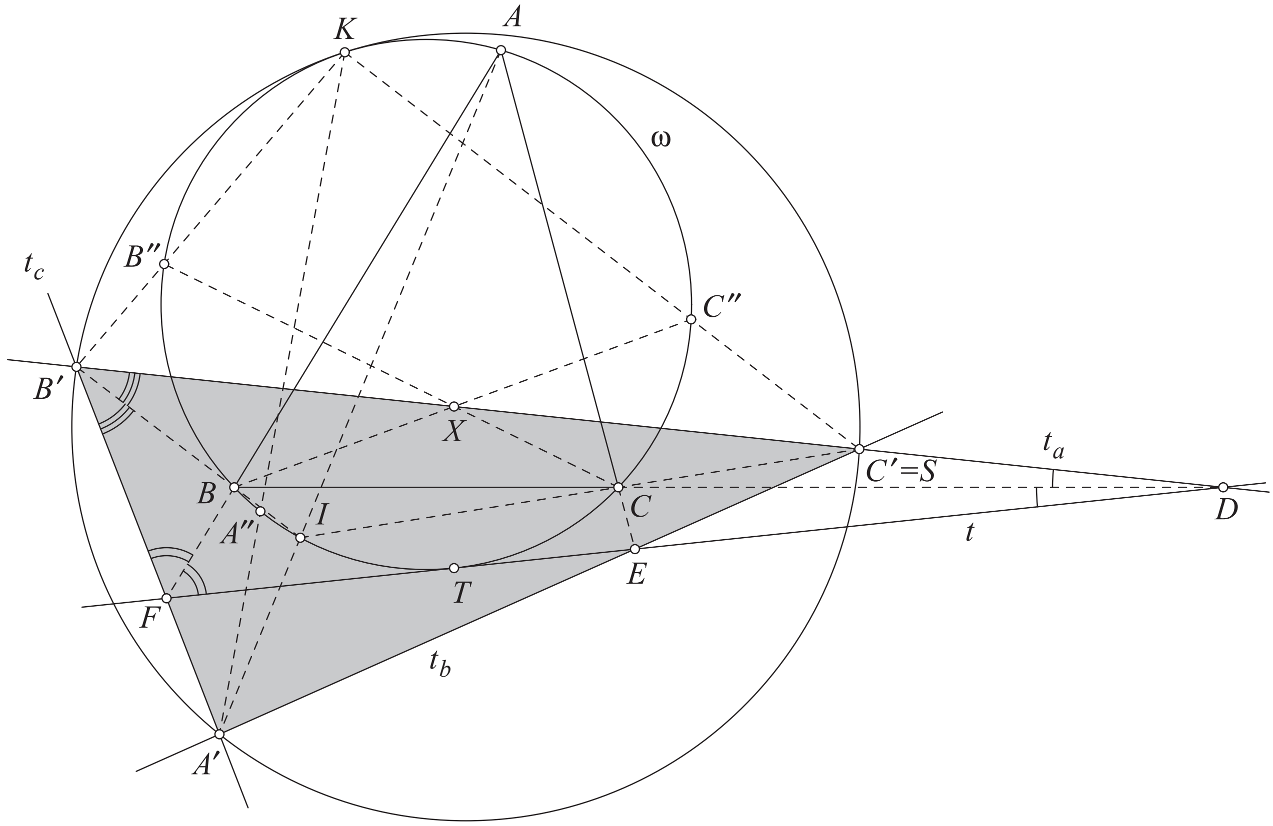
Comment. The collinearity of Z, F, and D may be shown in various more elementary ways. For instance, applying the sine theorem to the triangles DTZ and DUZ, one gets $\dfrac{\sin \angle DZT}{\sin \angle DZU} = \dfrac{\sin \angle DTZ}{\sin \angle DUZ}$; analogously, $\dfrac{\sin \angle FZW}{\sin \angle FZV} = \dfrac{\sin \angle FWZ}{\sin \angle FVZ}$. The right-hand sides are equal, hence so are the left-hand sides, which implies the collinearity of the points D, F, and Z.

There also exist purely synthetic proofs of this fact. E.g., let Q be the point of intersection of the circumcircles of the triangles ZTV and ZWU different from Z. Then QZ is the bisector of $\angle VQW$ since $\angle VQZ = \angle VTZ = \angle VUW = \angle ZQW$. Moreover, all these angles are equal to $\frac{1}{2}\angle VOW$, so $\angle VQW = \angle VOW$, hence the quadrilateral $VWOQ$ is cyclic. On the other hand, the points O, V, W lie on the circle with diameter OF due to the right angles; so Q also belongs to this circle. Since $FV = FW$, QF is also the bisector of $\angle VQW$, so F lies on QZ. Analogously, D lies on the same line.

G8

Let ABC be an acute triangle with circumcircle ω. Let t be a tangent line to ω. Let t_a, t_b, and t_c be the lines obtained by reflecting t in the lines BC, CA, and AB, respectively. Show that the circumcircle of the triangle determined by the lines t_a, t_b, and t_c is tangent to the circle ω.

To avoid a large case distinction, we will use the notion of *oriented angles*. Namely, for two lines ℓ and m, we denote by $\angle(\ell, m)$ the angle by which one may rotate ℓ anticlockwise to obtain a line parallel to m. Thus, all oriented angles are considered modulo 180°.



Solution 1. Denote by T the point of tangency of t and ω. Let $A' = t_b \cap t_c$, $B' = t_a \cap t_c$, $C' = t_a \cap t_b$. Introduce the point A'' on ω such that $TA = AA''$ ($A'' \neq T$ unless TA is a diameter). Define the points B'' and C'' in a similar way.

Since the points C and B are the midpoints of arcs TC'' and TB'', respectively, we have

$$\begin{aligned}\angle(t, B''C'') &= \angle(t, TC'') + \angle(TC'', B''C'') = 2\angle(t, TC) + 2\angle(TC'', BC'') \\ &= 2\big(\angle(t, TC) + \angle(TC, BC)\big) = 2\angle(t, BC) = \angle(t, t_a).\end{aligned}$$

It follows that t_a and $B''C''$ are parallel. Similarly, $t_b \parallel A''C''$ and $t_c \parallel A''B''$. Thus, either the triangles $A'B'C'$ and $A''B''C''$ are homothetic, or they are translates of each other. Now we will prove that they are in fact homothetic, and that the center K of the homothety belongs to ω. It would then follow that their circumcircles are also homothetic with respect to K and are therefore tangent at this point, as desired.

We need the two following claims.

Claim 1. The point of intersection X of the lines $B''C$ and BC'' lies on t_a.

Proof. Actually, the points X and T are symmetric about the line BC, since the lines CT and CB'' are symmetric about this line, as are the lines BT and BC''. □

Claim 2. The point of intersection I of the lines BB' and CC' lies on the circle ω.

Proof. We consider the case that t is not parallel to the sides of ABC; the other cases may be regarded as limit cases. Let $D = t \cap BC$, $E = t \cap AC$, and $F = t \cap AB$.

Due to symmetry, the line DB is one of the angle bisectors of the lines $B'D$ and FD; analogously, the line FB is one of the angle bisectors of the lines $B'F$ and DF. So B is either the incenter or one of the excenters of the triangle $B'DF$. In any case we have $\angle(BD, DF) + \angle(DF, FB) + \angle(B'B, B'D) = 90^\circ$, so

$$\angle(B'B, B'C') = \angle(B'B, B'D) = 90^\circ - \angle(BC, DF) - \angle(DF, BA) = 90^\circ - \angle(BC, AB).$$

Analogously, we get $\angle(C'C, B'C') = 90^\circ - \angle(BC, AC)$. Hence,

$$\angle(BI, CI) = \angle(B'B, B'C') + \angle(B'C', C'C) = \angle(BC, AC) - \angle(BC, AB) = \angle(AB, AC),$$

which means exactly that the points A, B, I, C are concyclic. □

Now we can complete the proof. Let K be the second intersection point of $B'B''$ and ω. Applying Pascal's theorem to hexagon $KB''CIBC''$ we get that the points $B' = KB'' \cap IB$ and $X = B''C \cap BC''$ are collinear with the intersection point S of CI and $C''K$. So $S = CI \cap B'X = C'$, and the points C', C'', K are collinear. Thus K is the intersection point of $B'B''$ and $C'C''$ which implies that K is the center of the homothety mapping $A'B'C'$ to $A''B''C''$, and it belongs to ω.

Solution 2. Define the points T, A', B', and C' in the same way as in the previous solution. Let X, Y, and Z be the symmetric images of T about the lines BC, CA, and AB, respectively. Note that the projections of T on these lines form a Simson line of T with respect to ABC, therefore the points X, Y, Z are also collinear. Moreover, we have $X \in B'C'$, $Y \in C'A'$, $Z \in A'B'$.

Denote $\alpha = \angle(t, TC) = \angle(BT, BC)$. Using the symmetry in the lines AC and BC, we get

$$\angle(BC, BX) = \angle(BT, BC) = \alpha \quad \text{and} \quad \angle(XC, XC') = \angle(t, TC) = \angle(YC, YC') = \alpha.$$

Since $\angle(XC, XC') = \angle(YC, YC')$, the points X, Y, C, C' lie on some circle ω_c. Define the circles ω_a and ω_b analogously. Let ω' be the circumcircle of triangle $A'B'C'$.

Now, applying MIQUEL's theorem to the four lines $A'B'$, $A'C'$, $B'C'$, and XY, we obtain that the circles ω', ω_a, ω_b, ω_c intersect at some point K. We will show that K lies on ω, and that the tangent lines to ω and ω' at this point coincide; this implies the problem statement.

Due to symmetry, we have $XB = TB = ZB$, so the point B is the midpoint of one of the arcs XZ of circle ω_b. Therefore $\angle(KB, KX) = \angle(XZ, XB)$. Analogously, $\angle(KX, KC) = \angle(XC, XY)$. Adding these equalities and using the symmetry in the line BC we get

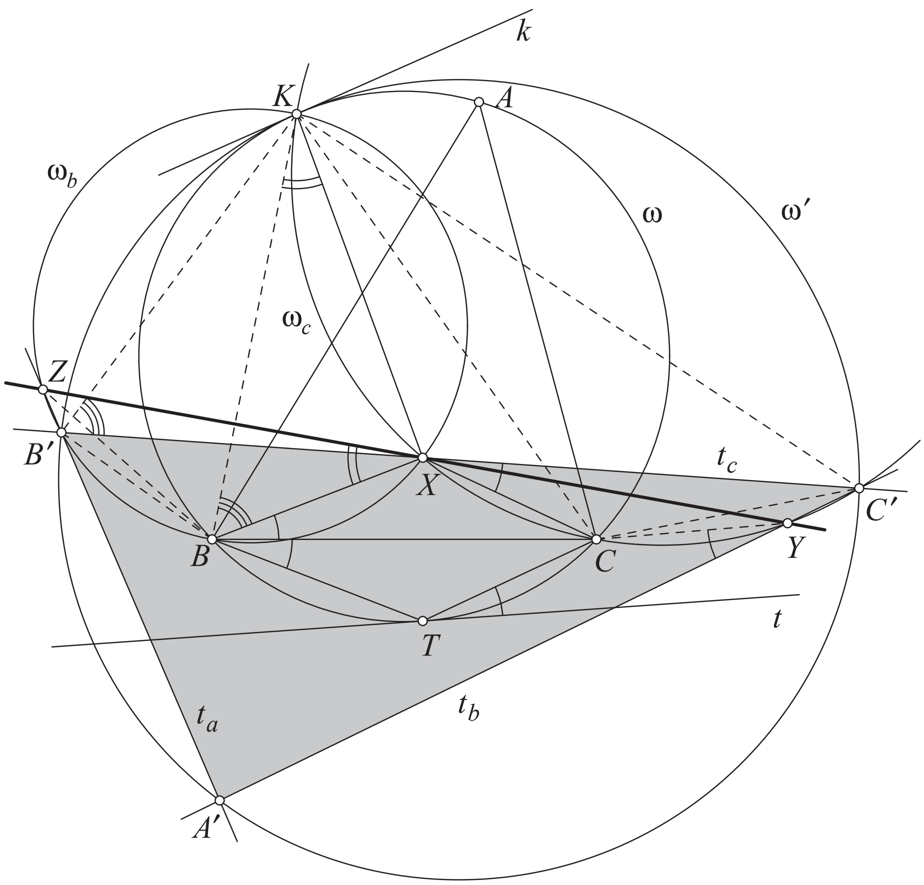
$$\angle(KB, KC) = \angle(XZ, XB) + \angle(XC, XZ) = \angle(XC, XB) = \angle(TB, TC).$$

Therefore, K lies on ω.

Next, let k be the tangent line to ω at K. We have

$$\begin{aligned}\angle(k, KC') &= \angle(k, KC) + \angle(KC, KC') = \angle(KB, BC) + \angle(XC, XC') \\ &= \big(\angle(KB, BX) - \angle(BC, BX)\big) + \alpha = \angle(KB', B'X) - \alpha + \alpha = \angle(KB', B'C'),\end{aligned}$$

which means exactly that k is tangent to ω'.



Comment. There exist various solutions combining the ideas from the two solutions presented above. For instance, one may define the point X as the reflection of T with respect to the line BC, and then introduce the point K as the second intersection point of the circumcircles of $BB'X$ and $CC'X$. Using the fact that BB' and CC' are the bisectors of $\angle(A'B', B'C')$ and $\angle(A'C', B'C')$ one can show successively that $K \in \omega$, $K \in \omega'$, and that the tangents to ω and ω' at K coincide.

N1

For any integer $d > 0$, let $f(d)$ be the smallest positive integer that has exactly d positive divisors (so for example we have $f(1) = 1$, $f(5) = 16$, and $f(6) = 12$). Prove that for every integer $k \geq 0$ the number $f(2^k)$ divides $f(2^{k+1})$.

Solution 1. For any positive integer n, let $d(n)$ be the number of positive divisors of n. Let $n = \prod_p p^{a(p)}$ be the prime factorization of n where p ranges over the prime numbers, the integers $a(p)$ are nonnegative and all but finitely many $a(p)$ are zero. Then we have $d(n) = \prod_p (a(p)+1)$. Thus, $d(n)$ is a power of 2 if and only if for every prime p there is a nonnegative integer $b(p)$ with $a(p) = 2^{b(p)} - 1 = 1 + 2 + 2^2 + \cdots + 2^{b(p)-1}$. We then have

$$n = \prod_p \prod_{i=0}^{b(p)-1} p^{2^i}, \qquad \text{and} \quad d(n) = 2^k \quad \text{with} \quad k = \sum_p b(p).$$

Let $\mathcal{S}$ be the set of all numbers of the form p^{2^r} with p prime and r a nonnegative integer. Then we deduce that $d(n)$ is a power of 2 if and only if n is the product of the elements of some finite subset $\mathcal{T}$ of $\mathcal{S}$ that satisfies the following condition: for all $t \in \mathcal{T}$ and $s \in \mathcal{S}$ with $s \mid t$ we have $s \in \mathcal{T}$. Moreover, if $d(n) = 2^k$ then the corresponding set $\mathcal{T}$ has k elements.

Note that the set $\mathcal{T}_k$ consisting of the smallest k elements from $\mathcal{S}$ obviously satisfies the condition above. Thus, given k, the smallest n with $d(n) = 2^k$ is the product of the elements of $\mathcal{T}_k$. This n is $f(2^k)$. Since obviously $\mathcal{T}_k \subset \mathcal{T}_{k+1}$, it follows that $f(2^k) \mid f(2^{k+1})$.

Solution 2. This is an alternative to the second part of the Solution 1. Suppose k is a nonnegative integer. From the first part of Solution 1 we see that $f(2^k) = \prod_p p^{a(p)}$ with $a(p) = 2^{b(p)} - 1$ and $\sum_p b(p) = k$. We now claim that for any two distinct primes p, q with $b(q) > 0$ we have

$$m = p^{2^{b(p)}} > q^{2^{b(q)-1}} = \ell. \tag{1}$$

To see this, note first that ℓ divides $f(2^k)$. With the first part of Solution 1 one can see that the integer $n = f(2^k)m/\ell$ also satisfies $d(n) = 2^k$. By the definition of $f(2^k)$ this implies that $n \geq f(2^k)$ so $m \geq \ell$. Since $p \neq q$ the inequality (1) follows.

Let the prime factorization of $f(2^{k+1})$ be given by $f(2^{k+1}) = \prod_p p^{r(p)}$ with $r(p) = 2^{s(p)} - 1$. Since we have $\sum_p s(p) = k+1 > k = \sum_p b(p)$ there is a prime p with $s(p) > b(p)$. For any prime $q \neq p$ with $b(q) > 0$ we apply inequality (1) twice and get

$$q^{2^{s(q)}} > p^{2^{s(p)-1}} \geq p^{2^{b(p)}} > q^{2^{b(q)-1}},$$

which implies $s(q) \geq b(q)$. It follows that $s(q) \geq b(q)$ for all primes q, so $f(2^k) \mid f(2^{k+1})$.

N2

Consider a polynomial $P(x) = (x + d_1)(x + d_2) \cdot \ldots \cdot (x + d_9)$, where $d_1, d_2, \ldots, d_9$ are nine distinct integers. Prove that there exists an integer N such that for all integers $x \geq N$ the number $P(x)$ is divisible by a prime number greater than 20.

Solution 1. Note that the statement of the problem is invariant under translations of x; hence without loss of generality we may suppose that the numbers $d_1, d_2, \ldots, d_9$ are positive.

The key observation is that there are only eight primes below 20, while $P(x)$ involves more than eight factors.

We shall prove that $N = d^8$ satisfies the desired property, where $d = \max\{d_1, d_2, \ldots, d_9\}$. Suppose for the sake of contradiction that there is some integer $x \geq N$ such that $P(x)$ is composed of primes below 20 only. Then for every index $i \in \{1, 2, \ldots, 9\}$ the number $x + d_i$ can be expressed as product of powers of the first 8 primes.

Since $x + d_i > x \geq d^8$ there is some prime power $f_i > d$ that divides $x + d_i$. Invoking the pigeonhole principle we see that there are two distinct indices i and j such that f_i and f_j are powers of the same prime number. For reasons of symmetry, we may suppose that $f_i \leq f_j$. Now both of the numbers $x + d_i$ and $x + d_j$ are divisible by f_i and hence so is their difference $d_i - d_j$. But as

$$0 < |d_i - d_j| \leq \max(d_i, d_j) \leq d < f_i,$$

this is impossible. Thereby the problem is solved.

Solution 2. Observe that for each index $i \in \{1, 2, \ldots, 9\}$ the product

$$D_i = \prod_{1 \leq j \leq 9, j \neq i} |d_i - d_j|$$

is positive. We claim that $N = \max\{D_1 - d_1, D_2 - d_2, \ldots, D_9 - d_9\} + 1$ satisfies the statement of the problem. Suppose there exists an integer $x \geq N$ such that all primes dividing $P(x)$ are smaller than 20. For each index i we reduce the fraction $(x + d_i)/D_i$ to lowest terms. Since $x + d_i > D_i$ the numerator of the fraction we thereby get cannot be 1, and hence it has to be divisible by some prime number $p_i < 20$.

By the pigeonhole principle, there are a prime number p and two distinct indices i and j such that $p_i = p_j = p$. Let p^{α_i} and p^{α_j} be the greatest powers of p dividing $x + d_i$ and $x + d_j$, respectively. Due to symmetry we may suppose $\alpha_i \leq \alpha_j$. But now p^{α_i} divides $d_i - d_j$ and hence also D_i, which means that all occurrences of p in the numerator of the fraction $(x + d_i)/D_i$ cancel out, contrary to the choice of $p = p_i$. This contradiction proves our claim.

Solution 3. Given a nonzero integer N as well as a prime number p we write $v_p(N)$ for the exponent with which p occurs in the prime factorization of $|N|$.

Evidently, if the statement of the problem were not true, then there would exist an infinite sequence (x_n) of positive integers tending to infinity such that for each $n \in \mathbb{Z}_+$ the integer $P(x_n)$ is not divisible by any prime number > 20. Observe that the numbers $-d_1, -d_2, \ldots, -d_9$ do not appear in this sequence.

Now clearly there exists a prime $p_1 < 20$ for which the sequence $v_{p_1}(x_n + d_1)$ is not bounded; thinning out the sequence (x_n) if necessary we may even suppose that

$$v_{p_1}(x_n + d_1) \longrightarrow \infty.$$

Repeating this argument eight more times we may similarly choose primes $p_2, \ldots, p_9 < 20$ and suppose that our sequence (x_n) has been thinned out to such an extent that $v_{p_i}(x_n + d_i) \longrightarrow \infty$ holds for $i = 2, \ldots, 9$ as well. In view of the pigeonhole principle, there are distinct indices i and j as well as a prime $p < 20$ such that $p_i = p_j = p$. Setting $k = v_p(d_i - d_j)$ there now has to be some n for which both $v_p(x_n + d_i)$ and $v_p(x_n + d_j)$ are greater than k. But now the numbers $x_n + d_i$ and $x_n + d_j$ are divisible by p^{k+1} whilst their difference $d_i - d_j$ is not – a contradiction.

Comment. This problem is supposed to be a relatively easy one, so one might consider adding the hypothesis that the numbers $d_1, d_2, \ldots, d_9$ be positive. Then certain merely technical issues are not going to arise while the main ideas required to solve the problems remain the same.

N3

Let $n \geq 1$ be an odd integer. Determine all functions f from the set of integers to itself such that for all integers x and y the difference $f(x) - f(y)$ divides $x^n - y^n$.

Answer. All functions f of the form $f(x) = \varepsilon x^d + c$, where ε is in $\{1, -1\}$, the integer d is a positive divisor of n, and c is an integer.

Solution. Obviously, all functions in the answer satisfy the condition of the problem. We will show that there are no other functions satisfying that condition.

Let f be a function satisfying the given condition. For each integer n, the function g defined by $g(x) = f(x) + n$ also satisfies the same condition. Therefore, by subtracting $f(0)$ from $f(x)$ we may assume that $f(0) = 0$.

For any prime p, the condition on f with $(x, y) = (p, 0)$ states that $f(p)$ divides p^n. Since the set of primes is infinite, there exist integers d and ε with $0 \leq d \leq n$ and $\varepsilon \in \{1, -1\}$ such that for infinitely many primes p we have $f(p) = \varepsilon p^d$. Denote the set of these primes by P. Since a function g satisfies the given condition if and only if $-g$ satisfies the same condition, we may suppose $\varepsilon = 1$.

The case $d = 0$ is easily ruled out, because 0 does not divide any nonzero integer. Suppose $d \geq 1$ and write n as $md + r$, where m and r are integers such that $m \geq 1$ and $0 \leq r \leq d - 1$. Let x be an arbitrary integer. For each prime p in P, the difference $f(p) - f(x)$ divides $p^n - x^n$. Using the equality $f(p) = p^d$, we get

$$p^n - x^n = p^r (p^d)^m - x^n \equiv p^r f(x)^m - x^n \equiv 0 \pmod{p^d - f(x)}$$

Since we have $r < d$, for large enough primes $p \in P$ we obtain

$$|p^r f(x)^m - x^n| < p^d - f(x).$$

Hence $p^r f(x)^m - x^n$ has to be zero. This implies $r = 0$ and $x^n = (x^d)^m = f(x)^m$. Since m is odd, we obtain $f(x) = x^d$.

Comment. If n is an even positive integer, then the functions f of the form

$$f(x) = \begin{cases} x^d + c \text{ for some integers,} \\ -x^d + c \text{ for the rest of integers,} \end{cases}$$

where d is a positive divisor of $n/2$ and c is an integer, also satisfy the condition of the problem. Together with the functions in the answer, they are all functions that satisfy the condition when n is even.

N4

For each positive integer k, let $t(k)$ be the largest odd divisor of k. Determine all positive integers a for which there exists a positive integer n such that all the differences

$$t(n+a)-t(n), \quad t(n+a+1)-t(n+1), \quad \ldots, \quad t(n+2a-1)-t(n+a-1)$$

are divisible by 4.

Answer. $a = 1$, 3, or 5.

Solution. A pair (a, n) satisfying the condition of the problem will be called a *winning pair*. It is straightforward to check that the pairs $(1, 1)$, $(3, 1)$, and $(5, 4)$ are winning pairs.

Now suppose that a is a positive integer not equal to 1, 3, and 5. We will show that there are no winning pairs (a, n) by distinguishing three cases.

Case 1: a is even. In this case we have $a = 2^\alpha d$ for some positive integer α and some odd d. Since $a \geq 2^\alpha$, for each positive integer n there exists an $i \in \{0, 1, \ldots, a-1\}$ such that $n + i = 2^{\alpha-1}e$, where e is some odd integer. Then we have $t(n+i) = t(2^{\alpha-1}e) = e$ and

$$t(n+a+i) = t(2^\alpha d + 2^{\alpha-1}e) = 2d + e \equiv e + 2 \pmod 4.$$

So we get $t(n+i) - t(n+a+i) \equiv 2 \pmod 4$, and (a, n) is not a winning pair.

Case 2: a is odd and $a > 8$. For each positive integer n, there exists an $i \in \{0, 1, \ldots, a-5\}$ such that $n + i = 2d$ for some odd d. We get

$$t(n+i) = d \not\equiv d + 2 = t(n+i+4) \pmod 4$$

and

$$t(n+a+i) = n+a+i \equiv n+a+i+4 = t(n+a+i+4) \pmod 4.$$

Therefore, the integers $t(n+a+i) - t(n+i)$ and $t(n+a+i+4) - t(n+i+4)$ cannot be both divisible by 4, and therefore there are no winning pairs in this case.

Case 3: $a = 7$. For each positive integer n, there exists an $i \in \{0, 1, \ldots, 6\}$ such that $n + i$ is either of the form $8k + 3$ or of the form $8k + 6$, where k is a nonnegative integer. But we have

$$t(8k+3) \equiv 3 \not\equiv 1 \equiv 4k + 5 = t(8k+3+7) \pmod 4$$

and

$$t(8k+6) = 4k+3 \equiv 3 \not\equiv 1 \equiv t(8k+6+7) \pmod 4.$$

Hence, there are no winning pairs of the form $(7, n)$.

N5

Let f be a function from the set of integers to the set of positive integers. Suppose that for any two integers m and n, the difference $f(m) - f(n)$ is divisible by $f(m-n)$. Prove that for all integers m, n with $f(m) \leq f(n)$ the number $f(n)$ is divisible by $f(m)$.

Solution 1. Suppose that x and y are two integers with $f(x) < f(y)$. We will show that $f(x) \mid f(y)$. By taking $m = x$ and $n = y$ we see that

$$f(x-y) \mid |f(x) - f(y)| = f(y) - f(x) > 0,$$

so $f(x-y) \leq f(y) - f(x) < f(y)$. Hence the number $d = f(x) - f(x-y)$ satisfies

$$-f(y) < -f(x-y) < d < f(x) < f(y).$$

Taking $m = x$ and $n = x - y$ we see that $f(y) \mid d$, so we deduce $d = 0$, or in other words $f(x) = f(x-y)$. Taking $m = x$ and $n = y$ we see that $f(x) = f(x-y) \mid f(x) - f(y)$, which implies $f(x) \mid f(y)$.

Solution 2. We split the solution into a sequence of claims; in each claim, the letters m and n denote arbitrary integers.

Claim 1. $f(n) \mid f(mn)$.

Proof. Since trivially $f(n) \mid f(1 \cdot n)$ and $f(n) \mid f((k+1)n) - f(kn)$ for all integers k, this is easily seen by using induction on m in both directions. □

Claim 2. $f(n) \mid f(0)$ and $f(n) = f(-n)$.

Proof. The first part follows by plugging $m = 0$ into Claim 1. Using Claim 1 twice with $m = -1$, we get $f(n) \mid f(-n) \mid f(n)$, from which the second part follows. □

From Claim 1, we get $f(1) \mid f(n)$ for all integers n, so $f(1)$ is the minimal value attained by f. Next, from Claim 2, the function f can attain only a finite number of values since all these values divide $f(0)$.

Now we prove the statement of the problem by induction on the number N_f of values attained by f. In the base case $N_f \leq 2$, we either have $f(0) \neq f(1)$, in which case these two numbers are the only values attained by f and the statement is clear, or we have $f(0) = f(1)$, in which case we have $f(1) \mid f(n) \mid f(0)$ for all integers n, so f is constant and the statement is obvious again.

For the induction step, assume that $N_f \geq 3$, and let a be the least positive integer with $f(a) > f(1)$. Note that such a number exists due to the symmetry of f obtained in Claim 2.

Claim 3. $f(n) \neq f(1)$ if and only if $a \mid n$.

Proof. Since $f(1) = \cdots = f(a-1) < f(a)$, the claim follows from the fact that

$$f(n) = f(1) \iff f(n+a) = f(1).$$

So it suffices to prove this fact.

Assume that $f(n) = f(1)$. Then $f(n+a) \mid f(a) - f(-n) = f(a) - f(n) > 0$, so $f(n+a) \leq f(a) - f(n) < f(a)$; in particular the difference $f(n+a) - f(n)$ is stricly smaller than $f(a)$. Furthermore, this difference is divisible by $f(a)$ and nonnegative since $f(n) = f(1)$ is the least value attained by f. So we have $f(n+a) - f(n) = 0$, as desired. For the converse direction we only need to remark that $f(n+a) = f(1)$ entails $f(-n-a) = f(1)$, and hence $f(n) = f(-n) = f(1)$ by the forward implication. □

We return to the induction step. So let us take two arbitrary integers m and n with $f(m) \leq f(n)$. If $a \nmid m$, then we have $f(m) = f(1) \mid f(n)$. On the other hand, suppose that $a \mid m$; then by Claim 3 $a \mid n$ as well. Now define the function $g(x) = f(ax)$. Clearly, g satisfies the conditions of the problem, but $N_g < N_f - 1$, since g does not attain $f(1)$. Hence, by the induction hypothesis, $f(m) = g(m/a) \mid g(n/a) = f(n)$, as desired.

Comment. After the fact that f attains a finite number of values has been established, there are several ways of finishing the solution. For instance, let $f(0) = b_1 > b_2 > \cdots > b_k$ be all these values. One may show (essentially in the same way as in Claim 3) that the set $S_i = \{n : f(n) \geq b_i\}$ consists exactly of all numbers divisible by some integer $a_i \geq 0$. One obviously has $a_i \mid a_{i-1}$, which implies $f(a_i) \mid f(a_{i-1})$ by Claim 1. So, $b_k \mid b_{k-1} \mid \cdots \mid b_1$, thus proving the problem statement.

Moreover, now it is easy to describe all functions satisfying the conditions of the problem. Namely, all these functions can be constructed as follows. Consider a sequence of nonnegative integers $a_1, a_2, \ldots, a_k$ and another sequence of positive integers $b_1, b_2, \ldots, b_k$ such that $|a_k| = 1$, $a_i \neq a_j$ and $b_i \neq b_j$ for all $1 \leq i < j \leq k$, and $a_i \mid a_{i-1}$ and $b_i \mid b_{i-1}$ for all $i = 2, \ldots, k$. Then one may introduce the function

$$f(n) = b_{i(n)}, \qquad \text{where} \quad i(n) = \min\{i : a_i \mid n\}.$$

These are all the functions which satisfy the conditions of the problem.

N6

Let $P(x)$ and $Q(x)$ be two polynomials with integer coefficients such that no nonconstant polynomial with rational coefficients divides both $P(x)$ and $Q(x)$. Suppose that for every positive integer n the integers $P(n)$ and $Q(n)$ are positive, and $2^{Q(n)} - 1$ divides $3^{P(n)} - 1$. Prove that $Q(x)$ is a constant polynomial.

Solution. First we show that there exists an integer d such that for all positive integers n we have $\gcd\bigl(P(n), Q(n)\bigr) \leq d$.

Since $P(x)$ and $Q(x)$ are coprime (over the polynomials with rational coefficients), Euclid's algorithm provides some polynomials $R_0(x), S_0(x)$ with rational coefficients such that $P(x)R_0(x) - Q(x)S_0(x) = 1$. Multiplying by a suitable positive integer d, we obtain polynomials $R(x) = d \cdot R_0(x)$ and $S(x) = d \cdot S_0(x)$ with integer coefficients for which $P(x)R(x) - Q(x)S(x) = d$. Then we have $\gcd\bigl(P(n), Q(n)\bigr) \leq d$ for any integer n.

To prove the problem statement, suppose that $Q(x)$ is not constant. Then the sequence $Q(n)$ is not bounded and we can choose a positive integer m for which

$$M = 2^{Q(m)} - 1 \geq 3^{\max\{P(1), P(2), \ldots, P(d)\}}. \tag{1}$$

Since $M = 2^{Q(n)} - 1 \mid 3^{P(n)} - 1$, we have $2, 3 \nmid M$. Let a and b be the multiplicative orders of 2 and 3 modulo M, respectively. Obviously, $a = Q(m)$ since the lower powers of 2 do not reach M. Since M divides $3^{P(m)} - 1$, we have $b \mid P(m)$. Then $\gcd(a, b) \leq \gcd\bigl(P(m), Q(m)\bigr) \leq d$. Since the expression $ax - by$ attains all integer values divisible by $\gcd(a, b)$ when x and y run over all nonnegative integer values, there exist some nonnegative integers x, y such that $1 \leq m + ax - by \leq d$.

By $Q(m + ax) \equiv Q(m) \pmod{a}$ we have

$$2^{Q(m+ax)} \equiv 2^{Q(m)} \equiv 1 \pmod{M}$$

and therefore

$$M \mid 2^{Q(m+ax)} - 1 \mid 3^{P(m+ax)} - 1.$$

Then, by $P(m + ax - by) \equiv P(m + ax) \pmod{b}$ we have

$$3^{P(m+ax-by)} \equiv 3^{P(m+ax)} \equiv 1 \pmod{M}.$$

Since $P(m + ax - by) > 0$ this implies $M \leq 3^{P(m+ax-by)} - 1$. But $P(m + ax - by)$ is listed among $P(1), P(2), \ldots, P(d)$, so

$$M < 3^{P(m+ax-by)} \leq 3^{\max\{P(1), P(2), \ldots, P(d)\}}$$

which contradicts (1).

Comment. We present another variant of the solution above.

Denote the degree of P by k and its leading coefficient by p. Consider any positive integer n and let $a = Q(n)$. Again, denote by b the multiplicative order of 3 modulo $2^a - 1$. Since $2^a - 1 \mid 3^{P(n)} - 1$, we have $b \mid P(n)$. Moreover, since $2^{Q(n+at)} - 1 \mid 3^{P(n+at)} - 1$ and $a = Q(n) \mid Q(n+at)$ for each positive integer t, we have $2^a - 1 \mid 3^{P(n+at)} - 1$, hence $b \mid P(n+at)$ as well.

Therefore, b divides $\gcd\{P(n+at) : t \geq 0\}$; hence it also divides the number

$$\sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}P(n+ai) = p \cdot k! \cdot a^k.$$

Finally, we get $b \mid \gcd\bigl(P(n), k! \cdot p \cdot Q(n)^k\bigr)$, which is bounded by the same arguments as in the beginning of the solution. So $3^b - 1$ is bounded, and hence $2^{Q(n)} - 1$ is bounded as well.

N7

Let p be an odd prime number. For every integer a, define the number

$$S_a = \frac{a}{1} + \frac{a^2}{2} + \cdots + \frac{a^{p-1}}{p-1}.$$

Let m and n be integers such that

$$S_3 + S_4 - 3S_2 = \frac{m}{n}.$$

Prove that p divides m.

Solution 1. For rational numbers p_1/q_1 and p_2/q_2 with the denominators q_1, q_2 not divisible by p, we write $p_1/q_1 \equiv p_2/q_2 \pmod p$ if the numerator $p_1q_2 - p_2q_1$ of their difference is divisible by p.

We start with finding an explicit formula for the residue of S_a modulo p. Note first that for every $k = 1, \ldots, p-1$ the number $\binom{p}{k}$ is divisible by p, and

$$\frac{1}{p}\binom{p}{k} = \frac{(p-1)(p-2)\cdots(p-k+1)}{k!} \equiv \frac{(-1)\cdot(-2)\cdots(-k+1)}{k!} = \frac{(-1)^{k-1}}{k} \pmod p$$

Therefore, we have

$$S_a = -\sum_{k=1}^{p-1} \frac{(-a)^k(-1)^{k-1}}{k} \equiv -\sum_{k=1}^{p-1}(-a)^k \cdot \frac{1}{p}\binom{p}{k} \pmod{p}.$$

The number on the right-hand side is integer. Using the binomial formula we express it as

$$-\sum_{k=1}^{p-1}(-a)^k \cdot \frac{1}{p}\binom{p}{k} = -\frac{1}{p}\left(-1-(-a)^p + \sum_{k=0}^{p}(-a)^k\binom{p}{k}\right) = \frac{(a-1)^p - a^p + 1}{p}$$

since p is odd. So, we have

$$S_a \equiv \frac{(a-1)^p - a^p + 1}{p} \pmod{p}.$$

Finally, using the obtained formula we get

$$\begin{aligned} S_3 + S_4 - 3S_2 &\equiv \frac{(2^p - 3^p + 1) + (3^p - 4^p + 1) - 3(1^p - 2^p + 1)}{p} \\ &= \frac{4 \cdot 2^p - 4^p - 4}{p} = -\frac{(2^p-2)^2}{p} \pmod{p}. \end{aligned}$$

By FERMAT's theorem, $p \mid 2^p - 2$, so $p^2 \mid (2^p-2)^2$ and hence $S_3 + S_4 - 3S_2 \equiv 0 \pmod{p}$.

Solution 2. One may solve the problem without finding an explicit formula for S_a. It is enough to find the following property.

Lemma. For every integer a, we have $S_{a+1} \equiv S_{-a} \pmod{p}$.

Proof. We expand S_{a+1} using the binomial formula as

$$S_{a+1} = \sum_{k=1}^{p-1} \frac{1}{k} \sum_{j=0}^{k} \binom{k}{j} a^j = \sum_{k=1}^{p-1} \left(\frac{1}{k} + \sum_{j=1}^{k} a^j \cdot \frac{1}{k}\binom{k}{j} \right) = \sum_{k=1}^{p-1} \frac{1}{k} + \sum_{j=1}^{p-1} a^j \sum_{k=j}^{p-1} \frac{1}{k}\binom{k}{j} a^k.$$

Note that $\frac{1}{k} + \frac{1}{p-k} = \frac{p}{k(p-k)} \equiv 0 \pmod{p}$ for all $1 \le k \le p-1$; hence the first sum vanishes modulo p. For the second sum, we use the relation $\frac{1}{k}\binom{k}{j} = \frac{1}{j}\binom{k-1}{j-1}$ to obtain

$$S_{a+1} \equiv \sum_{j=1}^{p-1} \frac{a^j}{j} \sum_{k=1}^{p-1} \binom{k-1}{j-1} \pmod{p}.$$

Finally, from the relation

$$\sum_{k=1}^{p-1} \binom{k-1}{j-1} = \binom{p-1}{j} = \frac{(p-1)(p-2)\dots(p-j)}{j!} \equiv (-1)^j \pmod{p}$$

we obtain

$$S_{a+1} \equiv \sum_{j=1}^{p-1} \frac{a^j(-1)^j}{j!} = S_{-a}. \qquad \square$$

Now we turn to the problem. Using the lemma we get

$$S_3 - 3S_2 \equiv S_{-2} - 3S_2 = \sum_{\substack{1 \le k \le p-1 \\ k \text{ is even}}} \frac{-2 \cdot 2^k}{k} + \sum_{\substack{1 \le k \le p-1 \\ k \text{ is odd}}} \frac{-4 \cdot 2^k}{k} \pmod{p}. \tag{1}$$

The first sum in (1) expands as

$$\sum_{\ell=1}^{(p-1)/2} \frac{-2 \cdot 2^{2\ell}}{2\ell} = -\sum_{\ell=1}^{(p-1)/2} \frac{4^\ell}{\ell}.$$

Next, using FERMAT's theorem, we expand the second sum in (1) as

$$-\sum_{\ell=1}^{(p-1)/2} \frac{2^{2\ell+1}}{2\ell-1} \equiv -\sum_{\ell=1}^{(p-1)/2} \frac{2^{p+2\ell}}{p+2\ell-1} = -\sum_{m=(p+1)/2}^{p-1} \frac{2 \cdot 4^m}{2m} = -\sum_{m=(p+1)/2}^{p-1} \frac{4^m}{m} \pmod{p}$$

(here we set $m = \ell + \frac{p-1}{2}$). Hence,

$$S_3 - 3S_2 \equiv -\sum_{\ell=1}^{(p-1)/2} \frac{4^\ell}{\ell} - \sum_{m=(p+1)/2}^{p-1} \frac{4^m}{m} = -S_4 \pmod{p}.$$

N8

Let k be a positive integer and set $n=2^k+1$. Prove that n is a prime number if and only if the following holds: there is a permutation $a_1,\ldots,a_{n-1}$ of the numbers $1,2,\ldots,n-1$ and a sequence of integers $g_1,g_2,\ldots,g_{n-1}$ such that n divides $g_i^{a_i}-a_{i+1}$ for every $i\in\{1,2,\ldots,n-1\}$, where we set $a_n=a_1$.

Solution. Let $N=\{1,2,\ldots,n-1\}$. For $a,b\in N$, we say that b *follows* a if there exists an integer g such that $b\equiv g^a \pmod n$ and denote this property as $a\to b$. This way we have a directed graph with N as set of vertices. If $a_1,\ldots,a_{n-1}$ is a permutation of $1,2,\ldots,n-1$ such that $a_1\to a_2\to\ldots\to a_{n-1}\to a_1$ then this is a *Hamiltonian cycle* in the graph.

Step I. First consider the case when n is composite. Let $n=p_1^{\alpha_1}\ldots p_s^{\alpha_s}$ be its prime factorization. All primes p_i are odd.

Suppose that $\alpha_i>1$ for some i. For all integers a,g with $a\ge 2$, we have $g^a\not\equiv p_i \pmod{p_i^2}$, because g^a is either divisible by p_i^2 or it is not divisible by p_i. It follows that in any Hamiltonian cycle p_i comes immediately after 1. The same argument shows that $2p_i$ also should come immediately after 1, which is impossible. Hence, there is no Hamiltonian cycle in the graph.

Now suppose that n is square-free. We have $n=p_1p_2\ldots p_s>9$ and $s\ge 2$. Assume that there exists a Hamiltonian cycle. There are $\frac{n-1}{2}$ even numbers in this cycle, and each number which follows one of them should be a quadratic residue modulo n. So, there should be at least $\frac{n-1}{2}$ nonzero quadratic residues modulo n. On the other hand, for each p_i there exist exactly $\frac{p_i+1}{2}$ quadratic residues modulo p_i; by the Chinese Remainder Theorem, the number of quadratic residues modulo n is exactly $\frac{p_1+1}{2}\cdot\frac{p_2+1}{2}\cdot\ldots\cdot\frac{p_s+1}{2}$, including 0. Then we have a contradiction by

$$\frac{p_1+1}{2}\cdot\frac{p_2+1}{2}\cdot\ldots\cdot\frac{p_s+1}{2}\le\frac{2p_1}{3}\cdot\frac{2p_2}{3}\cdot\ldots\cdot\frac{2p_s}{3}=\left(\frac{2}{3}\right)^s n\le\frac{4n}{9}<\frac{n-1}{2}.$$

This proves the "if"-part of the problem.

Step II. Now suppose that n is prime. For any $a\in N$, denote by $\nu_2(a)$ the exponent of 2 in the prime factorization of a, and let $\mu(a)=\max\{t\in[0,k]\mid 2^t\to a\}$.

Lemma. For any $a,b\in N$, we have $a\to b$ if and only if $\nu_2(a)\le\mu(b)$.

Proof. Let $\ell=\nu_2(a)$ and $m=\mu(b)$.

Suppose $\ell\le m$. Since b follows 2^m, there exists some g_0 such that $b\equiv g_0^{2^m} \pmod n$. By $\gcd(a,n-1)=2^\ell$ there exist some integers p and q such that $pa-q(n-1)=2^\ell$. Choosing $g=g_0^{2^{m-\ell}p}$ we have $g^a=g_0^{2^{m-\ell}pa}=g_0^{2^m+2^{m-\ell}q(n-1)}\equiv g_0^{2^m}\equiv b \pmod n$ by FERMAT's theorem. Hence, $a\to b$.

To prove the reverse statement, suppose that $a\to b$, so $b\equiv g^a \pmod n$ with some g. Then $b\equiv(g^{a/2^\ell})^{2^\ell}$, and therefore $2^\ell\to b$. By the definition of $\mu(b)$, we have $\mu(b)\ge\ell$. The lemma is proved. □

Now for every i with $0 \le i \le k$, let

$$\begin{aligned} A_i &= \{a \in N \mid \nu_2(a) = i\}, \\ B_i &= \{a \in N \mid \mu(a) = i\}, \\ \text{and } C_i &= \{a \in N \mid \mu(a) \ge i\} = B_i \cup B_{i+1} \cup \ldots \cup B_k. \end{aligned}$$

We claim that $|A_i| = |B_i|$ for all $0 \le i \le k$. Obviously we have $|A_i| = 2^{k-i-1}$ for all $i = 0, \ldots, k-1$, and $|A_k| = 1$. Now we determine $|C_i|$. We have $|C_0| = n-1$ and by Fermat's theorem we also have $C_k = \{1\}$, so $|C_k| = 1$. Next, notice that $C_{i+1} = \{x^2 \mod n \mid x \in C_i\}$. For every $a \in N$, the relation $x^2 \equiv a \pmod n$ has at most two solutions in N. Therefore we have $2|C_{i+1}| \le |C_i|$, with the equality achieved only if for every $y \in C_{i+1}$, there exist distinct elements $x, x' \in C_i$ such that $x^2 \equiv x'^2 \equiv y \pmod n$ (this implies $x + x' = n$). Now, since $2^k|C_k| = |C_0|$, we obtain that this equality should be achieved in each step. Hence $|C_i| = 2^{k-i}$ for $0 \le i \le k$, and therefore $|B_i| = 2^{k-i-1}$ for $0 \le i \le k-1$ and $|B_k| = 1$.

From the previous arguments we can see that for each $z \in C_i$ $(0 \le i < k)$ the equation $x^2 \equiv z^2 \pmod n$ has two solutions in C_i, so we have $n - z \in C_i$. Hence, for each $i = 0, 1, \ldots, k-1$, exactly half of the elements of C_i are odd. The same statement is valid for $B_i = C_i \setminus C_{i+1}$ for $0 \le i \le k-2$. In particular, each such B_i contains an odd number. Note that $B_k = \{1\}$ also contains an odd number, and $B_{k-1} = \{2^k\}$ since C_{k-1} consists of the two square roots of 1 modulo n.

Step III. Now we construct a Hamiltonian cycle in the graph. First, for each i with $0 \le i \le k$, connect the elements of A_i to the elements of B_i by means of an arbitrary bijection. After performing this for every i, we obtain a subgraph with all vertices having in-degree 1 and out-degree 1, so the subgraph is a disjoint union of cycles. If there is a unique cycle, we are done. Otherwise, we modify the subgraph in such a way that the previous property is preserved and the number of cycles decreases; after a finite number of steps we arrive at a single cycle.

For every cycle C, let $\lambda(C) = \min_{c \in C} \nu_2(c)$. Consider a cycle C for which $\lambda(C)$ is maximal. If $\lambda(C) = 0$, then for any other cycle C' we have $\lambda(C') = 0$. Take two arbitrary vertices $a \in C$ and $a' \in C'$ such that $\nu_2(a) = \nu_2(a') = 0$; let their direct successors be b and b', respectively. Then we can unify C and C' to a single cycle by replacing the edges $a \to b$ and $a' \to b'$ by $a \to b'$ and $a' \to b$.

Now suppose that $\lambda = \lambda(C) \ge 1$; let $a \in C \cap A_\lambda$. If there exists some $a' \in A_\lambda \setminus C$, then a' lies in another cycle C' and we can merge the two cycles in exactly the same way as above. So, the only remaining case is $A_\lambda \subset C$. Since the edges from A_λ lead to B_λ, we get also $B_\lambda \subset C$. If $\lambda \ne k-1$ then B_λ contains an odd number; this contradicts the assumption $\lambda(C) > 0$. Finally, if $\lambda = k-1$, then C contains 2^{k-1} which is the only element of A_{k-1}. Since $B_{k-1} = \{2^k\} = A_k$ and $B_k = \{1\}$, the cycle C contains the path $2^{k-1} \to 2^k \to 1$ and it contains an odd number again. This completes the proof of the "only if"-part of the problem.

Comment 1. The lemma and the fact $|A_i| = |B_i|$ together show that for every edge $a \to b$ of the Hamiltonian cycle, $\nu_2(a) = \mu(b)$ must hold. After this observation, the Hamiltonian cycle can be built in many ways. For instance, it is possible to select edges from A_i to B_i for $i = k, k-1, \dots, 1$ in such a way that they form disjoint paths; at the end all these paths will have odd endpoints. In the final step, the paths can be closed to form a unique cycle.

Comment 2. Step II is an easy consequence of some basic facts about the multiplicative group modulo the prime $n = 2^k + 1$. The Lemma follows by noting that this group has order 2^k, so the a-th powers are exactly the $2^{\nu_2(a)}$-th powers. Using the existence of a primitive root g modulo n one sees that the map from $\{1, 2, \dots, n-1\}$ to itself that sends a to $g^a \bmod n$ is a bijection that sends A_i to B_i for each $i \in \{0, \dots, k\}$.

Shortlisted Problems with Solutions

53rd International Mathematical Olympiad

Mar del Plata, Argentina 2012

Note of Confidentiality

The shortlisted problems should be kept strictly confidential until IMO 2013

Contributing Countries

The Organizing Committee and the Problem Selection Committee of IMO 2012 thank the following 40 countries for contributing 136 problem proposals:

Australia, Austria, Belarus, Belgium, Bulgaria, Canada, Cyprus, Czech Republic, Denmark, Estonia, Finland, France, Germany, Greece, Hong Kong, India, Iran, Ireland, Israel, Japan, Kazakhstan, Luxembourg, Malaysia, Montenegro, Netherlands, Norway, Pakistan, Romania, Russia, Serbia, Slovakia, Slovenia, South Africa, South Korea, Sweden, Thailand, Ukraine, United Kingdom, United States of America, Uzbekistan

Problem Selection Committee

Martín Avendaño
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Géza Kós
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Algebra

A1. Find all the functions $f : \mathbb{Z} \to \mathbb{Z}$ such that

$$f(a)^2 + f(b)^2 + f(c)^2 = 2f(a)f(b) + 2f(b)f(c) + 2f(c)f(a)$$

for all integers a, b, c satisfying $a + b + c = 0$.

A2. Let $\mathbb{Z}$ and $\mathbb{Q}$ be the sets of integers and rationals respectively.

a) Does there exist a partition of $\mathbb{Z}$ into three non-empty subsets A, B, C such that the sets $A + B$, $B + C$, $C + A$ are disjoint?

b) Does there exist a partition of $\mathbb{Q}$ into three non-empty subsets A, B, C such that the sets $A + B$, $B + C$, $C + A$ are disjoint?

Here $X + Y$ denotes the set $\{x + y \mid x \in X, y \in Y\}$, for $X, Y \subseteq \mathbb{Z}$ and $X, Y \subseteq \mathbb{Q}$.

A3. Let $a_2, \ldots, a_n$ be $n - 1$ positive real numbers, where $n \geq 3$, such that $a_2 a_3 \cdots a_n = 1$. Prove that

$$(1 + a_2)^2 (1 + a_3)^3 \cdots (1 + a_n)^n > n^n.$$

A4. Let f and g be two nonzero polynomials with integer coefficients and $\deg f > \deg g$. Suppose that for infinitely many primes p the polynomial $pf + g$ has a rational root. Prove that f has a rational root.

A5. Find all functions $f \colon \mathbb{R} \to \mathbb{R}$ that satisfy the conditions

$$f(1 + xy) - f(x + y) = f(x)f(y) \quad \text{for all } x, y \in \mathbb{R}$$

and $f(-1) \neq 0$.

A6. Let $f : \mathbb{N} \to \mathbb{N}$ be a function, and let f^m be f applied m times. Suppose that for every $n \in \mathbb{N}$ there exists a $k \in \mathbb{N}$ such that $f^{2k}(n) = n + k$, and let k_n be the smallest such k. Prove that the sequence $k_1, k_2, \ldots$ is unbounded.

A7. We say that a function $f : \mathbb{R}^k \to \mathbb{R}$ is a metapolynomial if, for some positive integers m and n, it can be represented in the form

$$f(x_1, \ldots, x_k) = \max_{i=1,\ldots,m} \min_{j=1,\ldots,n} P_{i,j}(x_1, \ldots, x_k)$$

where $P_{i,j}$ are multivariate polynomials. Prove that the product of two metapolynomials is also a metapolynomial.

Combinatorics

C1. Several positive integers are written in a row. Iteratively, Alice chooses two adjacent numbers x and y such that $x > y$ and x is to the left of y, and replaces the pair (x, y) by either $(y + 1, x)$ or $(x - 1, x)$. Prove that she can perform only finitely many such iterations.

C2. Let $n \geq 1$ be an integer. What is the maximum number of disjoint pairs of elements of the set $\{1, 2, \ldots, n\}$ such that the sums of the different pairs are different integers not exceeding n?

C3. In a 999×999 square table some cells are white and the remaining ones are red. Let T be the number of triples (C_1, C_2, C_3) of cells, the first two in the same row and the last two in the same column, with C_1 and C_3 white and C_2 red. Find the maximum value T can attain.

C4. Players A and B play a game with $N \geq 2012$ coins and 2012 boxes arranged around a circle. Initially A distributes the coins among the boxes so that there is at least 1 coin in each box. Then the two of them make moves in the order $B, A, B, A, \ldots$ by the following rules:

- On every move of his B passes 1 coin from every box to an adjacent box.
- On every move of hers A chooses several coins that were *not* involved in B's previous move and are in different boxes. She passes every chosen coin to an adjacent box.

Player A's goal is to ensure at least 1 coin in each box after every move of hers, regardless of how B plays and how many moves are made. Find the least N that enables her to succeed.

C5. The columns and the rows of a $3n \times 3n$ square board are numbered $1, 2, \ldots, 3n$. Every square (x, y) with $1 \leq x, y \leq 3n$ is colored asparagus, byzantium or citrine according as the modulo 3 remainder of $x + y$ is 0, 1 or 2 respectively. One token colored asparagus, byzantium or citrine is placed on each square, so that there are $3n^2$ tokens of each color.

Suppose that one can permute the tokens so that each token is moved to a distance of at most d from its original position, each asparagus token replaces a byzantium token, each byzantium token replaces a citrine token, and each citrine token replaces an asparagus token. Prove that it is possible to permute the tokens so that each token is moved to a distance of at most $d + 2$ from its original position, and each square contains a token with the same color as the square.

C6. Let k and n be fixed positive integers. In the liar's guessing game, Amy chooses integers x and N with $1 \leq x \leq N$. She tells Ben what N is, but not what x is. Ben may then repeatedly ask Amy whether $x \in S$ for arbitrary sets S of integers. Amy will always answer with *yes* or *no*, but she might lie. The only restriction is that she can lie at most k times in a row. After he has asked as many questions as he wants, Ben must specify a set of at most n positive integers. If x is in this set he wins; otherwise, he loses. Prove that:

a) If $n \geq 2^k$ then Ben can always win.

b) For sufficiently large k there exist $n \geq 1.99^k$ such that Ben cannot guarantee a win.

C7. There are given 2^{500} points on a circle labeled $1, 2, \ldots, 2^{500}$ in some order. Prove that one can choose 100 pairwise disjoint chords joining some of these points so that the 100 sums of the pairs of numbers at the endpoints of the chosen chords are equal.

Geometry

G1. In the triangle ABC the point J is the center of the excircle opposite to A. This excircle is tangent to the side BC at M, and to the lines AB and AC at K and L respectively. The lines LM and BJ meet at F, and the lines KM and CJ meet at G. Let S be the point of intersection of the lines AF and BC, and let T be the point of intersection of the lines AG and BC. Prove that M is the midpoint of ST.

G2. Let $ABCD$ be a cyclic quadrilateral whose diagonals AC and BD meet at E. The extensions of the sides AD and BC beyond A and B meet at F. Let G be the point such that $ECGD$ is a parallelogram, and let H be the image of E under reflection in AD. Prove that D, H, F, G are concyclic.

G3. In an acute triangle ABC the points D, E and F are the feet of the altitudes through A, B and C respectively. The incenters of the triangles AEF and BDF are I_1 and I_2 respectively; the circumcenters of the triangles ACI_1 and BCI_2 are O_1 and O_2 respectively. Prove that I_1I_2 and O_1O_2 are parallel.

G4. Let ABC be a triangle with $AB \neq AC$ and circumcenter O. The bisector of $\angle BAC$ intersects BC at D. Let E be the reflection of D with respect to the midpoint of BC. The lines through D and E perpendicular to BC intersect the lines AO and AD at X and Y respectively. Prove that the quadrilateral $BXCY$ is cyclic.

G5. Let ABC be a triangle with $\angle BCA = 90°$, and let C_0 be the foot of the altitude from C. Choose a point X in the interior of the segment CC_0, and let K, L be the points on the segments AX, BX for which $BK = BC$ and $AL = AC$ respectively. Denote by M the intersection of AL and BK. Show that $MK = ML$.

G6. Let ABC be a triangle with circumcenter O and incenter I. The points D, E and F on the sides BC, CA and AB respectively are such that $BD + BF = CA$ and $CD + CE = AB$. The circumcircles of the triangles BFD and CDE intersect at $P \neq D$. Prove that $OP = OI$.

G7. Let $ABCD$ be a convex quadrilateral with non-parallel sides BC and AD. Assume that there is a point E on the side BC such that the quadrilaterals $ABED$ and $AECD$ are circumscribed. Prove that there is a point F on the side AD such that the quadrilaterals $ABCF$ and $BCDF$ are circumscribed if and only if AB is parallel to CD.

G8. Let ABC be a triangle with circumcircle ω and ℓ a line without common points with ω. Denote by P the foot of the perpendicular from the center of ω to ℓ. The side-lines BC, CA, AB intersect ℓ at the points X, Y, Z different from P. Prove that the circumcircles of the triangles AXP, BYP and CZP have a common point different from P or are mutually tangent at P.

Number Theory

N1. Call admissible a set A of integers that has the following property:

If $x, y \in A$ (possibly $x = y$) then $x^2 + kxy + y^2 \in A$ for every integer k.

Determine all pairs m, n of nonzero integers such that the only admissible set containing both m and n is the set of all integers.

N2. Find all triples (x, y, z) of positive integers such that $x \le y \le z$ and

$$x^3(y^3 + z^3) = 2012(xyz + 2).$$

N3. Determine all integers $m \ge 2$ such that every n with $\frac{m}{3} \le n \le \frac{m}{2}$ divides the binomial coefficient $\binom{n}{m-2n}$.

N4. An integer a is called friendly if the equation $(m^2 + n)(n^2 + m) = a(m - n)^3$ has a solution over the positive integers.

a) Prove that there are at least 500 friendly integers in the set $\{1, 2, \ldots, 2012\}$.

b) Decide whether $a = 2$ is friendly.

N5. For a nonnegative integer n define $rad(n) = 1$ if $n = 0$ or $n = 1$, and $rad(n) = p_1 p_2 \cdots p_k$ where $p_1 < p_2 < \cdots < p_k$ are all prime factors of n. Find all polynomials $f(x)$ with nonnegative integer coefficients such that $rad(f(n))$ divides $rad(f(n^{rad(n)}))$ for every nonnegative integer n.

N6. Let x and y be positive integers. If $x^{2^n} - 1$ is divisible by $2^n y + 1$ for every positive integer n, prove that $x = 1$.

N7. Find all $n \in \mathbb{N}$ for which there exist nonnegative integers $a_1, a_2, \ldots, a_n$ such that

$$\frac{1}{2^{a_1}} + \frac{1}{2^{a_2}} + \cdots + \frac{1}{2^{a_n}} = \frac{1}{3^{a_1}} + \frac{2}{3^{a_2}} + \cdots + \frac{n}{3^{a_n}} = 1.$$

N8. Prove that for every prime $p > 100$ and every integer r there exist two integers a and b such that p divides $a^2 + b^5 - r$.

Algebra

A1. Find all the functions $f : \mathbb{Z} \to \mathbb{Z}$ such that

$$f(a)^2 + f(b)^2 + f(c)^2 = 2f(a)f(b) + 2f(b)f(c) + 2f(c)f(a)$$

for all integers a, b, c satisfying $a + b + c = 0$.

Solution. The substitution $a = b = c = 0$ gives $3f(0)^2 = 6f(0)^2$, hence

$$f(0) = 0. \tag{1}$$

The substitution $b = -a$ and $c = 0$ gives $((f(a) - f(-a))^2 = 0$. Hence f is an even function:

$$f(a) = f(-a) \quad \text{for all } a \in \mathbb{Z}. \tag{2}$$

Now set $b = a$ and $c = -2a$ to obtain $2f(a)^2 + f(2a)^2 = 2f(a)^2 + 4f(a)f(2a)$. Hence

$$f(2a) = 0 \ \text{ or } \ f(2a) = 4f(a) \quad \text{for all } a \in \mathbb{Z}. \tag{3}$$

If $f(r) = 0$ for some $r \geq 1$ then the substitution $b = r$ and $c = -a-r$ gives $(f(a+r)-f(a))^2 = 0$. So f is periodic with period r, i. e.

$$f(a + r) = f(a) \quad \text{for all } a \in \mathbb{Z}.$$

In particular, if $f(1) = 0$ then f is constant, thus $f(a) = 0$ for all $a \in \mathbb{Z}$. This function clearly satisfies the functional equation. For the rest of the analysis, we assume $f(1) = k \neq 0$.

By (3) we have $f(2) = 0$ or $f(2) = 4k$. If $f(2) = 0$ then f is periodic of period 2, thus $f(\text{even}) = 0$ and $f(\text{odd}) = k$. This function is a solution for every k. We postpone the verification; for the sequel assume $f(2) = 4k \neq 0$.

By (3) again, we have $f(4) = 0$ or $f(4) = 16k$. In the first case f is periodic of period 4, and $f(3) = f(-1) = f(1) = k$, so we have $f(4n) = 0$, $f(4n+1) = f(4n+3) = k$, and $f(4n+2) = 4k$ for all $n \in \mathbb{Z}$. This function is a solution too, which we justify later. For the rest of the analysis, we assume $f(4) = 16k \neq 0$.

We show now that $f(3) = 9k$. In order to do so, we need two substitutions:

$$a = 1, b = 2, c = -3 \Longrightarrow f(3)^2 - 10kf(3) + 9k^2 = 0 \Longrightarrow f(3) \in \{k, 9k\},$$
$$a = 1, b = 3, c = -4 \Longrightarrow f(3)^2 - 34kf(3) + 225k^2 = 0 \Longrightarrow f(3) \in \{9k, 25k\}.$$

Therefore $f(3) = 9k$, as claimed. Now we prove inductively that the only remaining function is $f(x) = kx^2$, $x \in \mathbb{Z}$. We proved this for $x = 0, 1, 2, 3, 4$. Assume that $n \geq 4$ and that $f(x) = kx^2$ holds for all integers $x \in [0, n]$. Then the substitutions $a = n$, $b = 1$, $c = -n-1$ and $a = n-1$, $b = 2$, $c = -n-1$ lead respectively to

$$f(n + 1) \in \{k(n + 1)^2, k(n - 1)^2\} \qquad \text{and} \qquad f(n + 1) \in \{k(n + 1)^2, k(n - 3)^2\}.$$

Since $k(n - 1)^2 \neq k(n - 3)^2$ for $n \neq 2$, the only possibility is $f(n + 1) = k(n + 1)^2$. This completes the induction, so $f(x) = kx^2$ for all $x \geq 0$. The same expression is valid for negative values of x since f is even. To verify that $f(x) = kx^2$ is actually a solution, we need to check the identity $a^4 + b^4 + (a + b)^4 = 2a^2b^2 + 2a^2(a + b)^2 + 2b^2(a + b)^2$, which follows directly by expanding both sides.

Therefore the only possible solutions of the functional equation are the constant function $f_1(x) = 0$ and the following functions:

$$f_2(x) = kx^2 \qquad f_3(x) = \begin{cases} 0 & x \text{ even} \\ k & x \text{ odd} \end{cases} \qquad f_4(x) = \begin{cases} 0 & x \equiv 0 \pmod 4 \\ k & x \equiv 1 \pmod 2 \\ 4k & x \equiv 2 \pmod 4 \end{cases}$$

for any non-zero integer k. The verification that they are indeed solutions was done for the first two. For f_3 note that if $a + b + c = 0$ then either a, b, c are all even, in which case $f(a) = f(b) = f(c) = 0$, or one of them is even and the other two are odd, so both sides of the equation equal $2k^2$. For f_4 we use similar parity considerations and the symmetry of the equation, which reduces the verification to the triples $(0, k, k)$, $(4k, k, k)$, $(0, 0, 0)$, $(0, 4k, 4k)$. They all satisfy the equation.

Comment. We used several times the same fact: For any $a, b \in \mathbb{Z}$ the functional equation is a quadratic equation in $f(a+b)$ whose coefficients depend on $f(a)$ and $f(b)$:

$$f(a+b)^2 - 2(f(a) + f(b))f(a+b) + (f(a) - f(b))^2 = 0.$$

Its discriminant is $16f(a)f(b)$. Since this value has to be non-negative for any $a, b \in \mathbb{Z}$, we conclude that either f or $-f$ is always non-negative. Also, if f is a solution of the functional equation, then $-f$ is also a solution. Therefore we can assume $f(x) \geq 0$ for all $x \in \mathbb{Z}$. Now, the two solutions of the quadratic equation are

$$f(a+b) \in \left\{ \left(\sqrt{f(a)} + \sqrt{f(b)}\right)^2, \left(\sqrt{f(a)} - \sqrt{f(b)}\right)^2 \right\} \quad \text{for all } a, b \in \mathbb{Z}.$$

The computation of $f(3)$ from $f(1)$, $f(2)$ and $f(4)$ that we did above follows immediately by setting $(a, b) = (1, 2)$ and $(a, b) = (1, -4)$. The inductive step, where $f(n+1)$ is derived from $f(n)$, $f(n-1)$, $f(2)$ and $f(1)$, follows immediately using $(a, b) = (n, 1)$ and $(a, b) = (n-1, 2)$.

A2. Let $\mathbb{Z}$ and $\mathbb{Q}$ be the sets of integers and rationals respectively.

a) Does there exist a partition of $\mathbb{Z}$ into three non-empty subsets A, B, C such that the sets $A+B$, $B+C$, $C+A$ are disjoint?

b) Does there exist a partition of $\mathbb{Q}$ into three non-empty subsets A, B, C such that the sets $A+B$, $B+C$, $C+A$ are disjoint?

Here $X+Y$ denotes the set $\{x+y \mid x \in X, y \in Y\}$, for $X, Y \subseteq \mathbb{Z}$ and $X, Y \subseteq \mathbb{Q}$.

Solution 1. a) The residue classes modulo 3 yield such a partition:

$$A = \{3k \mid k \in \mathbb{Z}\}, \quad B = \{3k+1 \mid k \in \mathbb{Z}\}, \quad C = \{3k+2 \mid k \in \mathbb{Z}\}.$$

b) The answer is *no*. Suppose that $\mathbb{Q}$ can be partitioned into non-empty subsets A, B, C as stated. Note that for all $a \in A$, $b \in B$, $c \in C$ one has

$$a+b-c \in C, \quad b+c-a \in A, \quad c+a-b \in B. \tag{1}$$

Indeed $a+b-c \notin A$ as $(A+B) \cap (A+C) = \emptyset$, and similarly $a+b-c \notin B$, hence $a+b-c \in C$. The other two relations follow by symmetry. Hence $A+B \subset C+C$, $B+C \subset A+A$, $C+A \subset B+B$.

The opposite inclusions also hold. Let $a, a' \in A$ and $b \in B$, $c \in C$ be arbitrary. By (1) $a'+c-b \in B$, and since $a \in A$, $c \in C$, we use (1) again to obtain

$$a+a'-b = a+(a'+c-b)-c \in C.$$

So $A+A \subset B+C$ and likewise $B+B \subset C+A$, $C+C \subset A+B$. In summary

$$B+C = A+A, \qquad C+A = B+B, \qquad A+B = C+C.$$

Furthermore suppose that $0 \in A$ without loss of generality. Then $B = \{0\}+B \subset A+B$ and $C = \{0\}+C \subset A+C$. So, since $B+C$ is disjoint with $A+B$ and $A+C$, it is also disjoint with B and C. Hence $B+C$ is contained in $\mathbb{Z} \setminus (B \cup C) = A$. Because $B+C = A+A$, we obtain $A+A \subset A$. On the other hand $A = \{0\}+A \subset A+A$, implying $A = A+A = B+C$.

Therefore $A+B+C = A+A+A = A$, and now $B+B = C+A$ and $C+C = A+B$ yield $B+B+B = A+B+C = A$, $C+C+C = A+B+C = A$. In particular if $r \in \mathbb{Q} = A \cup B \cup C$ is arbitrary then $3r \in A$.

However such a conclusion is impossible. Take any $b \in B$ $(B \neq \emptyset)$ and let $r = b/3 \in \mathbb{Q}$. Then $b = 3r \in A$ which is a contradiction.

Solution 2. We prove that the example for $\mathbb{Z}$ from the first solution is unique, and then use this fact to solve part b).

Let $\mathbb{Z} = A \cup B \cup C$ be a partition of $\mathbb{Z}$ with $A, B, C \neq \emptyset$ and $A+B$, $B+C$, $C+A$ disjoint. We need the relations (1) which clearly hold for $\mathbb{Z}$. Fix two consecutive integers from different sets, say $b \in B$ and $c = b+1 \in C$. For every $a \in A$ we have, in view of (1), $a-1 = a+b-c \in C$ and $a+1 = a+c-b \in B$. So every $a \in A$ is preceded by a number from C and followed by a number from B.

In particular there are pairs of the form $c, c+1$ with $c \in C$, $c+1 \in A$. For such a pair and any $b \in B$ analogous reasoning shows that each $b \in B$ is preceded by a number from A and followed by a number from C. There are also pairs $b, b-1$ with $b \in B$, $b-1 \in A$. We use them in a similar way to prove that each $c \in C$ is preceded by a number from B and followed by a number from A.

By putting the observations together we infer that A, B, C are the three congruence classes modulo 3. Observe that all multiples of 3 are in the set of the partition that contains 0.

Now we turn to part b). Suppose that there is a partition of $\mathbb{Q}$ with the given properties. Choose three rationals $r_i = p_i/q_i$ from the three sets A, B, C, $i = 1, 2, 3$, and set $N = 3q_1q_2q_3$.

Let $S \subset \mathbb{Q}$ be the set of fractions with denominators N (irreducible or not). It is obtained through multiplication of every integer by the constant $1/N$, hence closed under sums and differences. Moreover, if we identify each $k \in \mathbb{Z}$ with $k/N \in S$ then S is essentially the set $\mathbb{Z}$ with respect to addition. The numbers r_i belong to S because

$$r_1 = \frac{3p_1q_2q_3}{N}, \qquad r_2 = \frac{3p_2q_3q_1}{N}, \qquad r_3 = \frac{3p_3q_1q_2}{N}.$$

The partition $\mathbb{Q} = A \cup B \cup C$ of $\mathbb{Q}$ induces a partition $S = A' \cup B' \cup C'$ of S, with $A' = A \cap S$, $B' = B \cap S$, $C' = C \cap S$. Clearly $A' + B'$, $B' + C'$, $C' + A'$ are disjoint, so this partition has the properties we consider.

By the uniqueness of the example for $\mathbb{Z}$ the sets A', B', C' are the congruence classes modulo 3, multiplied by $1/N$. Also all multiples of $3/N$ are in the same set, A', B' or C'. This holds for r_1, r_2, r_3 in particular as they are all multiples of $3/N$. However r_1, r_2, r_3 are in different sets A', B', C' since they were chosen from different sets A, B, C. The contradiction ends the proof.

Comment. The uniqueness of the example for $\mathbb{Z}$ can also be deduced from the argument in the first solution.

A3. Let $a_2, \ldots, a_n$ be $n-1$ positive real numbers, where $n \geq 3$, such that $a_2 a_3 \cdots a_n = 1$. Prove that

$$(1+a_2)^2(1+a_3)^3 \cdots (1+a_n)^n > n^n.$$

Solution. The substitution $a_2 = \frac{x_2}{x_1}$, $a_3 = \frac{x_3}{x_2}$, $\ldots$, $a_n = \frac{x_1}{x_{n-1}}$ transforms the original problem into the inequality

$$(x_1+x_2)^2(x_2+x_3)^3 \cdots (x_{n-1}+x_1)^n > n^n x_1^2 x_2^3 \cdots x_{n-1}^n \tag{$*$}$$

for all $x_1, \ldots, x_{n-1} > 0$. To prove this, we use the AM-GM inequality for each factor of the left-hand side as follows:

$$\begin{array}{rcccl}
(x_1+x_2)^2 & & & \geq & 2^2 x_1 x_2 \\
(x_2+x_3)^3 & = & \left(2\left(\frac{x_2}{2}\right)+x_3\right)^3 & \geq & 3^3\left(\frac{x_2}{2}\right)^2 x_3 \\
(x_3+x_4)^4 & = & \left(3\left(\frac{x_3}{3}\right)+x_4\right)^4 & \geq & 4^4\left(\frac{x_3}{3}\right)^3 x_4 \\
& \vdots & \vdots & \vdots & \\
(x_{n-1}+x_1)^n & = & \left((n-1)\left(\frac{x_{n-1}}{n-1}\right)+x_1\right)^n & \geq & n^n\left(\frac{x_{n-1}}{n-1}\right)^{n-1} x_1.
\end{array}$$

Multiplying these inequalities together gives (*), with inequality sign $\geq$ instead of $>$. However for the equality to occur it is necessary that $x_1 = x_2$, $x_2 = 2x_3$, $\ldots$, $x_{n-1} = (n-1)x_1$, implying $x_1 = (n-1)!x_1$. This is impossible since $x_1 > 0$ and $n \geq 3$. Therefore the inequality is strict.

Comment. One can avoid the substitution $a_i = x_i/x_{i-1}$. Apply the weighted AM-GM inequality to each factor $(1+a_k)^k$, with the same weights like above, to obtain

$$(1+a_k)^k = \left((k-1)\frac{1}{k-1}+a_k\right)^k \geq \frac{k^k}{(k-1)^{k-1}} a_k.$$

Multiplying all these inequalities together gives

$$(1+a_2)^2(1+a_3)^3 \cdots (1+a_n)^n \geq n^n a_2 a_3 \cdots a_n = n^n.$$

The same argument as in the proof above shows that the equality cannot be attained.

A4. Let f and g be two nonzero polynomials with integer coefficients and $\deg f > \deg g$. Suppose that for infinitely many primes p the polynomial $pf + g$ has a rational root. Prove that f has a rational root.

Solution 1. Since $\deg f > \deg g$, we have $|g(x)/f(x)| < 1$ for sufficiently large x; more precisely, there is a real number R such that $|g(x)/f(x)| < 1$ for all x with $|x| > R$. Then for all such x and all primes p we have

$$\left|pf(x) + g(x)\right| \geq \left|f(x)\right| \left(p - \frac{|g(x)|}{|f(x)|}\right) > 0.$$

Hence all real roots of the polynomials $pf + g$ lie in the interval $[-R, R]$.

Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$ and $g(x) = b_m x^m + b_{m-1}x^{m-1} + \cdots + b_0$ where $n > m$, $a_n \neq 0$ and $b_m \neq 0$. Upon replacing $f(x)$ and $g(x)$ by $a_n^{n-1} f(x/a_n)$ and $a_n^{n-1} g(x/a_n)$ respectively, we reduce the problem to the case $a_n = 1$. In other words one can assume that f is monic. Then the leading coefficient of $pf + g$ is p, and if $r = u/v$ is a rational root of $pf + g$ with $(u, v) = 1$ and $v > 0$, then either $v = 1$ or $v = p$.

First consider the case when $v = 1$ infinitely many times. If $v = 1$ then $|u| \leq R$, so there are only finitely many possibilities for the integer u. Therefore there exist distinct primes p and q for which we have the same value of u. Then the polynomials $pf + g$ and $qf + g$ share this root, implying $f(u) = g(u) = 0$. So in this case f and g have an integer root in common.

Now suppose that $v = p$ infinitely many times. By comparing the exponent of p in the denominators of $pf(u/p)$ and $g(u/p)$ we get $m = n - 1$ and $pf(u/p) + g(u/p) = 0$ reduces to an equation of the form

$$\left(u^n + a_{n-1}pu^{n-1} + \ldots + a_0 p^n\right) + \left(b_{n-1}u^{n-1} + b_{n-2}pu^{n-2} + \ldots + b_0 p^{n-1}\right) = 0.$$

The equation above implies that $u^n + b_{n-1}u^{n-1}$ is divisible by p and hence, since $(u, p) = 1$, we have $u + b_{n-1} = pk$ with some integer k. On the other hand all roots of $pf + g$ lie in the interval $[-R, R]$, so that

$$\frac{|pk - b_{n-1}|}{p} = \frac{|u|}{p} < R,$$
$$|k| < R + \frac{|b_{n-1}|}{p} < R + |b_{n-1}|.$$

Therefore the integer k can attain only finitely many values. Hence there exists an integer k such that the number $\frac{pk - b_{n-1}}{p} = k - \frac{b_{n-1}}{p}$ is a root of $pf + g$ for infinitely many primes p. For these primes we have

$$f\left(k - b_{n-1}\frac{1}{p}\right) + \frac{1}{p} g\left(k - b_{n-1}\frac{1}{p}\right) = 0.$$

So the equation

$$f\left(k - b_{n-1}x\right) + xg\left(k - b_{n-1}x\right) = 0 \tag{1}$$

has infinitely many solutions of the form $x = 1/p$. Since the left-hand side is a polynomial, this implies that (1) is a polynomial identity, so it holds for all real x. In particular, by substituting $x = 0$ in (1) we get $f(k) = 0$. Thus the integer k is a root of f.

In summary the monic polynomial f obtained after the initial reduction always has an integer root. Therefore the original polynomial f has a rational root.

Solution 2. Analogously to the first solution, there exists a real number R such that the complex roots of all polynomials of the form $pf+g$ lie in the disk $|z| \le R$.

For each prime p such that $pf+g$ has a rational root, by GAUSS' lemma $pf+g$ is the product of two integer polynomials, one with degree 1 and the other with degree $\deg f - 1$. Since p is a prime, the leading coefficient of one of these factors divides the leading coefficient of f. Denote that factor by h_p.

By narrowing the set of the primes used we can assume that all polynomials h_p have the same degree and the same leading coefficient. Their complex roots lie in the disk $|z| \le R$, hence VIETA's formulae imply that all coefficients of all polynomials h_p form a bounded set. Since these coefficients are integers, there are only finitely many possible polynomials h_p. Hence there is a polynomial h such that $h_p = h$ for infinitely many primes p.

Finally, if p and q are distinct primes with $h_p = h_q = h$ then h divides $(p-q)f$. Since $\deg h = 1$ or $\deg h = \deg f - 1$, in both cases f has a rational root.

Comment. Clearly the polynomial h is a common factor of f and g. If $\deg h = 1$ then f and g share a rational root. Otherwise $\deg h = \deg f - 1$ forces $\deg g = \deg f - 1$ and g divides f over the rationals.

Solution 3. Like in the first solution, there is a real number R such that the real roots of all polynomials of the form $pf+g$ lie in the interval $[-R, R]$.

Let $p_1 < p_2 < \cdots$ be an infinite sequence of primes so that for every index k the polynomial $p_k f + g$ has a rational root r_k. The sequence $r_1, r_2, \ldots$ is bounded, so it has a convergent subsequence $r_{k_1}, r_{k_2}, \ldots$. Now replace the sequences $(p_1, p_2, \ldots)$ and $(r_1, r_2, \ldots)$ by $(p_{k_1}, p_{k_2}, \ldots)$ and $(r_{k_1}, r_{k_2}, \ldots)$; after this we can assume that the sequence $r_1, r_2, \ldots$ is convergent. Let $\alpha = \lim\limits_{k\to\infty} r_k$. We show that α is a rational root of f.

Over the interval $[-R, R]$, the polynomial g is bounded, $|g(x)| \le M$ with some fixed M. Therefore

$$|f(r_k)| = \left| f(r_k) - \frac{p_k f(r_k) + g(r_k)}{p_k} \right| = \frac{|g(r_k)|}{p_k} \le \frac{M}{p_k} \to 0,$$

and

$$f(\alpha) = f\Big(\lim_{k\to\infty} r_k \Big) = \lim_{k\to\infty} f(r_k) = 0.$$

So α is a root of f indeed.

Now let u_k, v_k be relative prime integers for which $r_k = \frac{u_k}{v_k}$. Let a be the leading coefficient of f, let $b = f(0)$ and $c = g(0)$ be the constant terms of f and g, respectively. The leading coefficient of the polynomial $p_k f + g$ is $p_k a$, its constant term is $p_k b + c$. So v_k divides $p_k a$ and u_k divides $p_k b + c$. Let $p_k b + c = u_k e_k$ (if $p_k b + c = u_k = 0$ then let $e_k = 1$).

We prove that α is rational by using the following fact. *Let (p_n) and (q_n) be sequences of integers such that the sequence (p_n/q_n) converges. If (p_n) or (q_n) is bounded then* $\lim(p_n/q_n)$ *is rational.*

Case 1: There is an infinite subsequence (k_n) of indices such that v_{k_n} divides a. Then (v_{k_n}) is bounded, so $\alpha = \lim_{n\to\infty}(u_{k_n}/v_{k_n})$ is rational.

Case 2: There is an infinite subsequence (k_n) of indices such that v_{k_n} does not divide a. For such indices we have $v_{k_n} = p_{k_n} d_{k_n}$ where d_{k_n} is a divisor of a. Then

$$\alpha = \lim_{n\to\infty} \frac{u_{k_n}}{v_{k_n}} = \lim_{n\to\infty} \frac{p_{k_n} b + c}{p_{k_n} d_{k_n} e_{k_n}} = \lim_{n\to\infty} \frac{b}{d_{k_n} e_{k_n}} + \lim_{n\to\infty} \frac{c}{p_{k_n} d_{k_n} e_{k_n}} = \lim_{n\to\infty} \frac{b}{d_{k_n} e_{k_n}}.$$

Because the numerator b in the last limit is bounded, α is rational.

A5. Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ that satisfy the conditions

$$f(1+xy) - f(x+y) = f(x)f(y) \quad \text{for all } x, y \in \mathbb{R}$$

and $f(-1) \neq 0$.

Solution. The only solution is the function $f(x) = x - 1$, $x \in \mathbb{R}$.

We set $g(x) = f(x) + 1$ and show that $g(x) = x$ for all real x. The conditions take the form

$$g(1+xy) - g(x+y) = \big(g(x) - 1\big)\big(g(y) - 1\big) \quad \text{for all } x, y \in \mathbb{R} \text{ and } g(-1) \neq 1. \tag{1}$$

Denote $C = g(-1) - 1 \neq 0$. Setting $y = -1$ in (1) gives

$$g(1-x) - g(x-1) = C(g(x) - 1). \tag{2}$$

Set $x = 1$ in (2) to obtain $C(g(1) - 1) = 0$. Hence $g(1) = 1$ as $C \neq 0$. Now plugging in $x = 0$ and $x = 2$ yields $g(0) = 0$ and $g(2) = 2$ respectively.

We pass on to the key observations

$$g(x) + g(2-x) = 2 \quad \text{for all } x \in \mathbb{R}, \tag{3}$$
$$g(x+2) - g(x) = 2 \quad \text{for all } x \in \mathbb{R}. \tag{4}$$

Replace x by $1 - x$ in (2), then change x to $-x$ in the resulting equation. We obtain the relations $g(x) - g(-x) = C(g(1-x) - 1)$, $g(-x) - g(x) = C(g(1+x) - 1)$. Then adding them up leads to $C(g(1-x) + g(1+x) - 2) = 0$. Thus $C \neq 0$ implies (3).

Let u, v be such that $u + v = 1$. Apply (1) to the pairs (u, v) and $(2-u, 2-v)$:

$$g(1+uv) - g(1) = \big(g(u) - 1\big)\big(g(v) - 1\big), \quad g(3+uv) - g(3) = \big(g(2-u) - 1\big)\big(g(2-v) - 1\big).$$

Observe that the last two equations have equal right-hand sides by (3). Hence $u + v = 1$ implies

$$g(uv+3) - g(uv+1) = g(3) - g(1).$$

Each $x \leq 5/4$ is expressible in the form $x = uv + 1$ with $u + v = 1$ (the quadratic function $t^2 - t + (x-1)$ has real roots for $x \leq 5/4$). Hence $g(x+2) - g(x) = g(3) - g(1)$ whenever $x \leq 5/4$. Because $g(x) = x$ holds for $x = 0, 1, 2$, setting $x = 0$ yields $g(3) = 3$. This proves (4) for $x \leq 5/4$. If $x > 5/4$ then $-x < 5/4$ and so $g(2-x) - g(-x) = 2$ by the above. On the other hand (3) gives $g(x) = 2 - g(2-x)$, $g(x+2) = 2 - g(-x)$, so that $g(x+2) - g(x) = g(2-x) - g(-x) = 2$. Thus (4) is true for all $x \in \mathbb{R}$.

Now replace x by $-x$ in (3) to obtain $g(-x) + g(2+x) = 2$. In view of (4) this leads to $g(x) + g(-x) = 0$, i. e. $g(-x) = -g(x)$ for all x. Taking this into account, we apply (1) to the pairs $(-x, y)$ and $(x, -y)$:

$$g(1-xy) - g(-x+y) = \big(g(x) + 1\big)\big(1 - g(y)\big), \quad g(1-xy) - g(x-y) = \big(1 - g(x)\big)\big(g(y) + 1\big).$$

Adding up yields $g(1-xy) = 1 - g(x)g(y)$. Then $g(1+xy) = 1 + g(x)g(y)$ by (3). Now the original equation (1) takes the form $g(x+y) = g(x) + g(y)$. Hence g is additive.

By additvity $g(1+xy) = g(1) + g(xy) = 1 + g(xy)$; since $g(1+xy) = 1 + g(x)g(y)$ was shown above, we also have $g(xy) = g(x)g(y)$ (g is multiplicative). In particular $y = x$ gives $g(x^2) = g(x)^2 \geq 0$ for all x, meaning that $g(x) \geq 0$ for $x \geq 0$. Since g is additive and bounded from below on $[0, +\infty)$, it is linear; more exactly $g(x) = g(1)x = x$ for all $x \in \mathbb{R}$.

In summary $f(x) = x - 1$, $x \in \mathbb{R}$. It is straightforward that this function satisfies the requirements.

Comment. There are functions that satisfy the given equation but vanish at -1, for instance the constant function 0 and $f(x) = x^2 - 1$, $x \in \mathbb{R}$.

A6. Let $f : \mathbb{N} \to \mathbb{N}$ be a function, and let f^m be f applied m times. Suppose that for every $n \in \mathbb{N}$ there exists a $k \in \mathbb{N}$ such that $f^{2k}(n) = n + k$, and let k_n be the smallest such k. Prove that the sequence $k_1, k_2, \ldots$ is unbounded.

Solution. We restrict attention to the set

$$S = \{1, f(1), f^2(1), \ldots\}.$$

Observe that S is unbounded because for every number n in S there exists a $k > 0$ such that $f^{2k}(n) = n + k$ is in S. Clearly f maps S into itself; moreover f is injective on S. Indeed if $f^i(1) = f^j(1)$ with $i \neq j$ then the values $f^m(1)$ start repeating periodically from some point on, and S would be finite.

Define $g : S \to S$ by $g(n) = f^{2k_n}(n) = n + k_n$. We prove that g is injective too. Suppose that $g(a) = g(b)$ with $a < b$. Then $a + k_a = f^{2k_a}(a) = f^{2k_b}(b) = b + k_b$ implies $k_a > k_b$. So, since f is injective on S, we obtain

$$f^{2(k_a - k_b)}(a) = b = a + (k_a - k_b).$$

However this contradicts the minimality of k_a as $0 < k_a - k_b < k_a$.

Let T be the set of elements of S that are not of the form $g(n)$ with $n \in S$. Note that $1 \in T$ by $g(n) > n$ for $n \in S$, so T is non-empty. For each $t \in T$ denote $C_t = \{t, g(t), g^2(t), \ldots\}$; call C_t the chain starting at t. Observe that distinct chains are disjoint because g is injective. Each $n \in S \backslash T$ has the form $n = g(n')$ with $n' < n$, $n' \in S$. Repeated applications of the same observation show that $n \in C_t$ for some $t \in T$, i. e. S is the disjoint union of the chains C_t.

If $f^n(1)$ is in the chain C_t starting at $t = f^{n_t}(1)$ then $n = n_t + 2a_1 + \cdots + 2a_j$ with

$$f^n(1) = g^j(f^{n_t}(1)) = f^{2a_j}(f^{2a_{j-1}}(\cdots f^{2a_1}(f^{n_t}(1)))) = f^{n_t}(1) + a_1 + \cdots + a_j.$$

Hence

$$f^n(1) = f^{n_t}(1) + \frac{n - n_t}{2} = t + \frac{n - n_t}{2}. \tag{1}$$

Now we show that T is infinite. We argue by contradiction. Suppose that there are only finitely many chains $C_{t_1}, \ldots, C_{t_r}$, starting at $t_1 < \cdots < t_r$. Fix N. If $f^n(1)$ with $1 \leq n \leq N$ is in C_t then $f^n(1) = t + \frac{n-n_t}{2} \leq t_r + \frac{N}{2}$ by (1). But then the $N + 1$ distinct natural numbers $1, f(1), \ldots, f^N(1)$ are all less than $t_r + \frac{N}{2}$ and hence $N + 1 \leq t_r + \frac{N}{2}$. This is a contradiction if N is sufficiently large, and hence T is infinite.

To complete the argument, choose any k in $\mathbb{N}$ and consider the $k + 1$ chains starting at the first $k + 1$ numbers in T. Let t be the greatest one among these numbers. Then each of the chains in question contains a number not exceeding t, and at least one of them does not contain any number among $t+1, \ldots, t+k$. So there is a number n in this chain such that $g(n) - n > k$, i. e. $k_n > k$. In conclusion $k_1, k_2, \ldots$ is unbounded.

A7. We say that a function $f : \mathbb{R}^k \to \mathbb{R}$ is a metapolynomial if, for some positive integers m and n, it can be represented in the form

$$f(x_1, \ldots, x_k) = \max_{i=1,\ldots,m} \min_{j=1,\ldots,n} P_{i,j}(x_1, \ldots, x_k)$$

where $P_{i,j}$ are multivariate polynomials. Prove that the product of two metapolynomials is also a metapolynomial.

Solution. We use the notation $f(x) = f(x_1, \ldots, x_k)$ for $x = (x_1, \ldots, x_k)$ and $[m] = \{1, 2, \ldots, m\}$. Observe that if a metapolynomial $f(x)$ admits a representation like the one in the statement for certain positive integers m and n, then they can be replaced by any $m' \geq m$ and $n' \geq n$. For instance, if we want to replace m by $m+1$ then it is enough to define $P_{m+1,j}(x) = P_{m,j}(x)$ and note that repeating elements of a set do not change its maximum nor its minimum. So one can assume that any two metapolynomials are defined with the same m and n. We reserve letters P and Q for polynomials, so every function called $P, P_{i,j}, Q, Q_{i,j}, \ldots$ is a polynomial function.

We start with a lemma that is useful to change expressions of the form $\min\max f_{i,j}$ to ones of the form $\max\min g_{i,j}$.

Lemma. Let $\{a_{i,j}\}$ be real numbers, for all $i \in [m]$ and $j \in [n]$. Then

$$\min_{i\in[m]} \max_{j\in[n]} a_{i,j} = \max_{j_1,\ldots,j_m\in[n]} \min_{i\in[m]} a_{i,j_i},$$

where the max in the right-hand side is over all vectors $(j_1, \ldots, j_m)$ with $j_1, \ldots, j_m \in [n]$.

Proof. We can assume for all i that $a_{i,n} = \max\{a_{i,1}, \ldots, a_{i,n}\}$ and $a_{m,n} = \min\{a_{1,n}, \ldots, a_{m,n}\}$. The left-hand side is $= a_{m,n}$ and hence we need to prove the same for the right-hand side. If $(j_1, j_2, \ldots, j_m) = (n, n, \ldots, n)$ then $\min\{a_{1,j_1}, \ldots, a_{m,j_m}\} = \min\{a_{1,n}, \ldots, a_{m,n}\} = a_{m,n}$ which implies that the right-hand side is $\geq a_{m,n}$. It remains to prove the opposite inequality and this is equivalent to $\min\{a_{1,j_1}, \ldots, a_{m,j_m}\} \leq a_{m,n}$ for all possible $(j_1, j_2, \ldots, j_m)$. This is true because $\min\{a_{1,j_1}, \ldots, a_{m,j_m}\} \leq a_{m,j_m} \leq a_{m,n}$. $\square$

We need to show that the family $\mathcal{M}$ of metapolynomials is closed under multiplication, but it turns out easier to prove more: that it is also closed under addition, maxima and minima.

First we prove the assertions about the maxima and the minima. If $f_1, \ldots, f_r$ are metapolynomials, assume them defined with the same m and n. Then

$$f = \max\{f_1, \ldots, f_r\} = \max\{\max_{i\in[m]} \min_{j\in[n]} P^1_{i,j}, \ldots, \max_{i\in[m]} \min_{j\in[n]} P^r_{i,j}\} = \max_{s\in[r],i\in[m]} \min_{j\in[n]} P^s_{i,j}.$$

It follows that $f = \max\{f_1, \ldots, f_r\}$ is a metapolynomial. The same argument works for the minima, but first we have to replace *min max* by *max min*, and this is done via the lemma.

Another property we need is that if $f = \max\min P_{i,j}$ is a metapolynomial then so is $-f$. Indeed, $-f = \min(-\min P_{i,j}) = \min\max P_{i,j}$.

To prove $\mathcal{M}$ is closed under addition let $f = \max\min P_{i,j}$ and $g = \max\min Q_{i,j}$. Then

$$f(x) + g(x) = \max_{i\in[m]} \min_{j\in[n]} P_{i,j}(x) + \max_{i\in[m]} \min_{j\in[n]} Q_{i,j}(x)$$

$$= \max_{i_1,i_2\in[m]} (\min_{j\in[n]} P_{i_1,j}(x) + \min_{j\in[n]} Q_{i_2,j}(x)) = \max_{i_1,i_2\in[m]} \min_{j_1,j_2\in[n]} \big(P_{i_1,j_1}(x) + Q_{i_2,j_2}(x)\big),$$

and hence $f(x) + g(x)$ is a metapolynomial.

We proved that $\mathcal{M}$ is closed under sums, maxima and minima, in particular any function that can be expressed by sums, max, min, polynomials or even metapolynomials is in $\mathcal{M}$.

We would like to proceed with multiplication along the same lines like with addition, but there is an essential difference. In general the product of the maxima of two sets is not equal to the maximum of the product of the sets. We need to deal with the fact that $a < b$ and $c < d$ do not imply $ac < bd$. However this is true for $a, b, c, d \geq 0$.

In view of this we decompose each function $f(x)$ into its positive part $f^+(x) = \max\{f(x), 0\}$ and its negative part $f^-(x) = \max\{0, -f(x)\}$. Note that $f = f^+ - f^-$ and $f^+, f^- \in \mathcal{M}$ if $f \in \mathcal{M}$. The whole problem reduces to the claim that if f and g are metapolynomials with $f, g \geq 0$ then fg it is also a metapolynomial.

Assuming this claim, consider arbitrary $f, g \in \mathcal{M}$. We have

$$fg = (f^+ - f^-)(g^+ - g^-) = f^+g^+ - f^+g^- - f^-g^+ + f^-g^-,$$

and hence $fg \in \mathcal{M}$. Indeed, $\mathcal{M}$ is closed under addition, also $f^+g^+, f^+g^-, f^-g^+, f^-g^- \in \mathcal{M}$ because $f^+, f^-, g^+, g^- \geq 0$.

It remains to prove the claim. In this case $f, g \geq 0$, and one can try to repeat the argument for the sum. More precisely, let $f = \max\min P_{ij} \geq 0$ and $g = \max\min Q_{ij} \geq 0$. Then

$$fg = \max\min P_{i,j} \cdot \max\min Q_{i,j} = \max\min P^+_{i,j} \cdot \max\min Q^+_{i,j} = \max\min P^+_{i_1,j_1} \cdot Q^+_{i_2,j_2}.$$

Hence it suffices to check that $P^+Q^+ \in \mathcal{M}$ for any pair of polynomials P and Q. This reduces to the identity

$$u^+v^+ = \max\{0, \min\{uv, u, v\}, \min\{uv, uv^2, u^2v\}, \min\{uv, u, u^2v\}, \min\{uv, uv^2, v\}\},$$

with u replaced by $P(x)$ and v replaced by $Q(x)$. The formula is proved by a case-by-case analysis. If $u \leq 0$ or $v \leq 0$ then both sides equal 0. In case $u, v \geq 0$, the right-hand side is clearly $\leq uv$. To prove the opposite inequality we use that uv equals

$$\begin{array}{ll}
\min\{uv, u, v\} & \text{if } 0 \leq u, v \leq 1, \\
\min\{uv, uv^2, u^2v\} & \text{if } 1 \leq u, v, \\
\min\{uv, u, u^2v\} & \text{if } 0 \leq v \leq 1 \leq u, \\
\min\{uv, uv^2, v\} & \text{if } 0 \leq u \leq 1 \leq v.
\end{array}$$

Comment. The case $k = 1$ is simpler and can be solved by proving that a function $f : \mathbb{R} \to \mathbb{R}$ is a metapolynomial if and only if it is a piecewise polinomial (and continuos) function.

It is enough to prove that all such functions are metapolynomials, and this easily reduces to the following case. Given a polynomial $P(x)$ with $P(0) = 0$, the function f defined by $f(x) = P(x)$ for $x \geq 0$ and 0 otherwise is a metapolynomial. For this last claim, it suffices to prove that $(x^+)^n$ is a metapolynomial, and this follows from the formula $(x^+)^n = \max\{0, \min\{x^{n-1}, x^n\}, \min\{x^n, x^{n+1}\}\}$.

Combinatorics

C1. Several positive integers are written in a row. Iteratively, Alice chooses two adjacent numbers x and y such that $x > y$ and x is to the left of y, and replaces the pair (x, y) by either $(y+1, x)$ or $(x-1, x)$. Prove that she can perform only finitely many such iterations.

Solution 1. Note first that the allowed operation does not change the maximum M of the initial sequence. Let $a_1, a_2, \ldots, a_n$ be the numbers obtained at some point of the process. Consider the sum

$$S = a_1 + 2a_2 + \cdots + na_n.$$

We claim that S increases by a positive integer amount with every operation. Let the operation replace the pair (a_i, a_{i+1}) by a pair (c, a_i), where $a_i > a_{i+1}$ and $c = a_{i+1}+1$ or $c = a_i - 1$. Then the new and the old value of S differ by $d = (ic + (i+1)a_i) - (ia_i + (i+1)a_{i+1}) = a_i - a_{i+1} + i(c - a_{i+1})$. The integer d is positive since $a_i - a_{i+1} \geq 1$ and $c - a_{i+1} \geq 0$.

On the other hand $S \leq (1+2+\cdots+n)M$ as $a_i \leq M$ for all $i = 1, \ldots, n$. Since S increases by at least 1 at each step and never exceeds the constant $(1+2+\cdots+n)M$, the process stops after a finite number of iterations.

Solution 2. Like in the first solution note that the operations do not change the maximum M of the initial sequence. Now consider the reverse lexicographical order for n-tuples of integers. We say that $(x_1, \ldots, x_n) < (y_1, \ldots, y_n)$ if $x_n < y_n$, or if $x_n = y_n$ and $x_{n-1} < y_{n-1}$, or if $x_n = y_n$, $x_{n-1} = y_{n-1}$ and $x_{n-2} < y_{n-2}$, etc. Each iteration creates a sequence that is greater than the previous one with respect to this order, and no sequence occurs twice during the process. On the other hand there are finitely many possible sequences because their terms are always positive integers not exceeding M. Hence the process cannot continue forever.

Solution 3. Let the current numbers be $a_1, a_2, \ldots, a_n$. Define the *score* s_i of a_i as the number of a_j's that are less than a_i. Call the sequence $s_1, s_2, \ldots, s_n$ the score sequence of $a_1, a_2, \ldots, a_n$.

Let us say that a sequence $x_1, \ldots, x_n$ dominates a sequence $y_1, \ldots, y_n$ if the first index i with $x_i \neq y_i$ is such that $x_i < y_i$. We show that after each operation the new score sequence dominates the old one. Score sequences do not repeat, and there are finitely many possibilities for them, no more than $(n-1)^n$. Hence the process will terminate.

Consider an operation that replaces (x, y) by (a, x), with $a = y+1$ or $a = x-1$. Suppose that x was originally at position i. For each $j < i$ the score s_j does not increase with the change because $y \leq a$ and $x \leq x$. If s_j decreases for some $j < i$ then the new score sequence dominates the old one. Assume that s_j stays the same for all $j < i$ and consider s_i. Since $x > y$ and $y \leq a \leq x$, we see that s_i decreases by at least 1. This concludes the proof.

Comment. All three proofs work if x and y are not necessarily adjacent, and if the pair (x, y) is replaced by any pair (a, x), with a an integer satisfying $y \leq a \leq x$. There is nothing special about the "weights" $1, 2, \ldots, n$ in the definition of $S = \sum_{i=1}^{n} ia_i$ from the first solution. For any sequence $w_1 < w_2 < \cdots < w_n$ of positive integers, the sum $\sum_{i=1}^{n} w_i a_i$ increases by at least 1 with each operation.

Consider the same problem, but letting Alice replace the pair (x, y) by (a, x), where a is any positive integer less than x. The same conclusion holds in this version, i. e. the process stops eventually. The solution using the reverse lexicographical order works without any change. The first solution would require a special set of weights like $w_i = M^i$ for $i = 1, \ldots, n$.

Comment. The first and the second solutions provide upper bounds for the number of possible operations, respectively of order Mn^2 and M^n where M is the maximum of the original sequence. The upper bound $(n-1)^n$ in the third solution does not depend on M.

C2. Let $n \geq 1$ be an integer. What is the maximum number of disjoint pairs of elements of the set $\{1, 2, \ldots, n\}$ such that the sums of the different pairs are different integers not exceeding n?

Solution. Consider x such pairs in $\{1, 2, \ldots, n\}$. The sum S of the $2x$ numbers in them is at least $1+2+\cdots+2x$ since the pairs are disjoint. On the other hand $S \leq n+(n-1)+\cdots+(n-x+1)$ because the sums of the pairs are different and do not exceed n. This gives the inequality

$$\frac{2x(2x+1)}{2} \leq nx - \frac{x(x-1)}{2},$$

which leads to $x \leq \frac{2n-1}{5}$. Hence there are at most $\left\lfloor \frac{2n-1}{5} \right\rfloor$ pairs with the given properties.

We show a construction with exactly $\left\lfloor \frac{2n-1}{5} \right\rfloor$ pairs. First consider the case $n = 5k+3$ with $k \geq 0$, where $\left\lfloor \frac{2n-1}{5} \right\rfloor = 2k+1$. The pairs are displayed in the following table.

Pairs	$3k+1$	$3k$	$\cdots$	$2k+2$	$4k+2$	$4k+1$	$\cdots$	$3k+3$	$3k+2$
	2	4	$\cdots$	$2k$	1	3	$\cdots$	$2k-1$	$2k+1$
Sums	$3k+3$	$3k+4$	$\cdots$	$4k+2$	$4k+3$	$4k+4$	$\cdots$	$5k+2$	$5k+3$

The $2k+1$ pairs involve all numbers from 1 to $4k+2$; their sums are all numbers from $3k+3$ to $5k+3$. The same construction works for $n = 5k+4$ and $n = 5k+5$ with $k \geq 0$. In these cases the required number $\left\lfloor \frac{2n-1}{5} \right\rfloor$ of pairs equals $2k+1$ again, and the numbers in the table do not exceed $5k+3$. In the case $n = 5k+2$ with $k \geq 0$ one needs only $2k$ pairs. They can be obtained by ignoring the last column of the table (thus removing $5k+3$). Finally, $2k$ pairs are also needed for the case $n = 5k+1$ with $k \geq 0$. Now it suffices to ignore the last column of the table and then subtract 1 from each number in the first row.

Comment. The construction above is not unique. For instance, the following table shows another set of $2k+1$ pairs for the cases $n = 5k+3$, $n = 5k+4$, and $n = 5k+5$.

Pairs	1	2	$\cdots$	k	$k+1$	$k+2$	$\cdots$	$2k+1$
	$4k+1$	$4k-1$	$\cdots$	$2k+3$	$4k+2$	$4k$	$\cdots$	$2k+2$
Sums	$4k+2$	$4k+1$	$\cdots$	$3k+3$	$5k+3$	$5k+2$	$\cdots$	$4k+3$

The table for the case $n = 5k+2$ would be the same, with the pair $(k+1, 4k+2)$ removed. For the case $n = 5k+1$ remove the last column and subtract 2 from each number in the second row.

C3. In a 999×999 square table some cells are white and the remaining ones are red. Let T be the number of triples (C_1, C_2, C_3) of cells, the first two in the same row and the last two in the same column, with C_1 and C_3 white and C_2 red. Find the maximum value T can attain.

Solution. We prove that in an $n \times n$ square table there are at most $\frac{4n^4}{27}$ such triples.

Let row i and column j contain a_i and b_j white cells respectively, and let R be the set of red cells. For every red cell (i, j) there are $a_i b_j$ admissible triples (C_1, C_2, C_3) with $C_2 = (i, j)$, therefore

$$T = \sum_{(i,j)\in R} a_i b_j.$$

We use the inequality $2ab \leq a^2 + b^2$ to obtain

$$T \leq \frac{1}{2} \sum_{(i,j)\in R} (a_i^2 + b_j^2) = \frac{1}{2}\sum_{i=1}^{n}(n - a_i)a_i^2 + \frac{1}{2}\sum_{j=1}^{n}(n - b_j)b_j^2.$$

This is because there are $n - a_i$ red cells in row i and $n - b_j$ red cells in column j. Now we maximize the right-hand side.

By the AM-GM inequality we have

$$(n - x)x^2 = \frac{1}{2}(2n - 2x) \cdot x \cdot x \leq \frac{1}{2}\left(\frac{2n}{3}\right)^3 = \frac{4n^3}{27},$$

with equality if and only if $x = \frac{2n}{3}$. By putting everything together, we get

$$T \leq \frac{n}{2}\frac{4n^3}{27} + \frac{n}{2}\frac{4n^3}{27} = \frac{4n^4}{27}.$$

If $n = 999$ then any coloring of the square table with $x = \frac{2n}{3} = 666$ white cells in each row and column attains the maximum as all inequalities in the previous argument become equalities. For example color a cell (i, j) white if $i - j \equiv 1, 2, \ldots, 666 \pmod{999}$, and red otherwise.

Therefore the maximum value T can attain is $T = \frac{4\cdot 999^4}{27}$.

Comment. One can obtain a better preliminary estimate with the Cauchy-Schwarz inequality:

$$T = \sum_{(i,j)\in R} a_i b_j \leq \left(\sum_{(i,j)\in R} a_i^2\right)^{\frac{1}{2}} \cdot \left(\sum_{(i,j)\in R} b_j^2\right)^{\frac{1}{2}} = \left(\sum_{i=1}^{n}(n - a_i)a_i^2\right)^{\frac{1}{2}} \cdot \left(\sum_{j=1}^{n}(n - b_j)b_j^2\right)^{\frac{1}{2}}.$$

It can be used to reach the same conclusion.

C4. Players A and B play a game with $N \geq 2012$ coins and 2012 boxes arranged around a circle. Initially A distributes the coins among the boxes so that there is at least 1 coin in each box. Then the two of them make moves in the order $B, A, B, A, \ldots$ by the following rules:

- On every move of his B passes 1 coin from every box to an adjacent box.
- On every move of hers A chooses several coins that were *not* involved in B's previous move and are in different boxes. She passes every chosen coin to an adjacent box.

Player A's goal is to ensure at least 1 coin in each box after every move of hers, regardless of how B plays and how many moves are made. Find the least N that enables her to succeed.

Solution. We argue for a general $n \geq 7$ instead of 2012 and prove that the required minimum N is $2n-2$. For $n = 2012$ this gives $N_{\text{min}} = 4022$.

a) If $N = 2n-2$ player A can achieve her goal. Let her start the game with a *regular* distribution: $n-2$ boxes with 2 coins and 2 boxes with 1 coin. Call the boxes of the two kinds *red* and *white* respectively. We claim that on her first move A can achieve a regular distribution again, regardless of B's first move M. She acts according as the following situation S occurs after M or not: *The initial distribution contains a red box R with 2 white neighbors, and R receives no coins from them on move M.*

Suppose that S does not occur. Exactly one of the coins c_1 and c_2 in a given red box X is involved in M, say c_1. If M passes c_1 to the right neighbor of X, let A pass c_2 to its left neighbor, and vice versa. By doing so with all red boxes A performs a legal move M'. Thus M and M' combined move the 2 coins of every red box in opposite directions. Hence after M and M' are complete each neighbor of a red box X contains exactly 1 coin that was initially in X. So each box with a red neighbor is non-empty after M'. If initially there is a box X with 2 white neighbors (X is red and unique) then X receives a coin from at least one of them on move M since S does not occur. Such a coin is not involved in M', so X is also non-empty after M'. Furthermore each box Y has given away its initial content after M and M'. A red neighbor of Y adds 1 coin to it; a white neighbor adds at most 1 coin because it is not involved in M'. Hence each box contains 1 or 2 coins after M'. Because $N = 2n-2$, such a distribution is regular.

Now let S occur after move M. Then A leaves untouched the exceptional red box R. With all remaining red boxes she proceeds like in the previous case, thus making a legal move M''. Box R receives no coins from its neighbors on either move, so there is 1 coin in it after M''. Like above M and M'' combined pass exactly 1 coin from every red box different from R to each of its neighbors. Every box except R has a red neighbor different from R, hence all boxes are non-empty after M''. Next, each box Y except R loses its initial content after M and M''. A red neighbor of Y adds at most 1 coin to it; a white neighbor also adds at most 1 coin as it does not participate in M''. Thus each box has 1 or 2 coins after M'', and the obtained distribution is regular.

Player A can apply the described strategy indefinitely, so $N = 2n-2$ enables her to succeed.

b) For $N \leq 2n-3$ player B can achieve an empty box after some move of A. Let α be a set of ℓ consecutive boxes containing a total of $N(\alpha)$ coins. We call α an *arc* if $\ell \leq n-2$ and $N(\alpha) \leq 2\ell - 3$. Note that $\ell \geq 2$ by the last condition. Moreover if both extremes of α are non-empty boxes then $N(\alpha) \geq 2$, so that $N(\alpha) \leq 2\ell - 3$ implies $\ell \geq 3$. Observe also that if an extreme X of α has more than 1 coin then ignoring X yields a shorter arc. It follows that every arc contains an arc whose extremes have at most 1 coin each.

Given a clockwise labeling $1, 2, \ldots, n$ of the boxes, suppose that boxes $1, 2, \ldots, \ell$ form an arc α, with $\ell \leq n-2$ and $N(\alpha) \leq 2\ell - 3$. Suppose also that all $n \geq 7$ boxes are non-empty. Then B can move so that an arc α' with $N(\alpha') < N(\alpha)$ will appear after any response of A.

One may assume exactly 1 coin in boxes 1 and ℓ by a previous remark. Let B pass 1 coin in counterclockwise direction from box 1 and box n, and in clockwise direction from each remaining box. This leaves $N(\alpha)-2$ coins in the boxes of α. In addition, due to $3 \le \ell \le n-2$, box ℓ has exactly 1 coin c, the one received from box $\ell - 1$.

Let player A's next move M pass $k \le 2$ coins to boxes $1, 2, \dots, \ell$ from the remaining ones. Only boxes 1 and ℓ can receive such coins, at most 1 each. If $k < 2$ then after move M boxes $1, 2, \dots, \ell$ form an arc α' with $N(\alpha') < N(\alpha)$. If $k = 2$ then M adds a coin to box ℓ. Also M does not move coin c from ℓ because c is involved in the previous move of B. In summary boxes $1, 2, \dots, \ell$ contain $N(\alpha)$ coins like before, so they form an arc. However there are 2 coins now in the extreme ℓ of the arc. Ignore ℓ to obtain a shorter arc α' with $N(\alpha') < N(\alpha)$.

Consider any initial distribution without empty boxes. Since $N \le 2n-3$, there are at least 3 boxes in it with exactly 1 coin. It follows from $n \ge 7$ that some 2 of them are the extremes of an arc α. Hence B can make the move described above, which leads to an arc α' with $N(\alpha') < N(\alpha)$ after A's response. If all boxes in the new distribution are non-empty he can repeat the same, and so on. Because $N(\alpha)$ cannot decrease indefinitely, an empty box will occur after some move of A.

C5. The columns and the rows of a $3n \times 3n$ square board are numbered $1, 2, \ldots, 3n$. Every square (x, y) with $1 \leq x, y \leq 3n$ is colored asparagus, byzantium or citrine according as the modulo 3 remainder of $x + y$ is 0, 1 or 2 respectively. One token colored asparagus, byzantium or citrine is placed on each square, so that there are $3n^2$ tokens of each color.

Suppose that one can permute the tokens so that each token is moved to a distance of at most d from its original position, each asparagus token replaces a byzantium token, each byzantium token replaces a citrine token, and each citrine token replaces an asparagus token. Prove that it is possible to permute the tokens so that each token is moved to a distance of at most $d + 2$ from its original position, and each square contains a token with the same color as the square.

Solution. Without loss of generality it suffices to prove that the A-tokens can be moved to distinct A-squares in such a way that each A-token is moved to a distance at most $d + 2$ from its original place. This means we need a perfect matching between the $3n^2$ A-squares and the $3n^2$ A-tokens such that the distance in each pair of the matching is at most $d + 2$.

To find the matching, we construct a bipartite graph. The A-squares will be the vertices in one class of the graph; the vertices in the other class will be the A-tokens.

Split the board into 3×1 horizontal triminos; then each trimino contains exactly one A-square. Take a permutation π of the tokens which moves A-tokens to B-tokens, B-tokens to C-tokens, and C-tokens to A-tokens, in each case to a distance at most d. For each A-square S, and for each A-token T, connect S and T by an edge if T, $\pi(T)$ or $\pi^{-1}(T)$ is on the trimino containing S. We allow multiple edges; it is even possible that the same square and the same token are connected with three edges. Obviously the lengths of the edges in the graph do not exceed $d + 2$. By length of an edge we mean the distance between the A-square and the A-token it connects.

Each A-token T is connected with the three A-squares whose triminos contain T, $\pi(T)$ and $\pi^{-1}(T)$. Therefore in the graph all tokens are of degree 3. We show that the same is true for the A-squares. Let S be an arbitrary A-square, and let T_1, T_2, T_3 be the three tokens on the trimino containing S. For $i = 1, 2, 3$, if T_i is an A-token, then S is connected with T_i; if T_i is a B-token then S is connected with $\pi^{-1}(T_i)$; finally, if T_i is a C-token then S is connected with $\pi(T_i)$. Hence in the graph the A-squares also are of degree 3.

Since the A-squares are of degree 3, from every set $\mathcal{S}$ of A-squares exactly $3|\mathcal{S}|$ edges start. These edges end in at least $|\mathcal{S}|$ tokens because the A-tokens also are of degree 3. Hence every set $\mathcal{S}$ of A-squares has at least $|\mathcal{S}|$ neighbors among the A-tokens.

Therefore, by Hall's marriage theorem, the graph contains a perfect matching between the two vertex classes. So there is a perfect matching between the A-squares and A-tokens with edges no longer than $d + 2$. It follows that the tokens can be permuted as specified in the problem statement.

Comment 1. In the original problem proposal the board was infinite and there were only two colors. Having n colors for some positive integer n was an option; we chose $n = 3$. Moreover, we changed the board to a finite one to avoid dealing with infinite graphs (although Hall's theorem works in the infinite case as well).

With only two colors Hall's theorem is not needed. In this case we split the board into 2×1 dominos, and in the resulting graph all vertices are of degree 2. The graph consists of disjoint cycles with even length and infinite paths, so the existence of the matching is trivial.

Having more than three colors would make the problem statement more complicated, because we need a matching between every two color classes of tokens. However, this would not mean a significant increase in difficulty.

Comment 2. According to Wikipedia, the color *asparagus* (hexadecimal code `#87A96B`) is a tone of green that is named after the vegetable. Crayola created this color in 1993 as one of the 16 to be named in the Name The Color Contest. *Byzantium* (`#702963`) is a dark tone of purple. Its first recorded use as a color name in English was in 1926. *Citrine* (`#E4D00A`) is variously described as yellow, greenish-yellow, brownish-yellow or orange. The first known use of citrine as a color name in English was in the 14th century.

C6. Let k and n be fixed positive integers. In the liar's guessing game, Amy chooses integers x and N with $1 \leq x \leq N$. She tells Ben what N is, but not what x is. Ben may then repeatedly ask Amy whether $x \in S$ for arbitrary sets S of integers. Amy will always answer with *yes* or *no*, but she might lie. The only restriction is that she can lie at most k times in a row. After he has asked as many questions as he wants, Ben must specify a set of at most n positive integers. If x is in this set he wins; otherwise, he loses. Prove that:

a) If $n \geq 2^k$ then Ben can always win.

b) For sufficiently large k there exist $n \geq 1.99^k$ such that Ben cannot guarantee a win.

Solution. Consider an answer $A \in \{yes, no\}$ to a question of the kind "Is x in the set S?" We say that A is inconsistent with a number i if $A = yes$ and $i \notin S$, or if $A = no$ and $i \in S$. Observe that an answer inconsistent with the target number x is a lie.

a) Suppose that Ben has determined a set T of size m that contains x. This is true initially with $m = N$ and $T = \{1, 2, \ldots, N\}$. For $m > 2^k$ we show how Ben can find a number $y \in T$ that is different from x. By performing this step repeatedly he can reduce T to be of size $2^k \leq n$ and thus win.

Since only the size $m > 2^k$ of T is relevant, assume that $T = \{0, 1, \ldots, 2^k, \ldots, m-1\}$. Ben begins by asking repeatedly whether x is 2^k. If Amy answers *no* $k+1$ times in a row, one of these answers is truthful, and so $x \neq 2^k$. Otherwise Ben stops asking about 2^k at the first answer *yes*. He then asks, for each $i = 1, \ldots, k$, if the binary representation of x has a 0 in the ith digit. Regardless of what the k answers are, they are all inconsistent with a certain number $y \in \{0, 1, \ldots, 2^k - 1\}$. The preceding answer *yes* about 2^k is also inconsistent with y. Hence $y \neq x$. Otherwise the last $k+1$ answers are not truthful, which is impossible.

Either way, Ben finds a number in T that is different from x, and the claim is proven.

b) We prove that if $1 < \lambda < 2$ and $n = \left\lfloor (2-\lambda)\lambda^{k+1} \right\rfloor - 1$ then Ben cannot guarantee a win. To complete the proof, then it suffices to take λ such that $1.99 < \lambda < 2$ and k large enough so that

$$n = \left\lfloor (2-\lambda)\lambda^{k+1} \right\rfloor - 1 \geq 1.99^k.$$

Consider the following strategy for Amy. First she chooses $N = n+1$ and $x \in \{1, 2, \ldots, n+1\}$ arbitrarily. After every answer of hers Amy determines, for each $i = 1, 2, \ldots, n+1$, the number m_i of consecutive answers she has given by that point that are inconsistent with i. To decide on her next answer, she then uses the quantity

$$\phi = \sum_{i=1}^{n+1} \lambda^{m_i}.$$

No matter what Ben's next question is, Amy chooses the answer which minimizes ϕ.

We claim that with this strategy ϕ will always stay less than λ^{k+1}. Consequently no exponent m_i in ϕ will ever exceed k, hence Amy will never give more than k consecutive answers inconsistent with some i. In particular this applies to the target number x, so she will never lie more than k times in a row. Thus, given the claim, Amy's strategy is legal. Since the strategy does not depend on x in any way, Ben can make no deductions about x, and therefore he cannot guarantee a win.

It remains to show that $\phi < \lambda^{k+1}$ at all times. Initially each m_i is 0, so this condition holds in the beginning due to $1 < \lambda < 2$ and $n = \left\lfloor (2-\lambda)\lambda^{k+1} \right\rfloor - 1$. Suppose that $\phi < \lambda^{k+1}$ at some point, and Ben has just asked if $x \in S$ for some set S. According as Amy answers *yes* or *no*, the new value of ϕ becomes

$$\phi_1 = \sum_{i \in S} 1 + \sum_{i \notin S} \lambda^{m_i+1} \qquad \text{or} \qquad \phi_2 = \sum_{i \in S} \lambda^{m_i+1} + \sum_{i \notin S} 1.$$

Since Amy chooses the option minimizing ϕ, the new ϕ will equal $\min(\phi_1, \phi_2)$. Now we have

$$\min(\phi_1, \phi_2) \leq \frac{1}{2}(\phi_1 + \phi_2) = \frac{1}{2}\left(\sum_{i \in S}\left(1 + \lambda^{m_i+1}\right) + \sum_{i \notin S}\left(\lambda^{m_i+1} + 1\right)\right) = \frac{1}{2}(\lambda\phi + n + 1).$$

Because $\phi < \lambda^{k+1}$, the assumptions $\lambda < 2$ and $n = \left\lfloor (2-\lambda)\lambda^{k+1} \right\rfloor - 1$ lead to

$$\min(\phi_1, \phi_2) < \frac{1}{2}(\lambda^{k+2} + (2-\lambda)\lambda^{k+1}) = \lambda^{k+1}.$$

The claim follows, which completes the solution.

Comment. Given a fixed k, let $f(k)$ denote the minimum value of n for which Ben can guarantee a victory. The problem asks for a proof that for large k

$$1.99^k \leq f(k) \leq 2^k.$$

A computer search shows that $f(k) = 2, 3, 4, 7, 11, 17$ for $k = 1, 2, 3, 4, 5, 6$.

C7. There are given 2^{500} points on a circle labeled $1, 2, \ldots, 2^{500}$ in some order. Prove that one can choose 100 pairwise disjoint chords joining some of these points so that the 100 sums of the pairs of numbers at the endpoints of the chosen chords are equal.

Solution. The proof is based on the following general fact.

Lemma. In a graph G each vertex v has degree d_v. Then G contains an independent set S of vertices such that $|S| \geq f(G)$ where

$$f(G) = \sum_{v \in G} \frac{1}{d_v + 1}.$$

Proof. Induction on $n = |G|$. The base $n = 1$ is clear. For the inductive step choose a vertex v_0 in G of minimum degree d. Delete v_0 and all of its neighbors $v_1, \ldots, v_d$ and also all edges with endpoints $v_0, v_1, \ldots, v_d$. This gives a new graph G'. By the inductive assumption G' contains an independent set S' of vertices such that $|S'| \geq f(G')$. Since no vertex in S' is a neighbor of v_0 in G, the set $S = S' \cup \{v_0\}$ is independent in G.

Let d'_v be the degree of a vertex v in G'. Clearly $d'_v \leq d_v$ for every such vertex v, and also $d_{v_i} \geq d$ for all $i = 0, 1, \ldots, d$ by the minimal choice of v_0. Therefore

$$f(G') = \sum_{v \in G'} \frac{1}{d'_v + 1} \geq \sum_{v \in G'} \frac{1}{d_v + 1} = f(G) - \sum_{i=0}^{d} \frac{1}{d_{v_i} + 1} \geq f(G) - \frac{d+1}{d+1} = f(G) - 1.$$

Hence $|S| = |S'| + 1 \geq f(G') + 1 \geq f(G)$, and the induction is complete. $\square$

We pass on to our problem. For clarity denote $n = 2^{499}$ and draw all chords determined by the given $2n$ points. Color each chord with one of the colors $3, 4, \ldots, 4n - 1$ according to the sum of the numbers at its endpoints. Chords with a common endpoint have different colors. For each color c consider the following graph G_c. Its vertices are the chords of color c, and two chords are neighbors in G_c if they intersect. Let $f(G_c)$ have the same meaning as in the lemma for all graphs G_c.

Every chord ℓ divides the circle into two arcs, and one of them contains $m(\ell) \leq n - 1$ given points. (In particular $m(\ell) = 0$ if ℓ joins two consecutive points.) For each $i = 0, 1, \ldots, n - 2$ there are $2n$ chords ℓ with $m(\ell) = i$. Such a chord has degree at most i in the respective graph. Indeed let $A_1, \ldots, A_i$ be all points on either arc determined by a chord ℓ with $m(\ell) = i$ and color c. Every A_j is an endpoint of at most 1 chord colored c, $j = 1, \ldots, i$. Hence at most i chords of color c intersect ℓ.

It follows that for each $i = 0, 1, \ldots, n - 2$ the $2n$ chords ℓ with $m(\ell) = i$ contribute at least $\frac{2n}{i+1}$ to the sum $\sum_c f(G_c)$. Summation over $i = 0, 1, \ldots, n - 2$ gives

$$\sum_c f(G_c) \geq 2n \sum_{i=1}^{n-1} \frac{1}{i}.$$

Because there are $4n - 3$ colors in all, averaging yields a color c such that

$$f(G_c) \geq \frac{2n}{4n-3} \sum_{i=1}^{n-1} \frac{1}{i} > \frac{1}{2} \sum_{i=1}^{n-1} \frac{1}{i}.$$

By the lemma there are at least $\frac{1}{2} \sum_{i=1}^{n-1} \frac{1}{i}$ pairwise disjoint chords of color c, i. e. with the same sum c of the pairs of numbers at their endpoints. It remains to show that $\frac{1}{2} \sum_{i=1}^{n-1} \frac{1}{i} \geq 100$ for $n = 2^{499}$. Indeed we have

$$\sum_{i=1}^{n-1} \frac{1}{i} > \sum_{i=1}^{2^{400}} \frac{1}{i} = 1 + \sum_{k=1}^{400} \sum_{i=2^{k-1}+1}^{2^k} \frac{1}{i} > 1 + \sum_{k=1}^{400} \frac{2^{k-1}}{2^k} = 201 > 200.$$

This completes the solution.

Geometry

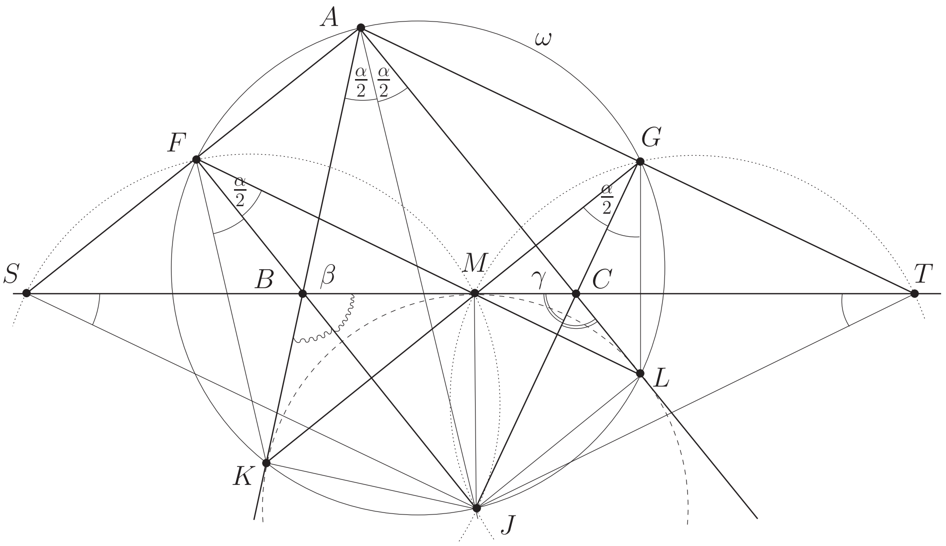
G1. In the triangle ABC the point J is the center of the excircle opposite to A. This excircle is tangent to the side BC at M, and to the lines AB and AC at K and L respectively. The lines LM and BJ meet at F, and the lines KM and CJ meet at G. Let S be the point of intersection of the lines AF and BC, and let T be the point of intersection of the lines AG and BC. Prove that M is the midpoint of ST.

Solution. Let $\alpha = \angle CAB$, $\beta = \angle ABC$ and $\gamma = \angle BCA$. The line AJ is the bisector of $\angle CAB$, so $\angle JAK = \angle JAL = \frac{\alpha}{2}$. By $\angle AKJ = \angle ALJ = 90^\circ$ the points K and L lie on the circle ω with diameter AJ.

The triangle KBM is isosceles as BK and BM are tangents to the excircle. Since BJ is the bisector of $\angle KBM$, we have $\angle MBJ = 90^\circ - \frac{\beta}{2}$ and $\angle BMK = \frac{\beta}{2}$. Likewise $\angle MCJ = 90^\circ - \frac{\gamma}{2}$ and $\angle CML = \frac{\gamma}{2}$. Also $\angle BMF = \angle CML$, therefore

$$\angle LFJ = \angle MBJ - \angle BMF = \left(90^\circ - \frac{\beta}{2}\right) - \frac{\gamma}{2} = \frac{\alpha}{2} = \angle LAJ.$$

Hence F lies on the circle ω. (By the angle computation, F and A are on the same side of BC.) Analogously, G also lies on ω. Since AJ is a diameter of ω, we obtain $\angle AFJ = \angle AGJ = 90^\circ$.



The lines AB and BC are symmetric with respect to the external bisector BF. Because $AF \perp BF$ and $KM \perp BF$, the segments SM and AK are symmetric with respect to BF, hence $SM = AK$. By symmetry $TM = AL$. Since AK and AL are equal as tangents to the excircle, it follows that $SM = TM$, and the proof is complete.

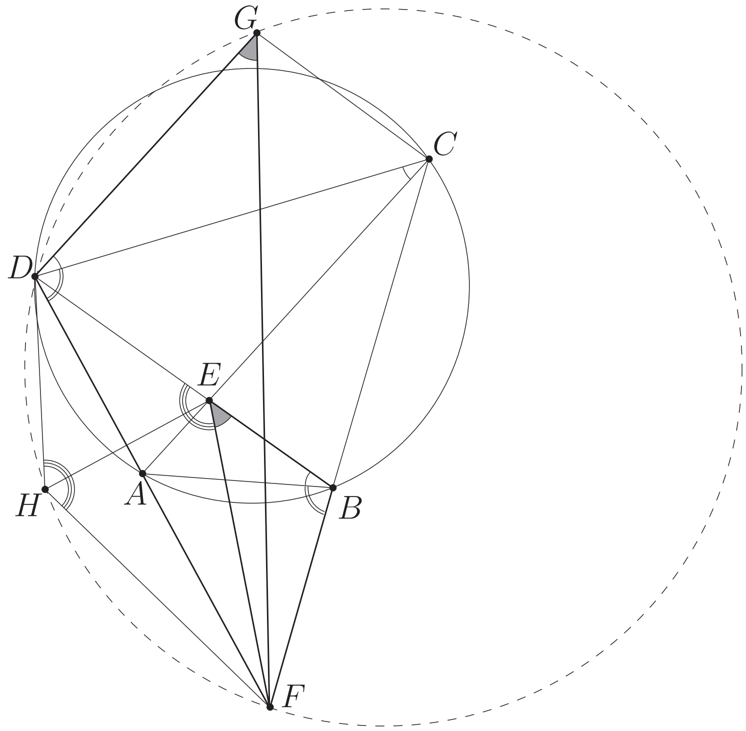
Comment. After discovering the circle $AFKJLG$, there are many other ways to complete the solution. For instance, from the cyclic quadrilaterals $JMFS$ and $JMGT$ one can find $\angle TSJ = \angle STJ = \frac{\alpha}{2}$. Another possibility is to use the fact that the lines AS and GM are parallel (both are perpendicular to the external angle bisector BJ), so $\frac{MS}{MT} = \frac{AG}{GT} = 1$.

G2. Let $ABCD$ be a cyclic quadrilateral whose diagonals AC and BD meet at E. The extensions of the sides AD and BC beyond A and B meet at F. Let G be the point such that $ECGD$ is a parallelogram, and let H be the image of E under reflection in AD. Prove that D, H, F, G are concyclic.

Solution. We show first that the triangles FDG and FBE are similar. Since $ABCD$ is cyclic, the triangles EAB and EDC are similar, as well as FAB and FCD. The parallelogram $ECGD$ yields $GD = EC$ and $\angle CDG = \angle DCE$; also $\angle DCE = \angle DCA = \angle DBA$ by inscribed angles. Therefore

$$\angle FDG = \angle FDC + \angle CDG = \angle FBA + \angle ABD = \angle FBE,$$
$$\frac{GD}{EB} = \frac{CE}{EB} = \frac{CD}{AB} = \frac{FD}{FB}.$$

It follows that FDG and FBE are similar, and so $\angle FGD = \angle FEB$.



Since H is the reflection of E with respect to FD, we conclude that

$$\angle FHD = \angle FED = 180^\circ - \angle FEB = 180^\circ - \angle FGD.$$

This proves that D, H, F, G are concyclic.

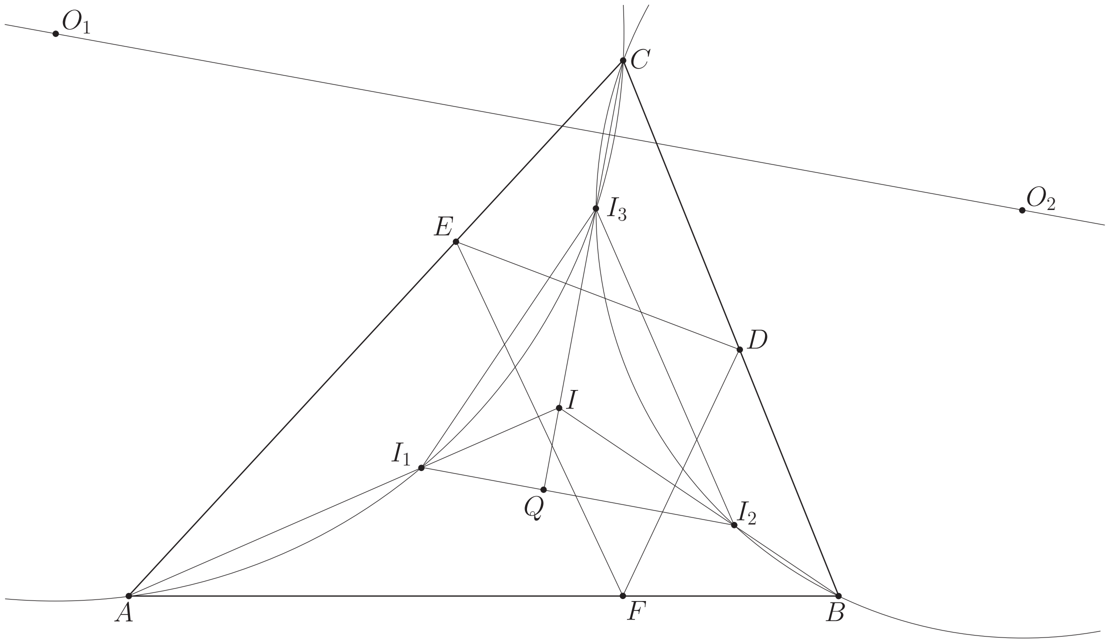
Comment. Points E and G are always in the half-plane determined by the line FD that contains B and C, but H is always in the other half-plane. In particular, $DHFG$ is cyclic if and only if $\angle FHD + \angle FGD = 180^\circ$.

G3. In an acute triangle ABC the points D, E and F are the feet of the altitudes through A, B and C respectively. The incenters of the triangles AEF and BDF are I_1 and I_2 respectively; the circumcenters of the triangles ACI_1 and BCI_2 are O_1 and O_2 respectively. Prove that I_1I_2 and O_1O_2 are parallel.

Solution. Let $\angle CAB = \alpha$, $\angle ABC = \beta$, $\angle BCA = \gamma$. We start by showing that A, B, I_1 and I_2 are concyclic. Since AI_1 and BI_2 bisect $\angle CAB$ and $\angle ABC$, their extensions beyond I_1 and I_2 meet at the incenter I of the triangle. The points E and F are on the circle with diameter BC, so $\angle AEF = \angle ABC$ and $\angle AFE = \angle ACB$. Hence the triangles AEF and ABC are similar with ratio of similitude $\frac{AE}{AB} = \cos\alpha$. Because I_1 and I are their incenters, we obtain $I_1A = IA\cos\alpha$ and $II_1 = IA - I_1A = 2IA\sin^2\frac{\alpha}{2}$. By symmetry $II_2 = 2IB\sin^2\frac{\beta}{2}$. The law of sines in the triangle ABI gives $IA\sin\frac{\alpha}{2} = IB\sin\frac{\beta}{2}$. Hence

$$II_1 \cdot IA = 2\left(IA\sin\tfrac{\alpha}{2}\right)^2 = 2\left(IB\sin\tfrac{\beta}{2}\right)^2 = II_2 \cdot IB.$$

Therefore A, B, I_1 and I_2 are concyclic, as claimed.



In addition $II_1 \cdot IA = II_2 \cdot IB$ implies that I has the same power with respect to the circles (ACI_1), (BCI_2) and (ABI_1I_2). Then CI is the radical axis of (ACI_1) and (BCI_2); in particular CI is perpendicular to the line of centers O_1O_2.

Now it suffices to prove that $CI \perp I_1I_2$. Let CI meet I_1I_2 at Q, then it is enough to check that $\angle II_1Q + \angle I_1IQ = 90°$. Since $\angle I_1IQ$ is external for the triangle ACI, we have

$$\angle II_1Q + \angle I_1IQ = \angle II_1Q + (\angle ACI + \angle CAI) = \angle II_1I_2 + \angle ACI + \angle CAI.$$

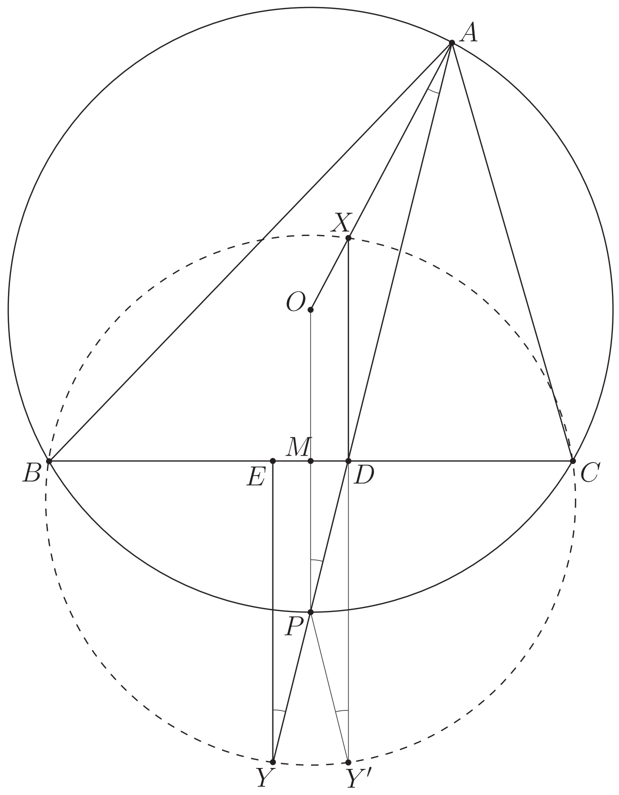
It remains to note that $\angle II_1I_2 = \frac{\beta}{2}$ from the cyclic quadrilateral ABI_1I_2, and $\angle ACI = \frac{\gamma}{2}$, $\angle CAI = \frac{\alpha}{2}$. Therefore $\angle II_1Q + \angle I_1IQ = \frac{\alpha}{2} + \frac{\beta}{2} + \frac{\gamma}{2} = 90°$, completing the proof.

Comment. It follows from the first part of the solution that the common point $I_3 \neq C$ of the circles (ACI_1) and (BCI_2) is the incenter of the triangle CDE.

G4. Let ABC be a triangle with $AB \neq AC$ and circumcenter O. The bisector of $\angle BAC$ intersects BC at D. Let E be the reflection of D with respect to the midpoint of BC. The lines through D and E perpendicular to BC intersect the lines AO and AD at X and Y respectively. Prove that the quadrilateral $BXCY$ is cyclic.

Solution. The bisector of $\angle BAC$ and the perpendicular bisector of BC meet at P, the midpoint of the minor arc $\widehat{BC}$ (they are different lines as $AB \neq AC$). In particular OP is perpendicular to BC and intersects it at M, the midpoint of BC.

Denote by Y' the reflexion of Y with respect to OP. Since $\angle BYC = \angle BY'C$, it suffices to prove that $BXCY'$ is cyclic.



We have

$$\angle XAP = \angle OPA = \angle EYP.$$

The first equality holds because $OA = OP$, and the second one because EY and OP are both perpendicular to BC and hence parallel. But $\{Y, Y'\}$ and $\{E, D\}$ are pairs of symmetric points with respect to OP, it follows that $\angle EYP = \angle DY'P$ and hence

$$\angle XAP = \angle DY'P = \angle XY'P.$$

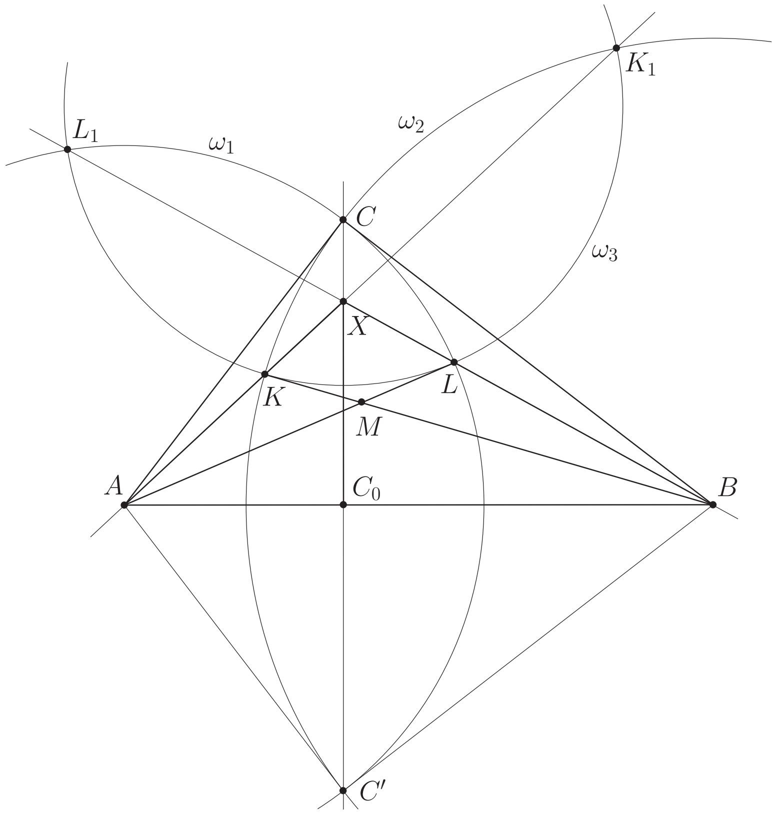
The last equation implies that $XAY'P$ is cyclic. By the powers of D with respect to the circles $(XAY'P)$ and $(ABPC)$ we obtain

$$XD \cdot DY' = AD \cdot DP = BD \cdot DC.$$

It follows that $BXCY'$ is cyclic, as desired.

G5. Let ABC be a triangle with $\angle BCA = 90^\circ$, and let C_0 be the foot of the altitude from C. Choose a point X in the interior of the segment CC_0, and let K, L be the points on the segments AX, BX for which $BK = BC$ and $AL = AC$ respectively. Denote by M the intersection of AL and BK. Show that $MK = ML$.

Solution. Let C' be the reflection of C in the line AB, and let ω_1 and ω_2 be the circles with centers A and B, passing through L and K respectively. Since $AC' = AC = AL$ and $BC' = BC = BK$, both ω_1 and ω_2 pass through C and C'. By $\angle BCA = 90^\circ$, AC is tangent to ω_2 at C, and BC is tangent to ω_1 at C. Let $K_1 \neq K$ be the second intersection of AX and ω_2, and let $L_1 \neq L$ be the second intersection of BX and ω_1.



By the powers of X with respect to ω_2 and ω_1,

$$XK \cdot XK_1 = XC \cdot XC' = XL \cdot XL_1,$$

so the points K_1, L, K, L_1 lie on a circle ω_3.

The power of A with respect to ω_2 gives

$$AL^2 = AC^2 = AK \cdot AK_1,$$

indicating that AL is tangent to ω_3 at L. Analogously, BK is tangent to ω_3 at K. Hence MK and ML are the two tangents from M to ω_3 and therefore $MK = ML$.

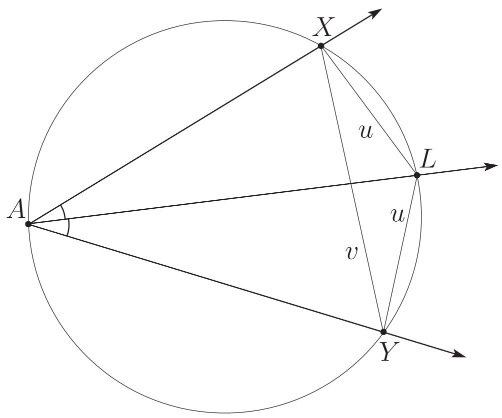
G6. Let ABC be a triangle with circumcenter O and incenter I. The points D, E and F on the sides BC, CA and AB respectively are such that $BD + BF = CA$ and $CD + CE = AB$. The circumcircles of the triangles BFD and CDE intersect at $P \neq D$. Prove that $OP = OI$.

Solution. By MIQUEL's theorem the circles $(AEF) = \omega_A$, $(BFD) = \omega_B$ and $(CDE) = \omega_C$ have a common point, for arbitrary points D, E and F on BC, CA and AB. So ω_A passes through the common point $P \neq D$ of ω_B and ω_C.

Let ω_A, ω_B and ω_C meet the bisectors AI, BI and CI at $A \neq A'$, $B \neq B'$ and $C \neq C'$ respectively. The key observation is that A', B' and C' do not depend on the particular choice of D, E and F, provided that $BD + BF = CA$, $CD + CE = AB$ and $AE + AF = BC$ hold true (the last equality follows from the other two). For a proof we need the following fact.

Lemma. Given is an angle with vertex A and measure α. A circle ω through A intersects the angle bisector at L and sides of the angle at X and Y. Then $AX + AY = 2AL \cos \frac{\alpha}{2}$.

Proof. Note that L is the midpoint of arc $\widehat{XLY}$ in ω and set $XL = YL = u$, $XY = v$. By PTOLEMY's theorem $AX \cdot YL + AY \cdot XL = AL \cdot XY$, which rewrites as $(AX + AY)u = AL \cdot v$. Since $\angle LXY = \frac{\alpha}{2}$ and $\angle XLY = 180^\circ - \alpha$, we have $v = 2\cos\frac{\alpha}{2}u$ by the law of sines, and the claim follows. $\square$



Apply the lemma to $\angle BAC = \alpha$ and the circle $\omega = \omega_A$, which intersects AI at A'. This gives $2AA' \cos\frac{\alpha}{2} = AE + AF = BC$; by symmetry analogous relations hold for BB' and CC'. It follows that A', B' and C' are independent of the choice of D, E and F, as stated.

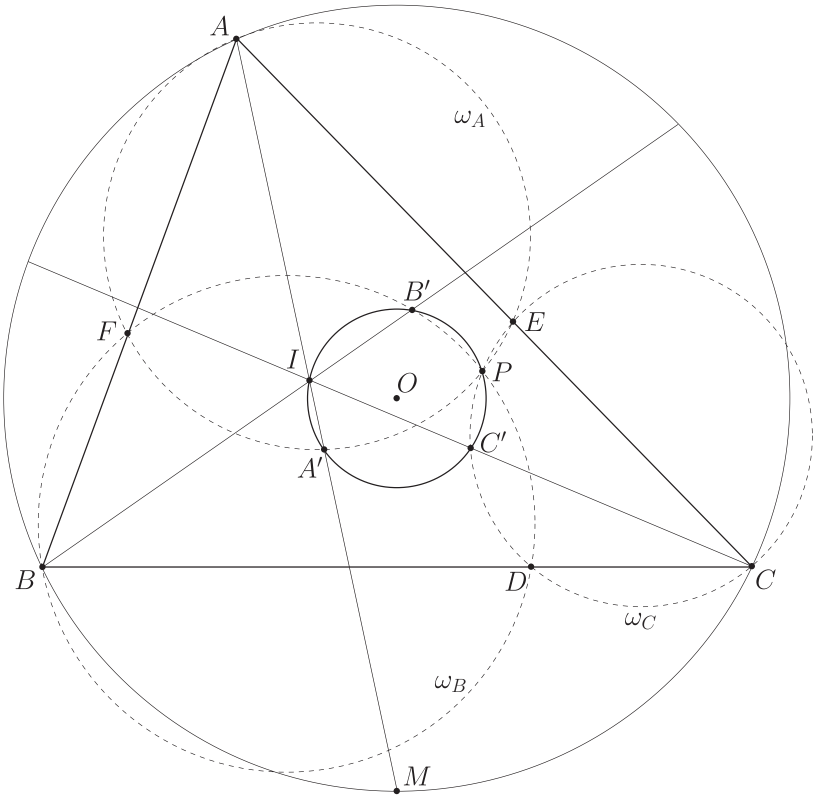
We use the lemma two more times with $\angle BAC = \alpha$. Let ω be the circle with diameter AI. Then X and Y are the tangency points of the incircle of ABC with AB and AC, and hence $AX = AY = \frac{1}{2}(AB + AC - BC)$. So the lemma yields $2AI \cos\frac{\alpha}{2} = AB + AC - BC$. Next, if ω is the circumcircle of ABC and AI intersects ω at $M \neq A$ then $\{X, Y\} = \{B, C\}$, and so $2AM \cos\frac{\alpha}{2} = AB + AC$ by the lemma. To summarize,

$$2AA' \cos\frac{\alpha}{2} = BC, \qquad 2AI \cos\frac{\alpha}{2} = AB + AC - BC, \qquad 2AM \cos\frac{\alpha}{2} = AB + AC. \qquad (*)$$

These equalities imply $AA' + AI = AM$, hence the segments AM and IA' have a common midpoint. It follows that I and A' are equidistant from the circumcenter O. By symmetry $OI = OA' = OB' = OC'$, so I, A', B', C' are on a circle centered at O.

To prove $OP = OI$, now it suffices to show that I, A', B', C' and P are concyclic. Clearly one can assume $P \neq I, A', B', C'$.

We use oriented angles to avoid heavy case distinction. The oriented angle between the lines l and m is denoted by $\angle(l, m)$. We have $\angle(l, m) = -\angle(m, l)$ and $\angle(l, m) + \angle(m, n) = \angle(l, n)$ for arbitrary lines l, m and n. Four distinct non-collinear points U, V, X, Y are concyclic if and only if $\angle(UX, VX) = \angle(UY, VY)$.



Suppose for the moment that A', B', P, I are distinct and noncollinear; then it is enough to check the equality $\angle(A'P, B'P) = \angle(A'I, B'I)$. Because A, F, P, A' are on the circle ω_A, we have $\angle(A'P, FP) = \angle(A'A, FA) = \angle(A'I, AB)$. Likewise $\angle(B'P, FP) = \angle(B'I, AB)$. Therefore

$$\angle(A'P, B'P) = \angle(A'P, FP) + \angle(FP, B'P) = \angle(A'I, AB) - \angle(B'I, AB) = \angle(A'I, B'I).$$

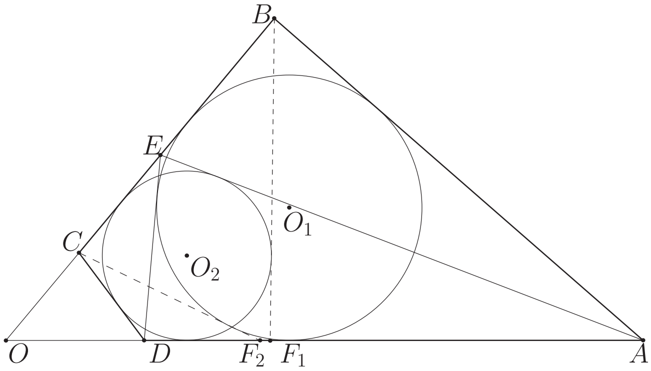
Here we assumed that $P \neq F$. If $P = F$ then $P \neq D, E$ and the conclusion follows similarly (use $\angle(A'F, B'F) = \angle(A'F, EF) + \angle(EF, DF) + \angle(DF, B'F)$ and inscribed angles in $\omega_A, \omega_B, \omega_C$).

There is no loss of generality in assuming A', B', P, I distinct and noncollinear. If ABC is an equilateral triangle then the equalities (*) imply that A', B', C', I, O and P coincide, so $OP = OI$. Otherwise at most one of A', B', C' coincides with I. If say $C' = I$ then $OI \perp CI$ by the previous reasoning. It follows that $A', B' \neq I$ and hence $A' \neq B'$. Finally A', B' and I are noncollinear because I, A', B', C' are concyclic.

Comment. The proposer remarks that the locus γ of the points P is an arc of the circle $(A'B'C'I)$. The reflection I' of I in O belongs to γ; it is obtained by choosing D, E and F to be the tangency points of the three excircles with their respective sides. The rest of the circle $(A'B'C'I)$, except I, can be included in γ by letting D, E and F vary on the extensions of the sides and assuming signed lengths. For instance if B is between C and D then the length BD must be taken with a negative sign. The incenter I corresponds to the limit case where D tends to infinity.

G7. Let $ABCD$ be a convex quadrilateral with non-parallel sides BC and AD. Assume that there is a point E on the side BC such that the quadrilaterals $ABED$ and $AECD$ are circumscribed. Prove that there is a point F on the side AD such that the quadrilaterals $ABCF$ and $BCDF$ are circumscribed if and only if AB is parallel to CD.

Solution. Let ω_1 and ω_2 be the incircles and O_1 and O_2 the incenters of the quadrilaterals $ABED$ and $AECD$ respectively. A point F with the stated property exists only if ω_1 and ω_2 are also the incircles of the quadrilaterals $ABCF$ and $BCDF$.



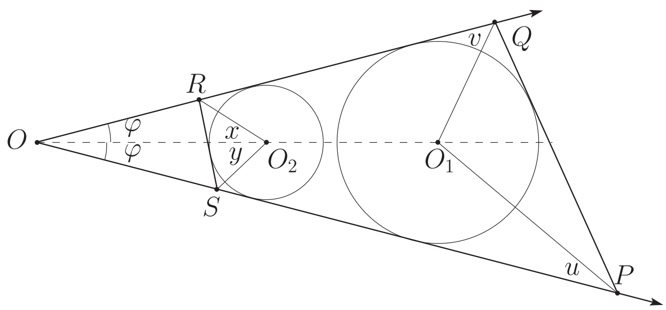
Let the tangents from B to ω_2 and from C to ω_1 (other than BC) meet AD at F_1 and F_2 respectively. We need to prove that $F_1 = F_2$ if and only if $AB \parallel CD$.

Lemma. The circles ω_1 and ω_2 with centers O_1 and O_2 are inscribed in an angle with vertex O. The points P, S on one side of the angle and Q, R on the other side are such that ω_1 is the incircle of the triangle PQO, and ω_2 is the excircle of the triangle RSO opposite to O. Denote $p = OO_1 \cdot OO_2$. Then exactly one of the following relations holds:

$$OP \cdot OR < p < OQ \cdot OS, \qquad OP \cdot OR > p > OQ \cdot OS, \qquad OP \cdot OR = p = OQ \cdot OS.$$

Proof. Denote $\angle OPO_1 = u$, $\angle OQO_1 = v$, $\angle OO_2R = x$, $\angle OO_2S = y$, $\angle POQ = 2\varphi$. Because PO_1, QO_1, RO_2, SO_2 are internal or external bisectors in the triangles PQO and RSO, we have

$$u + v = x + y \ (= 90^\circ - \varphi). \tag{1}$$



By the law of sines

$$\frac{OP}{OO_1} = \frac{\sin(u+\varphi)}{\sin u} \quad \text{and} \quad \frac{OO_2}{OR} = \frac{\sin(x+\varphi)}{\sin x}.$$

Therefore, since x, u and φ are acute,

$$OP \cdot OR \geq p \Leftrightarrow \frac{OP}{OO_1} \geq \frac{OO_2}{OR} \Leftrightarrow \sin x \sin(u+\varphi) \geq \sin u \sin(x+\varphi) \Leftrightarrow \sin(x-u) \geq 0 \Leftrightarrow x \geq u.$$

Thus $OP \cdot OR \geq p$ is equivalent to $x \geq u$, with $OP \cdot OR = p$ if and only if $x = u$.

Analogously, $p \geq OQ \cdot OS$ is equivalent to $v \geq y$, with $p = OQ \cdot OS$ if and only if $v = y$. On the other hand $x \geq u$ and $v \geq y$ are equivalent by (1), with $x = u$ if and only if $v = y$. The conclusion of the lemma follows from here. $\square$

Going back to the problem, apply the lemma to the quadruples $\{B, E, D, F_1\}$, $\{A, B, C, D\}$ and $\{A, E, C, F_2\}$. Assuming $OE \cdot OF_1 > p$, we obtain

$$OE \cdot OF_1 > p \;\Rightarrow\; OB \cdot OD < p \;\Rightarrow\; OA \cdot OC > p \;\Rightarrow\; OE \cdot OF_2 < p.$$

In other words, $OE \cdot OF_1 > p$ implies

$$OB \cdot OD < p < OA \cdot OC \quad \text{and} \quad OE \cdot OF_1 > p > OE \cdot OF_2.$$

Similarly, $OE \cdot OF_1 < p$ implies

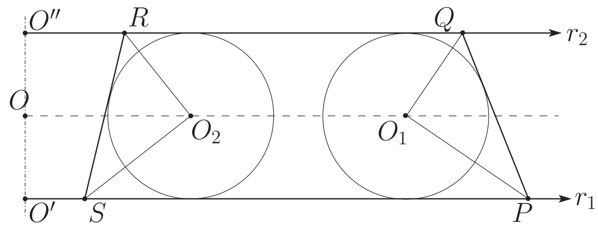
$$OB \cdot OD > p > OA \cdot OC \quad \text{and} \quad OE \cdot OF_1 < p < OE \cdot OF_2.$$

In these cases $F_1 \neq F_2$ and $OB \cdot OD \neq OA \cdot OC$, so the lines AB and CD are not parallel.

There remains the case $OE \cdot OF_1 = p$. Here the lemma leads to $OB \cdot OD = p = OA \cdot OC$ and $OE \cdot OF_1 = p = OE \cdot OF_2$. Therefore $F_1 = F_2$ and $AB \parallel CD$.

Comment. The conclusion is also true if BC and AD are parallel. One can prove a limit case of the lemma for the configuration shown in the figure below, where r_1 and r_2 are parallel rays starting at O' and O'', with $O'O'' \perp r_1, r_2$ and O the midpoint of $O'O''$. Two circles with centers O_1 and O_2 are inscribed in the strip between r_1 and r_2. The lines PQ and RS are tangent to the circles, with P, S on r_1, and Q, R on r_2, so that O, O_1 are on the same side of PQ and O, O_2 are on different sides of RS. Denote $s = OO_1 + OO_2$. Then exactly one of the following relations holds:

$$O'P + O''R < s < O''Q + O'S, \quad O'P + O''R > s > O''Q + O'S, \quad O'P + O''R = s = O''Q + O'S.$$

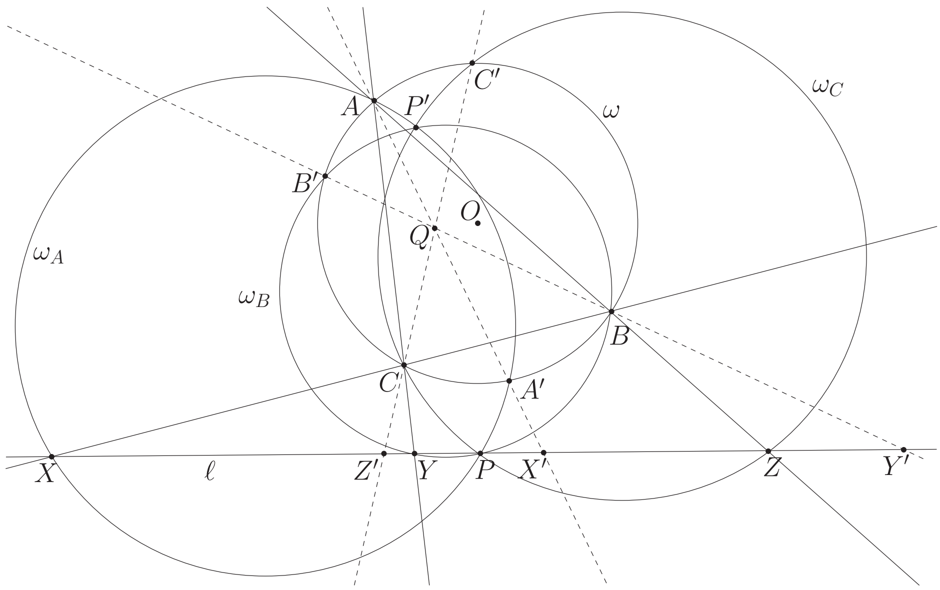


Once this is established, the proof of the original statement for $BC \parallel AD$ is analogous to the one in the intersecting case. One replaces products by sums of relevant segments.

G8. Let ABC be a triangle with circumcircle ω and ℓ a line without common points with ω. Denote by P the foot of the perpendicular from the center of ω to ℓ. The side-lines BC, CA, AB intersect ℓ at the points X, Y, Z different from P. Prove that the circumcircles of the triangles AXP, BYP and CZP have a common point different from P or are mutually tangent at P.

Solution 1. Let $\omega_A, \omega_B, \omega_C$ and ω be the circumcircles of triangles AXP, BYP, CZP and ABC respectively. The strategy of the proof is to construct a point Q with the same power with respect to the four circles. Then each of P and Q has the same power with respect to $\omega_A, \omega_B, \omega_C$ and hence the three circles are coaxial. In other words they have another common point P' or the three of them are tangent at P.

We first give a description of the point Q. Let $A' \neq A$ be the second intersection of ω and ω_A; define B' and C' analogously. We claim that AA', BB' and CC' have a common point. Once this claim is established, the point just constructed will be on the radical axes of the three pairs of circles $\{\omega, \omega_A\}, \{\omega, \omega_B\}, \{\omega, \omega_C\}$. Hence it will have the same power with respect to $\omega, \omega_A, \omega_B, \omega_C$.



We proceed to prove that AA', BB' and CC' intersect at one point. Let r be the circumradius of triangle ABC. Define the points X', Y', Z' as the intersections of AA', BB', CC' with ℓ. Observe that X', Y', Z' do exist. If AA' is parallel to ℓ then ω_A is tangent to ℓ; hence $X = P$ which is a contradiction. Similarly, BB' and CC' are not parallel to ℓ.

From the powers of the point X' with respect to the circles ω_A and ω we get

$$X'P \cdot (X'P + PX) = X'P \cdot X'X = X'A' \cdot X'A = X'O^2 - r^2,$$

hence

$$X'P \cdot PX = X'O^2 - r^2 - X'P^2 = OP^2 - r^2.$$

We argue analogously for the points Y' and Z', obtaining

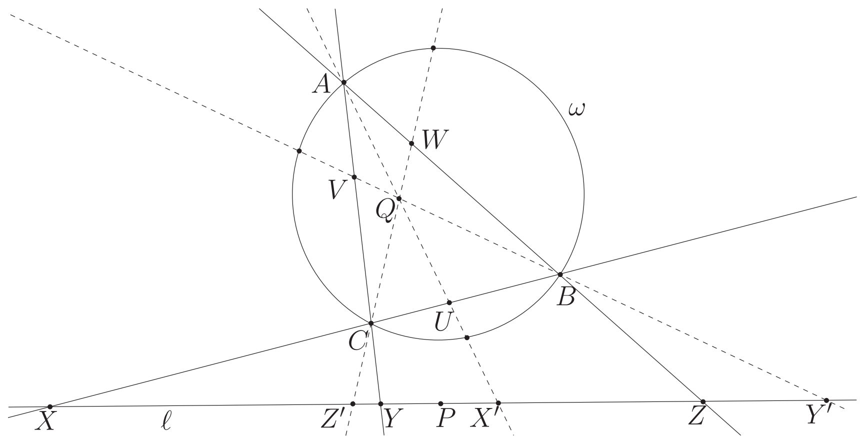
$$X'P \cdot PX = Y'P \cdot PY = Z'P \cdot PZ = OP^2 - r^2 = k^2. \tag{1}$$

In these computations all segments are regarded as directed segments. We keep the same convention for the sequel.

We prove that the lines AA', BB', CC' intersect at one point by Ceva's theorem. To avoid distracting remarks we interpret everything projectively, i. e. whenever two lines are parallel they meet at a point on the line at infinity.

Let U, V, W be the intersections of AA', BB', CC' with BC, CA, AB respectively. The idea is that although it is difficult to calculate the ratio $\frac{BU}{CU}$, it is easier to deal with the cross-ratio $\frac{BU}{CU}/\frac{BX}{CX}$ because we can send it to the line ℓ. With this in mind we apply MENELAUS' theorem to the triangle ABC and obtain $\frac{BX}{CX} \cdot \frac{CY}{AY} \cdot \frac{AZ}{BZ} = 1$. Hence Ceva's ratio can be expressed as

$$\frac{BU}{CU} \cdot \frac{CV}{AV} \cdot \frac{AW}{BW} = \frac{BU}{CU}\Big/\frac{BX}{CX} \cdot \frac{CV}{AV}\Big/\frac{CY}{AY} \cdot \frac{AW}{BW}\Big/\frac{AZ}{BZ}.$$



Project the line BC to ℓ from A. The cross-ratio between BC and UX equals the cross-ratio between ZY and $X'X$. Repeating the same argument with the lines CA and AB gives

$$\frac{BU}{CU} \cdot \frac{CV}{AV} \cdot \frac{AW}{BW} = \frac{ZX'}{YX'}\Big/\frac{ZX}{YX} \cdot \frac{XY'}{ZY'}\Big/\frac{XY}{ZY} \cdot \frac{YZ'}{XZ'}\Big/\frac{YZ}{XZ}$$

and hence

$$\frac{BU}{CU} \cdot \frac{CV}{AV} \cdot \frac{AW}{BW} = (-1) \cdot \frac{ZX'}{YX'} \cdot \frac{XY'}{ZY'} \cdot \frac{YZ'}{XZ'}.$$

The equations (1) reduce the problem to a straightforward computation on the line ℓ. For instance, the transformation $t \mapsto -k^2/t$ preserves cross-ratio and interchanges the points X, Y, Z with the points X', Y', Z'. Then

$$\frac{BU}{CU} \cdot \frac{CV}{AV} \cdot \frac{AW}{BW} = (-1) \cdot \frac{ZX'}{YX'}\Big/\frac{ZZ'}{YZ'} \cdot \frac{XY'}{ZY'}\Big/\frac{XZ'}{ZZ'} = -1.$$

We proved that CEVA's ratio equals -1, so AA', BB', CC' intersect at one point Q.

Comment 1. There is a nice projective argument to prove that AX', BY', CZ' intersect at one point. Suppose that ℓ and ω intersect at a pair of complex conjugate points D and E. Consider a projective transformation that takes D and E to $[i; 1, 0]$ and $[-i, 1, 0]$. Then ℓ is the line at infinity, and ω is a conic through the special points $[i; 1, 0]$ and $[-i, 1, 0]$, hence it is a circle. So one can assume that AX, BY, CZ are parallel to BC, CA, AB. The involution on ℓ taking X, Y, Z to X', Y', Z' and leaving D, E fixed is the involution changing each direction to its perpendicular one. Hence AX, BY, CZ are also perpendicular to AX', BY', CZ'.

It follows from the above that AX', BY', CZ' intersect at the orthocenter of triangle ABC.

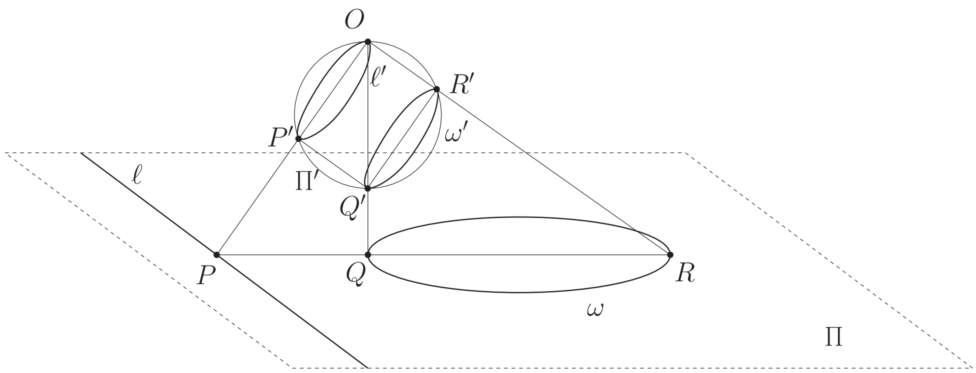
Comment 2. The restriction that the line ℓ does not intersect the circumcricle ω is unnecessary. The proof above works in general. In case ℓ intersects ω at D and E point P is the midpoint of DE, and some equations can be interpreted differently. For instance

$$X'P \cdot X'X = X'A' \cdot X'A = X'D \cdot X'E,$$

and hence the pairs $X'X$ and DE are harmonic conjugates. This means that X', Y', Z' are the harmonic conjugates of X, Y, Z with respect to the segment DE.

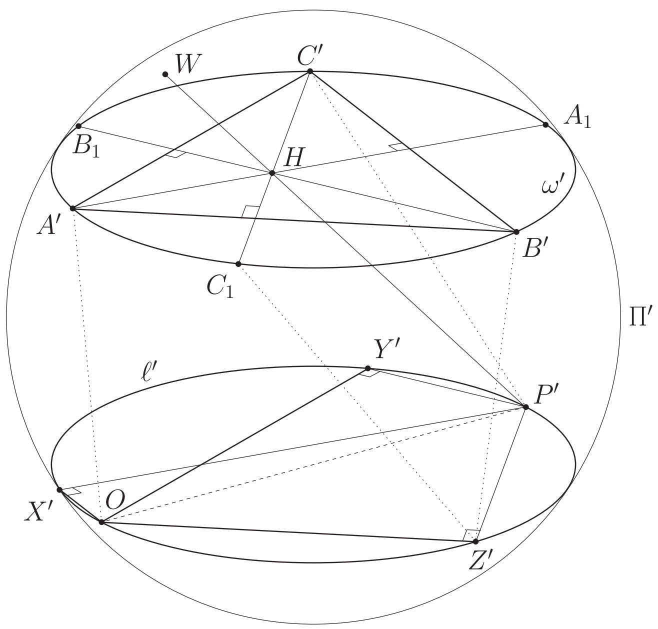
Solution 2. First we prove that there is an inversion in space that takes ℓ and ω to parallel circles on a sphere. Let QR be the diameter of ω whose extension beyond Q passes through P. Let Π be the plane carrying our objects. In space, choose a point O such that the line QO is perpendicular to Π and $\angle POR = 90^\circ$, and apply an inversion with pole O (the radius of the inversion does not matter). For any object $\mathcal{T}$ denote by $\mathcal{T}'$ the image of $\mathcal{T}$ under this inversion.

The inversion takes the plane Π to a sphere Π'. The lines in Π are taken to circles through O, and the circles in Π also are taken to circles on Π'.



Since the line ℓ and the circle ω are perpendicular to the plane OPQ, the circles ℓ' and ω' also are perpendicular to this plane. Hence, the planes of the circles ℓ' and ω' are parallel.

Now consider the circles $A'X'P'$, $B'Y'P'$ and $C'Z'P'$. We want to prove that either they have a common point (on Π'), different from P', or they are tangent to each other.



The point X' is the second intersection of the circles $B'C'O$ and ℓ', other than O. Hence, the lines OX' and $B'C'$ are coplanar. Moreover, they lie in the parallel planes of ℓ' and ω'. Therefore, OX' and $B'C'$ are parallel. Analogously, OY' and OZ' are parallel to $A'C'$ and $A'B'$.

Let A_1 be the second intersection of the circles $A'X'P'$ and ω', other than A'. The segments $A'A_1$ and $P'X'$ are coplanar, and therefore parallel. Now we know that $B'C'$ and $A'A_1$ are parallel to OX' and $X'P'$ respectively, but these two segments are perpendicular because OP' is a diameter in ℓ'. We found that $A'A_1$ and $B'C'$ are perpendicular, hence $A'A_1$ is the altitude in the triangle $A'B'C'$, starting from A.

Analogously, let B_1 and C_1 be the second intersections of ω' with the circles $B'P'Y'$ and $C'P'Z'$, other than B' and C' respectively. Then $B'B_1$ and $C'C_1$ are the other two altitudes in the triangle $A'B'C'$.

Let H be the orthocenter of the triangle $A'B'C'$. Let W be the second intersection of the line $P'H$ with the sphere Π', other than P'. The point W lies on the sphere Π', in the plane of the circle $A'P'X'$, so W lies on the circle $A'P'X'$. Similarly, W lies on the circles $B'P'Y'$ and $C'P'Z'$ as well; indeed W is the second common point of the three circles.

If the line $P'H$ is tangent to the sphere then W coincides with P', and $P'H$ is the common tangent of the three circles.

Number Theory

N1. Call admissible a set A of integers that has the following property:

If $x, y \in A$ (possibly $x = y$) then $x^2 + kxy + y^2 \in A$ for every integer k.

Determine all pairs m, n of nonzero integers such that the only admissible set containing both m and n is the set of all integers.

Solution. A pair of integers m, n fulfills the condition if and only if $gcd(m, n) = 1$. Suppose that $gcd(m, n) = d > 1$. The set

$$A = \{\ldots, -2d, -d, 0, d, 2d, \ldots\}$$

is admissible, because if d divides x and y then it divides $x^2 + kxy + y^2$ for every integer k. Also $m, n \in A$ and $A \neq \mathbb{Z}$.

Now let $gcd(m, n) = 1$, and let A be an admissible set containing m and n. We use the following observations to prove that $A = \mathbb{Z}$:

(i) $kx^2 \in A$ for every $x \in A$ and every integer k.

(ii) $(x + y)^2 \in A$ for all $x, y \in A$.

To justify (i) let $y = x$ in the definition of an admissible set; to justify (ii) let $k = 2$.

Since $gcd(m, n) = 1$, we also have $gcd(m^2, n^2) = 1$. Hence one can find integers a, b such that $am^2 + bn^2 = 1$. It follows from (i) that $am^2 \in A$ and $bn^2 \in A$. Now we deduce from (ii) that $1 = (am^2 + bn^2)^2 \in A$. But if $1 \in A$ then (i) implies $k \in A$ for every integer k.

N2. Find all triples (x, y, z) of positive integers such that $x \le y \le z$ and

$$x^3(y^3 + z^3) = 2012(xyz + 2).$$

Solution. First note that x divides $2012 \cdot 2 = 2^3 \cdot 503$. If $503 \mid x$ then the right-hand side of the equation is divisible by 503^3, and it follows that $503^2 \mid xyz + 2$. This is false as $503 \mid x$. Hence $x = 2^m$ with $m \in \{0, 1, 2, 3\}$. If $m \ge 2$ then $2^6 \mid 2012(xyz + 2)$. However the highest powers of 2 dividing 2012 and $xyz + 2 = 2^m yz + 2$ are 2^2 and 2^1 respectively. So $x = 1$ or $x = 2$, yielding the two equations

$$y^3 + z^3 = 2012(yz + 2), \qquad \text{and} \qquad y^3 + z^3 = 503(yz + 1).$$

In both cases the prime $503 = 3 \cdot 167 + 2$ divides $y^3 + z^3$. We claim that $503 \mid y + z$. This is clear if $503 \mid y$, so let $503 \nmid y$ and $503 \nmid z$. Then $y^{502} \equiv z^{502} \pmod{503}$ by Fermat's little theorem. On the other hand $y^3 \equiv -z^3 \pmod{503}$ implies $y^{3\cdot 167} \equiv -z^{3\cdot 167} \pmod{503}$, i. e. $y^{501} \equiv -z^{501} \pmod{503}$. It follows that $y \equiv -z \pmod{503}$ as claimed.

Therefore $y + z = 503k$ with $k \ge 1$. In view of $y^3 + z^3 = (y + z)\big((y - z)^2 + yz\big)$ the two equations take the form

$$k(y - z)^2 + (k - 4)yz = 8, \tag{1}$$
$$k(y - z)^2 + (k - 1)yz = 1. \tag{2}$$

In (1) we have $(k - 4)yz \le 8$, which implies $k \le 4$. Indeed if $k > 4$ then $1 \le (k - 4)yz \le 8$, so that $y \le 8$ and $z \le 8$. This is impossible as $y + z = 503k \ge 503$. Note next that $y^3 + z^3$ is even in the first equation. Hence $y + z = 503k$ is even too, meaning that k is even. Thus $k = 2$ or $k = 4$. Clearly (1) has no integer solutions for $k = 4$. If $k = 2$ then (1) takes the form $(y + z)^2 - 5yz = 4$. Since $y + z = 503k = 503 \cdot 2$, this leads to $5yz = 503^2 \cdot 2^2 - 4$. However $503^2 \cdot 2^2 - 4$ is not a multiple of 5. Therefore (1) has no integer solutions.

Equation (2) implies $0 \le (k - 1)yz \le 1$, so that $k = 1$ or $k = 2$. Also $0 \le k(y - z)^2 \le 1$, hence $k = 2$ only if $y = z$. However then $y = z = 1$, which is false in view of $y + z \ge 503$. Therefore $k = 1$ and (2) takes the form $(y - z)^2 = 1$, yielding $z - y = |y - z| = 1$. Combined with $k = 1$ and $y + z = 503k$, this leads to $y = 251$, $z = 252$.

In summary the triple $(2, 251, 252)$ is the only solution.

N3. Determine all integers $m \geq 2$ such that every n with $\frac{m}{3} \leq n \leq \frac{m}{2}$ divides the binomial coefficient $\binom{n}{m-2n}$.

Solution. The integers in question are all prime numbers.

First we check that all primes satisfy the condition, and even a stronger one. Namely, if p is a prime then every n with $1 \leq n \leq \frac{p}{2}$ divides $\binom{n}{p-2n}$. This is true for $p = 2$ where $n = 1$ is the only possibility. For an odd prime p take $n \in [1, \frac{p}{2}]$ and consider the following identity of binomial coefficients:

$$(p-2n) \cdot \binom{n}{p-2n} = n \cdot \binom{n-1}{p-2n-1}.$$

Since $p \geq 2n$ and p is odd, all factors are non-zero. If $d = gcd(p-2n, n)$ then d divides p, but $d \leq n < p$ and hence $d = 1$. It follows that $p-2n$ and n are relatively prime, and so the factor n in the right-hand side divides the binomial coefficient $\binom{n}{p-2n}$.

Next we show that no composite number m has the stated property. Consider two cases.

- If $m = 2k$ with $k > 1$, pick $n = k$. Then $\frac{m}{3} \leq n \leq \frac{m}{2}$ but $\binom{n}{m-2n} = \binom{k}{0} = 1$ is not divisible by $k > 1$.

- If m is odd then there exist an odd prime p and an integer $k \geq 1$ with $m = p(2k+1)$. Pick $n = pk$, then $\frac{m}{3} \leq n \leq \frac{m}{2}$ by $k \geq 1$. However

$$\frac{1}{n}\binom{n}{m-2n} = \frac{1}{pk}\binom{pk}{p} = \frac{(pk-1)(pk-2)\cdots(pk-(p-1))}{p!}$$

is not an integer, because p divides the denominator but not the numerator.

N4. An integer a is called friendly if the equation $(m^2+n)(n^2+m)=a(m-n)^3$ has a solution over the positive integers.

a) Prove that there are at least 500 friendly integers in the set $\{1,2,\ldots,2012\}$.

b) Decide whether $a=2$ is friendly.

Solution. a) Every a of the form $a=4k-3$ with $k\geq 2$ is friendly. Indeed the numbers $m=2k-1>0$ and $n=k-1>0$ satisfy the given equation with $a=4k-3$:

$$(m^2+n)(n^2+m)=\left((2k-1)^2+(k-1)\right)\left((k-1)^2+(2k-1)\right)=(4k-3)k^3=a(m-n)^3.$$

Hence $5,9,\ldots,2009$ are friendly and so $\{1,2,\ldots,2012\}$ contains at least 502 friendly numbers.

b) We show that $a=2$ is not friendly. Consider the equation with $a=2$ and rewrite its left-hand side as a difference of squares:

$$\frac{1}{4}\left((m^2+n+n^2+m)^2-(m^2+n-n^2-m)^2\right)=2(m-n)^3.$$

Since $m^2+n-n^2-m=(m-n)(m+n-1)$, we can further reformulate the equation as

$$(m^2+n+n^2+m)^2=(m-n)^2\left(8(m-n)+(m+n-1)^2\right).$$

It follows that $8(m-n)+(m+n-1)^2$ is a perfect square. Clearly $m>n$, hence there is an integer $s\geq 1$ such that

$$(m+n-1+2s)^2=8(m-n)+(m+n-1)^2.$$

Subtracting the squares gives $s(m+n-1+s)=2(m-n)$. Since $m+n-1+s>m-n$, we conclude that $s<2$. Therefore the only possibility is $s=1$ and $m=3n$. However then the left-hand side of the given equation (with $a=2$) is greater than $m^3=27n^3$, whereas its right-hand side equals $16n^3$. The contradiction proves that $a=2$ is not friendly.

Comment. A computer search shows that there are 561 friendly numbers in $\{1,2,\ldots,2012\}$.

N5. For a nonnegative integer n define $rad(n) = 1$ if $n = 0$ or $n = 1$, and $rad(n) = p_1 p_2 \cdots p_k$ where $p_1 < p_2 < \cdots < p_k$ are all prime factors of n. Find all polynomials $f(x)$ with nonnegative integer coefficients such that $rad(f(n))$ divides $rad(f(n^{rad(n)}))$ for every nonnegative integer n.

Solution 1. We are going to prove that $f(x) = ax^m$ for some nonnegative integers a and m. If $f(x)$ is the zero polynomial we are done, so assume that $f(x)$ has at least one positive coefficient. In particular $f(1) > 0$.

Let p be a prime number. The condition is that $f(n) \equiv 0 \pmod{p}$ implies

$$f(n^{rad(n)}) \equiv 0 \pmod{p}. \tag{1}$$

Since $rad(n^{rad(n)^k}) = rad(n)$ for all k, repeated applications of the preceding implication show that if p divides $f(n)$ then

$$f(n^{rad(n)^k}) \equiv 0 \pmod{p} \quad \text{for all } k.$$

The idea is to construct a prime p and a positive integer n such that $p-1$ divides n and p divides $f(n)$. In this case, for k large enough $p-1$ divides $rad(n)^k$. Hence if $(p,n) = 1$ then $n^{rad(n)^k} \equiv 1 \pmod{p}$ by FERMAT's little theorem, so that

$$f(1) \equiv f(n^{rad(n)^k}) \equiv 0 \pmod{p}. \tag{2}$$

Suppose that $f(x) = g(x)x^m$ with $g(0) \neq 0$. Let t be a positive integer, p any prime factor of $g(-t)$ and $n = (p-1)t$. So $p-1$ divides n and $f(n) = f((p-1)t) \equiv f(-t) \equiv 0 \pmod{p}$, hence either $(p,n) > 1$ or (2) holds. If $(p,(p-1)t) > 1$ then p divides t and $g(0) \equiv g(-t) \equiv 0 \pmod{p}$, meaning that p divides $g(0)$.

In conclusion we proved that each prime factor of $g(-t)$ divides $g(0)f(1) \neq 0$, and thus the set of prime factors of $g(-t)$ when t ranges through the positive integers is finite. This is known to imply that $g(x)$ is a constant polynomial, and so $f(x) = ax^m$.

Solution 2. Let $f(x)$ be a polynomial with integer coefficients (not necessarily nonnegative) such that $rad(f(n))$ divides $rad(f(n^{rad(n)}))$ for any nonnegative integer n. We give a complete description of all polynomials with this property. More precisely, we claim that if $f(x)$ is such a polynomial and ξ is a root of $f(x)$ then so is ξ^d for every positive integer d.

Therefore each root of $f(x)$ is zero or a root of unity. In particular, if a root of unity ξ is a root of $f(x)$ then $1 = \xi^d$ is a root too (for some positive integer d). In the original problem $f(x)$ has nonnegative coefficients. Then either $f(x)$ is the zero polynomial or $f(1) > 0$ and $\xi = 0$ is the only possible root. In either case $f(x) = ax^m$ with a and m nonnegative integers.

To prove the claim let ξ be a root of $f(x)$, and let $g(x)$ be an irreducible factor of $f(x)$ such that $g(\xi) = 0$. If 0 or 1 are roots of $g(x)$ then either $\xi = 0$ or $\xi = 1$ (because $g(x)$ is irreducible) and we are done. So assume that $g(0), g(1) \neq 0$. By decomposing d as a product of prime numbers, it is enough to consider the case $d = p$ prime. We argue for $p = 2$. Since $rad(2^k) = 2$ for every k, we have

$$rad(f(2^k)) \mid rad(f(2^{2k})).$$

Now we prove that $g(x)$ divides $f(x^2)$. Suppose that this is not the case. Then, since $g(x)$ is irreducible, there are integer-coefficient polynomials $a(x)$, $b(x)$ and an integer N such that

$$a(x)g(x) + b(x)f(x^2) = N. \tag{3}$$

Each prime factor p of $g(2^k)$ divides $f(2^k)$, so by $rad(f(2^k)) | rad(f(2^{2k}))$ it also divides $f(2^{2k})$. From the equation above with $x = 2^k$ it follows that p divides N.

In summary, each prime divisor of $g(2^k)$ divides N, for all $k \geq 0$. Let $p_1, \ldots, p_n$ be the odd primes dividing N, and suppose that

$$g(1) = 2^{\alpha} p_1^{\alpha_1} \cdots p_n^{\alpha_n}.$$

If k is divisible by $\varphi(p_1^{\alpha_1+1} \cdots p_n^{\alpha_n+1})$ then

$$2^k \equiv 1 \pmod{p_1^{\alpha_1+1} \cdots p_n^{\alpha_n+1}},$$

yielding

$$g(2^k) \equiv g(1) \pmod{p_1^{\alpha_1+1} \cdots p_n^{\alpha_n+1}}.$$

It follows that for each i the maximal power of p_i dividing $g(2^k)$ and $g(1)$ is the same, namely $p_i^{\alpha_i}$. On the other hand, for large enough k, the maximal power of 2 dividing $g(2^k)$ and $g(0) \neq 0$ is the same. From the above, for k divisible by $\varphi(p_1^{\alpha_1+1} \cdots p_n^{\alpha_n+1})$ and large enough, we obtain that $g(2^k)$ divides $g(0) \cdot g(1)$. This is impossible because $g(0), g(1) \neq 0$ are fixed and $g(2^k)$ is arbitrarily large.

In conclusion, $g(x)$ divides $f(x^2)$. Recall that ξ is a root of $f(x)$ such that $g(\xi) = 0$; then $f(\xi^2) = 0$, i. e. ξ^2 is a root of $f(x)$.

Likewise if ξ is a root of $f(x)$ and p an arbitrary prime then ξ^p is a root too. The argument is completely analogous, in the proof above just replace 2 by p and "odd prime" by "prime different from p."

Comment. The claim in the second solution can be proved by varying $n \pmod{p}$ in (1). For instance, we obtain

$$f(n^{rad(n+pk)}) \equiv 0 \pmod{p}$$

for every positive integer k. One can prove that if $(n,p) = 1$ then $rad(n+pk)$ runs through all residue classes $r \pmod{p-1}$ with $(r, p-1)$ squarefree. Hence if $f(n) \equiv 0 \pmod{p}$ then $f(n^r) \equiv 0 \pmod{p}$ for all integers r. This implies the claim by an argument leading to the identity (3).

N6. Let x and y be positive integers. If $x^{2^n}-1$ is divisible by $2^n y+1$ for every positive integer n, prove that $x=1$.

Solution. First we prove the following fact: For every positive integer y there exist infinitely many primes $p \equiv 3 \pmod 4$ such that p divides some number of the form $2^n y+1$.

Clearly it is enough to consider the case y odd. Let

$$2y+1=p_1^{e_1}\cdots p_r^{e_r}$$

be the prime factorization of $2y+1$. Suppose on the contrary that there are finitely many primes $p_{r+1},\dots,p_{r+s} \equiv 3 \pmod 4$ that divide some number of the form $2^n y+1$ but do not divide $2y+1$.

We want to find an n such that $p_i^{e_i} || 2^n y+1$ for $1 \le i \le r$ and $p_i \nmid 2^n y+1$ for $r+1 \le i \le r+s$. For this it suffices to take

$$n=1+\varphi(p_1^{e_1+1}\cdots p_r^{e_r+1} p_{r+1}^1 \cdots p_{r+s}^1),$$

because then

$$2^n y+1 \equiv 2y+1 \pmod{p_1^{e_1+1}\cdots p_r^{e_r+1} p_{r+1}^1 \cdots p_{r+s}^1}.$$

The last congruence means that $p_1^{e_1},\dots,p_r^{e_r}$ divide exactly $2^n y+1$ and no prime $p_{r+1},\dots,p_{r+s}$ divides $2^n y+1$. It follows that the prime factorization of $2^n y+1$ consists of the prime powers $p_1^{e_1},\dots,p_r^{e_r}$ and powers of primes $\equiv 1 \pmod 4$. Because y is odd, we obtain

$$2^n y+1 \equiv p_1^{e_1}\cdots p_r^{e_r} \equiv 2y+1 \equiv 3 \pmod 4.$$

This is a contradiction since $n>1$, and so $2^n y+1 \equiv 1 \pmod 4$.

Now we proceed to the problem. If p is a prime divisor of $2^n y+1$ the problem statement implies that $x^d \equiv 1 \pmod p$ for $d=2^n$. By FERMAT's little theorem the same congruence holds for $d=p-1$, so it must also hold for $d=(2^n,p-1)$. For $p \equiv 3 \pmod 4$ we have $(2^n,p-1)=2$, therefore in this case $x^2 \equiv 1 \pmod p$.

In summary, we proved that every prime $p \equiv 3 \pmod 4$ that divides some number of the form $2^n y+1$ also divides x^2-1. This is possible only if $x=1$, otherwise by the above x^2-1 would be a positive integer with infinitely many prime factors.

Comment. For each x and each odd prime p the maximal power of p dividing $x^{2^n}-1$ for some n is bounded and hence the same must be true for the numbers $2^n y+1$. We infer that p^2 divides $2^{p-1}-1$ for each prime divisor p of $2^n y+1$. However trying to reach a contradiction with this conclusion alone seems hopeless, since it is not even known if there are infinitely many primes p *without* this property.

N7. Find all $n \in \mathbb{N}$ for which there exist nonnegative integers $a_1, a_2, \ldots, a_n$ such that

$$\frac{1}{2^{a_1}} + \frac{1}{2^{a_2}} + \cdots + \frac{1}{2^{a_n}} = \frac{1}{3^{a_1}} + \frac{2}{3^{a_2}} + \cdots + \frac{n}{3^{a_n}} = 1.$$

Solution. Such numbers $a_1, a_2, \ldots, a_n$ exist if and only if $n \equiv 1 \pmod 4$ or $n \equiv 2 \pmod 4$.

Let $\sum_{k=1}^{n} \frac{k}{3^{a_k}} = 1$ with $a_1, a_2, \ldots, a_n$ nonnegative integers. Then $1 \cdot x_1 + 2 \cdot x_2 + \cdots + n \cdot x_n = 3^a$ with $x_1, \ldots, x_n$ powers of 3 and $a \geq 0$. The right-hand side is odd, and the left-hand side has the same parity as $1 + 2 + \cdots + n$. Hence the latter sum is odd, which implies $n \equiv 1, 2 \pmod 4$. Now we prove the converse.

Call *feasible* a sequence $b_1, b_2, \ldots, b_n$ if there are nonnegative integers $a_1, a_2, \ldots, a_n$ such that

$$\frac{1}{2^{a_1}} + \frac{1}{2^{a_2}} + \cdots + \frac{1}{2^{a_n}} = \frac{b_1}{3^{a_1}} + \frac{b_2}{3^{a_2}} + \cdots + \frac{b_n}{3^{a_n}} = 1.$$

Let b_k be a term of a feasible sequence $b_1, b_2, \ldots, b_n$ with exponents $a_1, a_2, \ldots, a_n$ like above, and let u, v be nonnegative integers with sum $3b_k$. Observe that

$$\frac{1}{2^{a_k+1}} + \frac{1}{2^{a_k+1}} = \frac{1}{2^{a_k}} \qquad \text{and} \qquad \frac{u}{3^{a_k+1}} + \frac{v}{3^{a_k+1}} = \frac{b_k}{3^{a_k}}.$$

It follows that the sequence $b_1, \ldots, b_{k-1}, u, v, b_{k+1}, \ldots, b_n$ is feasible. The exponents a_i are the same for the unchanged terms b_i, $i \neq k$; the new terms u, v have exponents $a_k + 1$.

We state the conclusion in reverse. If two terms u, v of a sequence are replaced by one term $\frac{u+v}{3}$ and the obtained sequence is feasible, then the original sequence is feasible too. Denote by α_n the sequence $1, 2, \ldots, n$. To show that α_n is feasible for $n \equiv 1, 2 \pmod 4$, we transform it by $n - 1$ replacements $\{u, v\} \mapsto \frac{u+v}{3}$ to the one-term sequence α_1. The latter is feasible, with $a_1 = 0$. Note that if m and $2m$ are terms of a sequence then $\{m, 2m\} \mapsto m$, so $2m$ can be ignored if necessary.

Let $n \geq 16$. We prove that α_n can be reduced to α_{n-12} by 12 operations. Write $n = 12k + r$ where $k \geq 1$ and $0 \leq r \leq 11$. If $0 \leq r \leq 5$ then the last 12 terms of α_n can be partitioned into 2 singletons $\{12k - 6\}$, $\{12k\}$ and the following 5 pairs:

$$\{12k - 6 - i, 12k - 6 + i\}, i = 1, \ldots, 5 - r; \qquad \{12k - j, 12k + j\}, j = 1, \ldots, r.$$

(There is only one kind of pairs if $r \in \{0, 5\}$.) One can ignore $12k - 6$ and $12k$ since α_n contains $6k - 3$ and $6k$. Furthermore the 5 operations $\{12k - 6 - i, 12k - 6 + i\} \mapsto 8k - 4$ and $\{12k - j, 12k + j\} \mapsto 8k$ remove the 10 terms in the pairs and bring in 5 new terms equal to $8k - 4$ or $8k$. All of these can be ignored too as $4k - 2$ and $4k$ are still present in the sequence. Indeed $4k \leq n - 12$ is equivalent to $8k \geq 12 - r$, which is true for $r \in \{4, 5\}$. And if $r \in \{0, 1, 2, 3\}$ then $n \geq 16$ implies $k \geq 2$, so $8k \geq 12 - r$ also holds. Thus α_n reduces to α_{n-12}.

The case $6 \leq r \leq 11$ is analogous. Consider the singletons $\{12k\}$, $\{12k + 6\}$ and the 5 pairs

$$\{12k - i, 12k + i\}, i = 1, \ldots, 11 - r; \qquad \{12k + 6 - j, 12k + 6 + j\}, j = 1, \ldots, r - 6.$$

Ignore the singletons like before, then remove the pairs via operations $\{12k - i, 12k + i\} \mapsto 8k$ and $\{12k + 6 - j, 12k + 6 + j\} \mapsto 8k + 4$. The 5 newly-appeared terms $8k$ and $8k + 4$ can be ignored too since $4k + 2 \leq n - 12$ (this follows from $k \geq 1$ and $r \geq 6$). We obtain α_{n-12} again.

The problem reduces to $2 \leq n \leq 15$. In fact $n \in \{2, 5, 6, 9, 10, 13, 14\}$ by $n \equiv 1, 2 \pmod 4$. The cases $n = 2, 6, 10, 14$ reduce to $n = 1, 5, 9, 13$ respectively because the last even term of α_n can be ignored. For $n = 5$ apply $\{4, 5\} \mapsto 3$, then $\{3, 3\} \mapsto 2$, then ignore the 2 occurrences of 2. For $n = 9$ ignore 6 first, then apply $\{5, 7\} \mapsto 4$, $\{4, 8\} \mapsto 4$, $\{3, 9\} \mapsto 4$. Now ignore the 3 occurrences of 4, then ignore 2. Finally $n = 13$ reduces to $n = 10$ by $\{11, 13\} \mapsto 8$ and ignoring 8 and 12. The proof is complete.

N8. Prove that for every prime $p > 100$ and every integer r there exist two integers a and b such that p divides $a^2 + b^5 - r$.

Solution 1. Throughout the solution, all congruence relations are meant modulo p.

Fix p, and let $\mathcal{P} = \{0, 1, \dots, p-1\}$ be the set of residue classes modulo p. For every $r \in \mathcal{P}$, let $S_r = \{(a, b) \in \mathcal{P} \times \mathcal{P} \colon a^2 + b^5 \equiv r\}$, and let $s_r = |S_r|$. Our aim is to prove $s_r > 0$ for all $r \in \mathcal{P}$.

We will use the well-known fact that for every residue class $r \in \mathcal{P}$ and every positive integer k, there are at most k values $x \in \mathcal{P}$ such that $x^k \equiv r$.

Lemma. Let N be the number of quadruples $(a, b, c, d) \in \mathcal{P}^4$ for which $a^2 + b^5 \equiv c^2 + d^5$. Then

$$N = \sum_{r \in \mathcal{P}} s_r^2 \tag{a}$$

and

$$N \leq p(p^2 + 4p - 4). \tag{b}$$

Proof. (a) For each residue class r there exist exactly s_r pairs (a, b) with $a^2 + b^5 \equiv r$ and s_r pairs (c, d) with $c^2 + d^5 \equiv r$. So there are s_r^2 quadruples with $a^2 + b^5 \equiv c^2 + d^5 \equiv r$. Taking the sum over all $r \in \mathcal{P}$, the statement follows.

(b) Choose an arbitrary pair $(b, d) \in \mathcal{P}$ and look for the possible values of a, c.

1. Suppose that $b^5 \equiv d^5$, and let k be the number of such pairs (b, d). The value b can be chosen in p different ways. For $b \equiv 0$ only $d = 0$ has this property; for the nonzero values of b there are at most 5 possible values for d. So we have $k \leq 1 + 5(p-1) = 5p - 4$.

The values a and c must satisfy $a^2 \equiv c^2$, so $a \equiv \pm c$, and there are exactly $2p - 1$ such pairs (a, c).

2. Now suppose $b^5 \not\equiv d^5$. In this case a and c must be distinct. By $(a-c)(a+c) = d^5 - b^5$, the value of $a - c$ uniquely determines $a + c$ and thus a and c as well. Hence, there are $p - 1$ suitable pairs (a, c).

Thus, for each of the k pairs (b, d) with $b^5 \equiv d^5$ there are $2p - 1$ pairs (a, c), and for each of the other $p^2 - k$ pairs (b, d) there are $p - 1$ pairs (a, c). Hence,

$$N = k(2p-1) + (p^2 - k)(p-1) = p^2(p-1) + kp \leq p^2(p-1) + (5p-4)p = p(p^2 + 4p - 4). \quad \square$$

To prove the statement of the problem, suppose that $S_r = \emptyset$ for some $r \in \mathcal{P}$; obviously $r \not\equiv 0$. Let $T = \{x^{10} \colon x \in \mathcal{P} \setminus \{0\}\}$ be the set of nonzero 10th powers modulo p. Since each residue class is the 10th power of at most 10 elements in $\mathcal{P}$, we have $|T| \geq \frac{p-1}{10} \geq 4$ by $p > 100$.

For every $t \in T$, we have $S_{tr} = \emptyset$. Indeed, if $(x, y) \in S_{tr}$ and $t \equiv z^{10}$ then

$$(z^{-5}x)^2 + (z^{-2}y)^5 \equiv t^{-1}(x^2 + y^5) \equiv r,$$

so $(z^{-5}x, z^{-2}y) \in S_r$. So, there are at least $\frac{p-1}{10} \geq 4$ empty sets among $S_1, \dots, S_{p-1}$, and there are at most $p - 4$ nonzero values among $s_0, s_2, \dots, s_{p-1}$. Then by the AM-QM inequality we obtain

$$N = \sum_{r \in \mathcal{P} \setminus rT} s_r^2 \geq \frac{1}{p-4} \left(\sum_{r \in \mathcal{P} \setminus rT} s_r \right)^2 = \frac{|\mathcal{P} \times \mathcal{P}|^2}{p-4} = \frac{p^4}{p-4} > p(p^2 + 4p - 4),$$

which is impossible by the lemma.

Solution 2. If $5 \nmid p-1$, then all modulo p residue classes are complete fifth powers and the statement is trivial. So assume that $p = 10k+1$ where $k \geq 10$. Let g be a primitive root modulo p.

We will use the following facts:

(F1) If some residue class x is not quadratic then $x^{(p-1)/2} \equiv -1 \pmod{p}$.

(F2) For every integer d, as a simple corollary of the summation formula for geometric progressions,
$$\sum_{i=0}^{2k-1} g^{5di} \equiv \begin{cases} 2k & \text{if } 2k \mid d \\ 0 & \text{if } 2k \nmid d \end{cases} \pmod{p}.$$

Suppose that, contrary to the statement, some modulo p residue class r cannot be expressed as a^2+b^5. Of course $r \not\equiv 0 \pmod{p}$. By (F1) we have $(r-b^5)^{(p-1)/2} = (r-b^5)^{5k} \equiv -1 \pmod{p}$ for all residue classes b.

For $t = 1, 2 \dots, k-1$ consider the sums
$$S(t) = \sum_{i=0}^{2k-1} \left(r - g^{5i}\right)^{5k} g^{5ti}.$$

By the indirect assumption and (F2),
$$S(t) = \sum_{i=0}^{2k-1} \left(r - (g^i)^5\right)^{5k} g^{5ti} \equiv \sum_{i=0}^{2k-1} (-1) g^{5ti} \equiv -\sum_{i=0}^{2k-1} g^{5ti} \equiv 0 \pmod{p}$$

because $2k$ cannot divide t.

On the other hand, by the binomial theorem,
$$S(t) = \sum_{i=0}^{2k-1} \left(\sum_{j=0}^{5k} \binom{5k}{j} r^{5k-j} \left(-g^{5i}\right)^j \right) g^{5ti} = \sum_{j=0}^{5k} (-1)^j \binom{5k}{j} r^{5k-j} \left(\sum_{i=0}^{2k-1} g^{5(j+t)i} \right) \equiv$$
$$\equiv \sum_{j=0}^{5k} (-1)^j \binom{5k}{j} r^{5k-j} \begin{cases} 2k & \text{if } 2k \mid j+t \\ 0 & \text{if } 2k \nmid j+t \end{cases} \pmod{p}.$$

Since $1 \leq j+t < 6k$, the number $2k$ divides $j+t$ only for $j = 2k-t$ and $j = 4k-t$. Hence,
$$0 \equiv S(t) \equiv (-1)^t \left(\binom{5k}{2k-t} r^{3k+t} + \binom{5k}{4k-t} r^{k+t} \right) \cdot 2k \pmod{p},$$
$$\binom{5k}{2k-t} r^{2k} + \binom{5k}{4k-t} \equiv 0 \pmod{p}.$$

Taking this for $t = 1, 2$ and eliminating r, we get
$$\begin{aligned} 0 &\equiv \binom{5k}{2k-2}\left(\binom{5k}{2k-1} r^{2k} + \binom{5k}{4k-1}\right) - \binom{5k}{2k-1}\left(\binom{5k}{2k-2} r^{2k} + \binom{5k}{4k-2}\right) \\ &= \binom{5k}{2k-2}\binom{5k}{4k-1} - \binom{5k}{2k-1}\binom{5k}{4k-2} \\ &= \frac{(5k)!^2}{(2k-1)!(3k+2)!(4k-1)!(k+2)!}\Big((2k-1)(k+2) - (3k+2)(4k-1)\Big) \\ &= \frac{-(5k)!^2 \cdot 2k(5k+1)}{(2k-1)!(3k+2)!(4k-1)!(k+2)!} \pmod{p}. \end{aligned}$$

But in the last expression none of the numbers is divisible by $p = 10k+1$, a contradiction.

Comment 1. The argument in the second solution is valid whenever $k \geq 3$, that is for all primes $p = 10k + 1$ except $p = 11$. This is an exceptional case when the statement is not true; $r = 7$ cannot be expressed as desired.

Comment 2. The statement is true in a more general setting: for every positive integer n, for all sufficiently large p, each residue class modulo p can be expressed as $a^2 + b^n$. Choosing $t = 3$ would allow using the Cauchy-Davenport theorem (together with some analysis on the case of equality).

In the literature more general results are known. For instance, the statement easily follows from the *Hasse-Weil bound*.

Shortlisted Problems with Solutions

54$^{\text{th}}$ International Mathematical Olympiad

Santa Marta, Colombia 2013

Note of Confidentiality

The Shortlisted Problems should be kept strictly confidential until IMO 2014.

Contributing Countries

The Organizing Committee and the Problem Selection Committee of IMO 2013 thank the following 50 countries for contributing 149 problem proposals.

Argentina, Armenia, Australia, Austria, Belgium, Belarus, Brazil, Bulgaria, Croatia, Cyprus, Czech Republic, Denmark, El Salvador, Estonia, Finland, France, Georgia, Germany, Greece, Hungary, India, Indonesia, Iran, Ireland, Israel, Italy, Japan, Latvia, Lithuania, Luxembourg, Malaysia, Mexico, Netherlands, Nicaragua, Pakistan, Panama, Poland, Romania, Russia, Saudi Arabia, Serbia, Slovenia, Sweden, Switzerland, Tajikistan, Thailand, Turkey, U.S.A., Ukraine, United Kingdom

Problem Selection Committee

Federico Ardila (chairman)
Ilya I. Bogdanov
Géza Kós
Carlos Gustavo Tamm de Araújo Moreira (Gugu)
Christian Reiher

Problems

Algebra

A1. Let n be a positive integer and let $a_1, \ldots, a_{n-1}$ be arbitrary real numbers. Define the sequences $u_0, \ldots, u_n$ and $v_0, \ldots, v_n$ inductively by $u_0 = u_1 = v_0 = v_1 = 1$, and

$$u_{k+1} = u_k + a_k u_{k-1}, \qquad v_{k+1} = v_k + a_{n-k} v_{k-1} \qquad \text{for } k = 1, \ldots, n-1.$$

Prove that $u_n = v_n$.

(France)

A2. Prove that in any set of 2000 distinct real numbers there exist two pairs $a > b$ and $c > d$ with $a \neq c$ or $b \neq d$, such that

$$\left| \frac{a-b}{c-d} - 1 \right| < \frac{1}{100000}.$$

(Lithuania)

A3. Let $\mathbb{Q}_{>0}$ be the set of positive rational numbers. Let $f \colon \mathbb{Q}_{>0} \to \mathbb{R}$ be a function satisfying the conditions

$$f(x)f(y) \geqslant f(xy) \quad \text{and} \quad f(x+y) \geqslant f(x) + f(y)$$

for all $x, y \in \mathbb{Q}_{>0}$. Given that $f(a) = a$ for some rational $a > 1$, prove that $f(x) = x$ for all $x \in \mathbb{Q}_{>0}$.

(Bulgaria)

A4. Let n be a positive integer, and consider a sequence $a_1, a_2, \ldots, a_n$ of positive integers. Extend it periodically to an infinite sequence $a_1, a_2, \ldots$ by defining $a_{n+i} = a_i$ for all $i \geqslant 1$. If

$$a_1 \leqslant a_2 \leqslant \cdots \leqslant a_n \leqslant a_1 + n$$

and

$$a_{a_i} \leqslant n + i - 1 \qquad \text{for } i = 1, 2, \ldots, n,$$

prove that

$$a_1 + \cdots + a_n \leqslant n^2.$$

(Germany)

A5. Let $\mathbb{Z}_{\geqslant 0}$ be the set of all nonnegative integers. Find all the functions $f \colon \mathbb{Z}_{\geqslant 0} \to \mathbb{Z}_{\geqslant 0}$ satisfying the relation

$$f(f(f(n))) = f(n+1) + 1$$

for all $n \in \mathbb{Z}_{\geqslant 0}$.

(Serbia)

A6. Let $m \neq 0$ be an integer. Find all polynomials $P(x)$ with real coefficients such that

$$(x^3 - mx^2 + 1)P(x+1) + (x^3 + mx^2 + 1)P(x-1) = 2(x^3 - mx + 1)P(x)$$

for all real numbers x.

(Serbia)

Combinatorics

C1. Let n be a positive integer. Find the smallest integer k with the following property: Given any real numbers $a_1, \ldots, a_d$ such that $a_1 + a_2 + \cdots + a_d = n$ and $0 \leqslant a_i \leqslant 1$ for $i = 1, 2, \ldots, d$, it is possible to partition these numbers into k groups (some of which may be empty) such that the sum of the numbers in each group is at most 1.

(Poland)

C2. In the plane, 2013 red points and 2014 blue points are marked so that no three of the marked points are collinear. One needs to draw k lines not passing through the marked points and dividing the plane into several regions. The goal is to do it in such a way that no region contains points of both colors.

Find the minimal value of k such that the goal is attainable for every possible configuration of 4027 points.

(Australia)

C3. A crazy physicist discovered a new kind of particle which he called an *imon*, after some of them mysteriously appeared in his lab. Some pairs of imons in the lab can be *entangled*, and each imon can participate in many entanglement relations. The physicist has found a way to perform the following two kinds of operations with these particles, one operation at a time.

(i) If some imon is entangled with an odd number of other imons in the lab, then the physicist can destroy it.

(ii) At any moment, he may double the whole family of imons in his lab by creating a copy I' of each imon I. During this procedure, the two copies I' and J' become entangled if and only if the original imons I and J are entangled, and each copy I' becomes entangled with its original imon I; no other entanglements occur or disappear at this moment.

Prove that the physicist may apply a sequence of such operations resulting in a family of imons, no two of which are entangled.

(Japan)

C4. Let n be a positive integer, and let A be a subset of $\{1, \ldots, n\}$. An *A-partition of n into k parts* is a representation of n as a sum $n = a_1 + \cdots + a_k$, where the parts $a_1, \ldots, a_k$ belong to A and are not necessarily distinct. The *number of different parts* in such a partition is the number of (distinct) elements in the set $\{a_1, a_2, \ldots, a_k\}$.

We say that an A-partition of n into k parts is *optimal* if there is no A-partition of n into r parts with $r < k$. Prove that any optimal A-partition of n contains at most $\sqrt[3]{6n}$ different parts.

(Germany)

C5. Let r be a positive integer, and let $a_0, a_1, \ldots$ be an infinite sequence of real numbers. Assume that for all nonnegative integers m and s there exists a positive integer $n \in [m+1, m+r]$ such that

$$a_m + a_{m+1} + \cdots + a_{m+s} = a_n + a_{n+1} + \cdots + a_{n+s}.$$

Prove that the sequence is periodic, i.e. there exists some $p \geqslant 1$ such that $a_{n+p} = a_n$ for all $n \geqslant 0$.

(India)

C6. In some country several pairs of cities are connected by direct two-way flights. It is possible to go from any city to any other by a sequence of flights. The *distance* between two cities is defined to be the least possible number of flights required to go from one of them to the other. It is known that for any city there are at most 100 cities at distance exactly three from it. Prove that there is no city such that more than 2550 other cities have distance exactly four from it.

(Russia)

C7. Let $n \geqslant 2$ be an integer. Consider all circular arrangements of the numbers $0, 1, \ldots, n$; the $n+1$ rotations of an arrangement are considered to be equal. A circular arrangement is called *beautiful* if, for any four distinct numbers $0 \leqslant a, b, c, d \leqslant n$ with $a+c=b+d$, the chord joining numbers a and c does not intersect the chord joining numbers b and d.

Let M be the number of beautiful arrangements of $0, 1, \ldots, n$. Let N be the number of pairs (x, y) of positive integers such that $x+y \leqslant n$ and $\gcd(x, y)=1$. Prove that

$$M = N + 1.$$

(Russia)

C8. Players A and B play a paintful game on the real line. Player A has a pot of paint with four units of black ink. A quantity p of this ink suffices to blacken a (closed) real interval of length p. In every round, player A picks some positive integer m and provides $1/2^m$ units of ink from the pot. Player B then picks an integer k and blackens the interval from $k/2^m$ to $(k+1)/2^m$ (some parts of this interval may have been blackened before). The goal of player A is to reach a situation where the pot is empty and the interval $[0, 1]$ is not completely blackened.

Decide whether there exists a strategy for player A to win in a finite number of moves.

(Austria)

Geometry

G1. Let ABC be an acute-angled triangle with orthocenter H, and let W be a point on side BC. Denote by M and N the feet of the altitudes from B and C, respectively. Denote by ω_1 the circumcircle of BWN, and let X be the point on ω_1 which is diametrically opposite to W. Analogously, denote by ω_2 the circumcircle of CWM, and let Y be the point on ω_2 which is diametrically opposite to W. Prove that X, Y and H are collinear.

(Thaliand)

G2. Let ω be the circumcircle of a triangle ABC. Denote by M and N the midpoints of the sides AB and AC, respectively, and denote by T the midpoint of the arc BC of ω not containing A. The circumcircles of the triangles AMT and ANT intersect the perpendicular bisectors of AC and AB at points X and Y, respectively; assume that X and Y lie inside the triangle ABC. The lines MN and XY intersect at K. Prove that $KA = KT$.

(Iran)

G3. In a triangle ABC, let D and E be the feet of the angle bisectors of angles A and B, respectively. A rhombus is inscribed into the quadrilateral $AEDB$ (all vertices of the rhombus lie on different sides of $AEDB$). Let φ be the non-obtuse angle of the rhombus. Prove that $\varphi \leqslant \max\{\angle BAC, \angle ABC\}$.

(Serbia)

G4. Let ABC be a triangle with $\angle B > \angle C$. Let P and Q be two different points on line AC such that $\angle PBA = \angle QBA = \angle ACB$ and A is located between P and C. Suppose that there exists an interior point D of segment BQ for which $PD = PB$. Let the ray AD intersect the circle ABC at $R \neq A$. Prove that $QB = QR$.

(Georgia)

G5. Let $ABCDEF$ be a convex hexagon with $AB = DE$, $BC = EF$, $CD = FA$, and $\angle A - \angle D = \angle C - \angle F = \angle E - \angle B$. Prove that the diagonals AD, BE, and CF are concurrent.

(Ukraine)

G6. Let the excircle of the triangle ABC lying opposite to A touch its side BC at the point A_1. Define the points B_1 and C_1 analogously. Suppose that the circumcentre of the triangle $A_1B_1C_1$ lies on the circumcircle of the triangle ABC. Prove that the triangle ABC is right-angled.

(Russia)

Number Theory

N1. Let $\mathbb{Z}_{>0}$ be the set of positive integers. Find all functions $f\colon \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ such that

$$m^2 + f(n) \mid mf(m) + n$$

for all positive integers m and n.

(Malaysia)

N2. Prove that for any pair of positive integers k and n there exist k positive integers $m_1, m_2, \ldots, m_k$ such that

$$1 + \frac{2^k - 1}{n} = \left(1 + \frac{1}{m_1}\right)\left(1 + \frac{1}{m_2}\right)\cdots\left(1 + \frac{1}{m_k}\right).$$

(Japan)

N3. Prove that there exist infinitely many positive integers n such that the largest prime divisor of $n^4 + n^2 + 1$ is equal to the largest prime divisor of $(n+1)^4 + (n+1)^2 + 1$.

(Belgium)

N4. Determine whether there exists an infinite sequence of nonzero digits $a_1, a_2, a_3, \ldots$ and a positive integer N such that for every integer $k > N$, the number $\overline{a_k a_{k-1} \ldots a_1}$ is a perfect square.

(Iran)

N5. Fix an integer $k \geqslant 2$. Two players, called Ana and Banana, play the following *game of numbers*: Initially, some integer $n \geqslant k$ gets written on the blackboard. Then they take moves in turn, with Ana beginning. A player making a move erases the number m just written on the blackboard and replaces it by some number m' with $k \leqslant m' < m$ that is coprime to m. The first player who cannot move anymore loses.

An integer $n \geqslant k$ is called *good* if Banana has a winning strategy when the initial number is n, and *bad* otherwise.

Consider two integers $n, n' \geqslant k$ with the property that each prime number $p \leqslant k$ divides n if and only if it divides n'. Prove that either both n and n' are good or both are bad.

(Italy)

N6. Determine all functions $f\colon \mathbb{Q} \longrightarrow \mathbb{Z}$ satisfying

$$f\left(\frac{f(x)+a}{b}\right) = f\left(\frac{x+a}{b}\right)$$

for all $x \in \mathbb{Q}$, $a \in \mathbb{Z}$, and $b \in \mathbb{Z}_{>0}$. (Here, $\mathbb{Z}_{>0}$ denotes the set of positive integers.)

(Israel)

N7. Let ν be an irrational positive number, and let m be a positive integer. A pair (a, b) of positive integers is called *good* if

$$a\lceil b\nu \rceil - b\lfloor a\nu \rfloor = m.$$

A good pair (a, b) is called *excellent* if neither of the pairs $(a-b, b)$ and $(a, b-a)$ is good. (As usual, by $\lfloor x \rfloor$ and $\lceil x \rceil$ we denote the integer numbers such that $x - 1 < \lfloor x \rfloor \leqslant x$ and $x \leqslant \lceil x \rceil < x + 1$.)

Prove that the number of excellent pairs is equal to the sum of the positive divisors of m.

(U.S.A.)

Solutions

Algebra

A1. Let n be a positive integer and let $a_1, \ldots, a_{n-1}$ be arbitrary real numbers. Define the sequences $u_0, \ldots, u_n$ and $v_0, \ldots, v_n$ inductively by $u_0 = u_1 = v_0 = v_1 = 1$, and

$$u_{k+1} = u_k + a_k u_{k-1}, \qquad v_{k+1} = v_k + a_{n-k} v_{k-1} \qquad \text{for } k = 1, \ldots, n-1.$$

Prove that $u_n = v_n$.

(France)

Solution 1. We prove by induction on k that

$$u_k = \sum_{\substack{0<i_1<\ldots<i_t<k,\\ i_{j+1}-i_j \geqslant 2}} a_{i_1} \ldots a_{i_t}. \tag{1}$$

Note that we have one trivial summand equal to 1 (which corresponds to $t = 0$ and the empty sequence, whose product is 1).

For $k = 0, 1$ the sum on the right-hand side only contains the empty product, so (1) holds due to $u_0 = u_1 = 1$. For $k \geqslant 1$, assuming the result is true for $0, 1, \ldots, k$, we have

$$\begin{aligned} u_{k+1} &= \sum_{\substack{0<i_1<\ldots<i_t<k,\\ i_{j+1}-i_j \geqslant 2}} a_{i_1} \ldots a_{i_t} + \sum_{\substack{0<i_1<\ldots<i_t<k-1,\\ i_{j+1}-i_j \geqslant 2}} a_{i_1} \ldots a_{i_t} \cdot a_k \\ &= \sum_{\substack{0<i_1<\ldots<i_t<k+1,\\ i_{j+1}-i_j \geqslant 2,\\ k \notin \{i_1,\ldots,i_t\}}} a_{i_1} \ldots a_{i_t} + \sum_{\substack{0<i_1<\ldots<i_t<k+1,\\ i_{j+1}-i_j \geqslant 2,\\ k \in \{i_1,\ldots,i_t\}}} a_{i_1} \ldots a_{i_t} \\ &= \sum_{\substack{0<i_1<\ldots<i_t<k+1,\\ i_{j+1}-i_j \geqslant 2}} a_{i_1} \ldots a_{i_t}, \end{aligned}$$

as required.

Applying (1) to the sequence $b_1, \ldots, b_n$ given by $b_k = a_{n-k}$ for $1 \leqslant k \leqslant n$, we get

$$v_k = \sum_{\substack{0<i_1<\ldots<i_t<k,\\ i_{j+1}-i_j \geqslant 2}} b_{i_1} \ldots b_{i_t} = \sum_{\substack{n>i_1>\ldots>i_t>n-k,\\ i_j-i_{j+1} \geqslant 2}} a_{i_1} \ldots a_{i_t}. \tag{2}$$

For $k = n$ the expressions (1) and (2) coincide, so indeed $u_n = v_n$.

Solution 2. Define recursively a sequence of multivariate polynomials by

$$P_0 = P_1 = 1, \qquad P_{k+1}(x_1, \ldots, x_k) = P_k(x_1, \ldots, x_{k-1}) + x_k P_{k-1}(x_1, \ldots, x_{k-2}),$$

so P_n is a polynomial in $n-1$ variables for each $n \geqslant 1$. Two easy inductive arguments show that

$$u_n = P_n(a_1, \ldots, a_{n-1}), \qquad v_n = P_n(a_{n-1}, \ldots, a_1),$$

so we need to prove $P_n(x_1,\dots,x_{n-1}) = P_n(x_{n-1},\dots,x_1)$ for every positive integer n. The cases $n = 1,2$ are trivial, and the cases $n = 3,4$ follow from $P_3(x,y) = 1+x+y$ and $P_4(x,y,z) = 1+x+y+z+xz$.

Now we proceed by induction, assuming that $n \geqslant 5$ and the claim hold for all smaller cases. Using $F(a,b)$ as an abbreviation for $P_{|a-b|+1}(x_a,\dots,x_b)$ (where the indices $a,\dots,b$ can be either in increasing or decreasing order),

$$\begin{aligned}F(n,1) &= F(n,2) + x_1F(n,3) = F(2,n) + x_1F(3,n)\\ &= \big(F(2,n-1) + x_nF(2,n-2)\big) + x_1\big(F(3,n-1) + x_nF(3,n-2)\big)\\ &= \big(F(n-1,2) + x_1F(n-1,3)\big) + x_n\big(F(n-2,2) + x_1F(n-2,3)\big)\\ &= F(n-1,1) + x_nF(n-2,1) = F(1,n-1) + x_nF(1,n-2)\\ &= F(1,n),\end{aligned}$$

as we wished to show.

Solution 3. Using matrix notation, we can rewrite the recurrence relation as

$$\begin{pmatrix} u_{k+1} \\ u_{k+1}-u_k \end{pmatrix} = \begin{pmatrix} u_k + a_ku_{k-1} \\ a_ku_{k-1} \end{pmatrix} = \begin{pmatrix} 1+a_k & -a_k \\ a_k & -a_k \end{pmatrix}\begin{pmatrix} u_k \\ u_k-u_{k-1} \end{pmatrix}$$

for $1 \leqslant k \leqslant n-1$, and similarly

$$(v_{k+1}; v_k - v_{k+1}) = \big(v_k + a_{n-k}v_{k-1}; -a_{n-k}v_{k-1}\big) = (v_k; v_{k-1} - v_k)\begin{pmatrix} 1+a_{n-k} & -a_{n-k} \\ a_{n-k} & -a_{n-k} \end{pmatrix}$$

for $1 \leqslant k \leqslant n-1$. Hence, introducing the 2×2 matrices $A_k = \begin{pmatrix} 1+a_k & -a_k \\ a_k & -a_k \end{pmatrix}$ we have

$$\begin{pmatrix} u_{k+1} \\ u_{k+1}-u_k \end{pmatrix} = A_k\begin{pmatrix} u_k \\ u_k-u_{k-1} \end{pmatrix} \quad \text{and} \quad (v_{k+1}; v_k - v_{k+1}) = (v_k; v_{k-1} - v_k)A_{n-k}.$$

for $1 \leqslant k \leqslant n-1$. Since $\binom{u_1}{u_1-u_0} = \binom{1}{0}$ and $(v_1; v_0 - v_1) = (1;0)$, we get

$$\begin{pmatrix} u_n \\ u_n-u_{n-1} \end{pmatrix} = A_{n-1}A_{n-2}\cdots A_1\cdot\begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad (v_n; v_{n-1}-v_n) = (1;0)\cdot A_{n-1}A_{n-2}\cdots A_1.$$

It follows that

$$(u_n) = (1;0)\begin{pmatrix} u_n \\ u_n-u_{n-1} \end{pmatrix} = (1;0)\cdot A_{n-1}A_{n-2}\cdots A_1\cdot\begin{pmatrix} 1 \\ 0 \end{pmatrix} = (v_n; v_{n-1}-v_n)\begin{pmatrix} 1 \\ 0 \end{pmatrix} = (v_n).$$

Comment 1. These sequences are related to the Fibonacci sequence; when $a_1 = \dots = a_{n-1} = 1$, we have $u_k = v_k = F_{k+1}$, the $(k+1)$st Fibonacci number. Also, for every positive integer k, the polynomial $P_k(x_1,\dots,x_{k-1})$ from Solution 2 is the sum of F_{k+1} monomials.

Comment 2. One may notice that the condition is equivalent to

$$\frac{u_{k+1}}{u_k} = 1 + \cfrac{a_k}{1 + \cfrac{a_{k-1}}{1 + \dots + \cfrac{a_2}{1+a_1}}} \qquad \text{and} \qquad \frac{v_{k+1}}{v_k} = 1 + \cfrac{a_{n-k}}{1 + \cfrac{a_{n-k+1}}{1 + \dots + \cfrac{a_{n-2}}{1+a_{n-1}}}}$$

so the problem claims that the corresponding continued fractions for u_n/u_{n-1} and v_n/v_{n-1} have the same numerator.

Comment 3. An alternative variant of the problem is the following.

> *Let n be a positive integer and let $a_1, \dots, a_{n-1}$ be arbitrary real numbers. Define the sequences $u_0, \dots, u_n$ and $v_0, \dots, v_n$ inductively by $u_0 = v_0 = 0$, $u_1 = v_1 = 1$, and*
> $$u_{k+1} = a_k u_k + u_{k-1}, \qquad v_{k+1} = a_{n-k} v_k + v_{k-1} \qquad \text{for } k = 1, \dots, n-1.$$
> *Prove that $u_n = v_n$.*

All three solutions above can be reformulated to prove this statement; one may prove

$$u_n = v_n = \sum_{\substack{0=i_0<i_1<\ldots<i_t=n, \\ i_{j+1}-i_j \text{ is odd}}} a_{i_1} \dots a_{i_{t-1}} \qquad \text{for } n > 0$$

or observe that

$$\begin{pmatrix} u_{k+1} \\ u_k \end{pmatrix} = \begin{pmatrix} a_k & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u_k \\ u_{k-1} \end{pmatrix} \quad \text{and} \quad (v_{k+1}; v_k) = (v_k; v_{k-1}) \begin{pmatrix} a_k & 1 \\ 1 & 0 \end{pmatrix}.$$

Here we have

$$\frac{u_{k+1}}{u_k} = a_k + \cfrac{1}{a_{k-1} + \cfrac{1}{a_{k-2} + \ldots + \cfrac{1}{a_1}}} = [a_k; a_{k-1}, \dots, a_1]$$

and

$$\frac{v_{k+1}}{v_k} = a_{n-k} + \cfrac{1}{a_{n-k+1} + \cfrac{1}{a_{n-k+2} + \ldots + \cfrac{1}{a_{n-1}}}} = [a_{n-k}; a_{n-k+1}, \dots, a_{n-1}],$$

so this alternative statement is equivalent to the known fact that the continued fractions $[a_{n-1}; a_{n-2}, \dots, a_1]$ and $[a_1; a_2, \dots, a_{n-1}]$ have the same numerator.

A2. Prove that in any set of 2000 distinct real numbers there exist two pairs $a > b$ and $c > d$ with $a \neq c$ or $b \neq d$, such that

$$\left|\frac{a-b}{c-d} - 1\right| < \frac{1}{100000}.$$

(Lithuania)

Solution. For any set S of $n = 2000$ distinct real numbers, let $D_1 \leqslant D_2 \leqslant \cdots \leqslant D_m$ be the distances between them, displayed with their multiplicities. Here $m = n(n-1)/2$. By rescaling the numbers, we may assume that the smallest distance D_1 between two elements of S is $D_1 = 1$. Let $D_1 = 1 = y - x$ for $x, y \in S$. Evidently $D_m = v - u$ is the difference between the largest element v and the smallest element u of S.

If $D_{i+1}/D_i < 1 + 10^{-5}$ for some $i = 1, 2, \ldots, m-1$ then the required inequality holds, because $0 \leqslant D_{i+1}/D_i - 1 < 10^{-5}$. Otherwise, the reverse inequality

$$\frac{D_{i+1}}{D_i} \geqslant 1 + \frac{1}{10^5}$$

holds for each $i = 1, 2, \ldots, m-1$, and therefore

$$v - u = D_m = \frac{D_m}{D_1} = \frac{D_m}{D_{m-1}} \cdots \frac{D_3}{D_2} \cdot \frac{D_2}{D_1} \geqslant \left(1 + \frac{1}{10^5}\right)^{m-1}.$$

From $m - 1 = n(n-1)/2 - 1 = 1000 \cdot 1999 - 1 > 19 \cdot 10^5$, together with the fact that for all $n \geqslant 1$, $\left(1 + \frac{1}{n}\right)^n \geqslant 1 + \binom{n}{1} \cdot \frac{1}{n} = 2$, we get

$$\left(1 + \frac{1}{10^5}\right)^{19 \cdot 10^5} = \left(\left(1 + \frac{1}{10^5}\right)^{10^5}\right)^{19} \geqslant 2^{19} = 2^9 \cdot 2^{10} > 500 \cdot 1000 > 2 \cdot 10^5,$$

and so $v - u = D_m > 2 \cdot 10^5$.

Since the distance of x to at least one of the numbers u, v is at least $(u-v)/2 > 10^5$, we have

$$|x - z| > 10^5.$$

for some $z \in \{u, v\}$. Since $y - x = 1$, we have either $z > y > x$ (if $z = v$) or $y > x > z$ (if $z = u$). If $z > y > x$, selecting $a = z, b = y, c = z$ and $d = x$ (so that $b \neq d$), we obtain

$$\left|\frac{a-b}{c-d} - 1\right| = \left|\frac{z-y}{z-x} - 1\right| = \left|\frac{x-y}{z-x}\right| = \frac{1}{z-x} < 10^{-5}.$$

Otherwise, if $y > x > z$, we may choose $a = y, b = z, c = x$ and $d = z$ (so that $a \neq c$), and obtain

$$\left|\frac{a-b}{c-d} - 1\right| = \left|\frac{y-z}{x-z} - 1\right| = \left|\frac{y-x}{x-z}\right| = \frac{1}{x-z} < 10^{-5}.$$

The desired result follows.

Comment. As the solution shows, the numbers 2000 and $\frac{1}{100000}$ appearing in the statement of the problem may be replaced by any $n \in \mathbb{Z}_{>0}$ and $\delta > 0$ satisfying

$$\delta(1+\delta)^{n(n-1)/2-1} > 2.$$

A3. Let $\mathbb{Q}_{>0}$ be the set of positive rational numbers. Let $f\colon \mathbb{Q}_{>0} \to \mathbb{R}$ be a function satisfying the conditions

$$f(x)f(y) \geqslant f(xy), \tag{1}$$
$$f(x+y) \geqslant f(x) + f(y) \tag{2}$$

for all $x, y \in \mathbb{Q}_{>0}$. Given that $f(a) = a$ for some rational $a > 1$, prove that $f(x) = x$ for all $x \in \mathbb{Q}_{>0}$.

(Bulgaria)

Solution. Denote by $\mathbb{Z}_{>0}$ the set of positive integers.

Plugging $x = 1$, $y = a$ into (1) we get $f(1) \geqslant 1$. Next, by an easy induction on n we get from (2) that

$$f(nx) \geqslant nf(x) \quad \text{for all } n \in \mathbb{Z}_{>0} \text{ and } x \in \mathbb{Q}_{>0}. \tag{3}$$

In particular, we have

$$f(n) \geqslant nf(1) \geqslant n \quad \text{for all } n \in \mathbb{Z}_{>0}. \tag{4}$$

From (1) again we have $f(m/n)f(n) \geqslant f(m)$, so $f(q) > 0$ for all $q \in \mathbb{Q}_{>0}$.

Now, (2) implies that f is strictly increasing; this fact together with (4) yields

$$f(x) \geqslant f(\lfloor x \rfloor) \geqslant \lfloor x \rfloor > x - 1 \quad \text{for all } x \geqslant 1.$$

By an easy induction we get from (1) that $f(x)^n \geqslant f(x^n)$, so

$$f(x)^n \geqslant f(x^n) > x^n - 1 \quad \Longrightarrow \quad f(x) \geqslant \sqrt[n]{x^n - 1} \quad \text{for all } x > 1 \text{ and } n \in \mathbb{Z}_{>0}.$$

This yields

$$f(x) \geqslant x \quad \text{for every } x > 1. \tag{5}$$

(Indeed, if $x > y > 1$ then $x^n - y^n = (x-y)(x^{n-1} + x^{n-2}y + \cdots + y^n) > n(x-y)$, so for a large n we have $x^n - 1 > y^n$ and thus $f(x) > y$.)

Now, (1) and (5) give $a^n = f(a)^n \geqslant f(a^n) \geqslant a^n$, so $f(a^n) = a^n$. Now, for $x > 1$ let us choose $n \in \mathbb{Z}_{>0}$ such that $a^n - x > 1$. Then by (2) and (5) we get

$$a^n = f(a^n) \geqslant f(x) + f(a^n - x) \geqslant x + (a^n - x) = a^n$$

and therefore $f(x) = x$ for $x > 1$. Finally, for every $x \in \mathbb{Q}_{>0}$ and every $n \in \mathbb{Z}_{>0}$, from (1) and (3) we get

$$nf(x) = f(n)f(x) \geqslant f(nx) \geqslant nf(x),$$

which gives $f(nx) = nf(x)$. Therefore $f(m/n) = f(m)/n = m/n$ for all $m, n \in \mathbb{Z}_{>0}$.

Comment. The condition $f(a) = a > 1$ is essential. Indeed, for $b \geqslant 1$ the function $f(x) = bx^2$ satisfies (1) and (2) for all $x, y \in \mathbb{Q}_{>0}$, and it has a unique fixed point $1/b \leqslant 1$.

A4. Let n be a positive integer, and consider a sequence $a_1, a_2, \dots, a_n$ of positive integers. Extend it periodically to an infinite sequence $a_1, a_2, \dots$ by defining $a_{n+i} = a_i$ for all $i \geqslant 1$. If

$$a_1 \leqslant a_2 \leqslant \cdots \leqslant a_n \leqslant a_1 + n \tag{1}$$

and

$$a_{a_i} \leqslant n + i - 1 \qquad \text{for } i = 1, 2, \dots, n, \tag{2}$$

prove that

$$a_1 + \cdots + a_n \leqslant n^2.$$

(Germany)

Solution 1. First, we claim that

$$a_i \leqslant n + i - 1 \qquad \text{for } i = 1, 2, \dots, n. \tag{3}$$

Assume contrariwise that i is the smallest counterexample. From $a_n \geqslant a_{n-1} \geqslant \cdots \geqslant a_i \geqslant n + i$ and $a_{a_i} \leqslant n + i - 1$, taking into account the periodicity of our sequence, it follows that

$$a_i \text{ cannot be congruent to } i, i+1, \dots, n-1, \text{or } n \pmod{n}. \tag{4}$$

Thus our assumption that $a_i \geqslant n + i$ implies the stronger statement that $a_i \geqslant 2n + 1$, which by $a_1 + n \geqslant a_n \geqslant a_i$ gives $a_1 \geqslant n + 1$. The minimality of i then yields $i = 1$, and (4) becomes contradictory. This establishes our first claim.

In particular we now know that $a_1 \leqslant n$. If $a_n \leqslant n$, then $a_1 \leqslant \cdots \leqslant \cdots a_n \leqslant n$ and the desired inequality holds trivially. Otherwise, consider the number t with $1 \leqslant t \leqslant n-1$ such that

$$a_1 \leqslant a_2 \leqslant \ldots \leqslant a_t \leqslant n < a_{t+1} \leqslant \ldots \leqslant a_n. \tag{5}$$

Since $1 \leqslant a_1 \leqslant n$ and $a_{a_1} \leqslant n$ by (2), we have $a_1 \leqslant t$ and hence $a_n \leqslant n + t$. Therefore if for each positive integer i we let b_i be the number of indices $j \in \{t+1, \dots, n\}$ satisfying $a_j \geqslant n + i$, we have

$$b_1 \geqslant b_2 \geqslant \ldots \geqslant b_t \geqslant b_{t+1} = 0.$$

Next we claim that $a_i + b_i \leqslant n$ for $1 \leqslant i \leqslant t$. Indeed, by $n + i - 1 \geqslant a_{a_i}$ and $a_i \leqslant n$, each j with $a_j \geqslant n + i$ (thus $a_j > a_{a_i}$) belongs to $\{a_i + 1, \dots, n\}$, and for this reason $b_i \leqslant n - a_i$.

It follows from the definition of the b_is and (5) that

$$a_{t+1} + \ldots + a_n \leqslant n(n-t) + b_1 + \ldots + b_t.$$

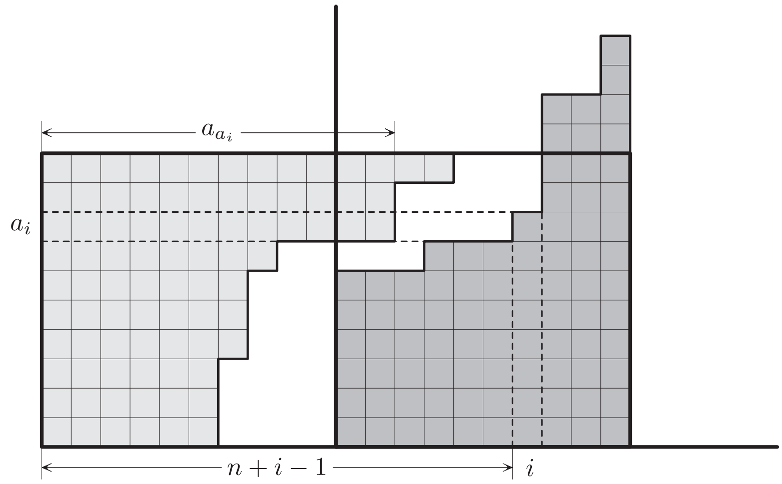
Adding $a_1 + \ldots + a_t$ to both sides and using that $a_i + b_i \leqslant n$ for $1 \leqslant i \leqslant t$, we get

$$a_1 + a_2 + \cdots + a_n \leqslant n(n-t) + nt = n^2$$

as we wished to prove.

Solution 2. In the first quadrant of an infinite grid, consider the increasing "staircase" obtained by shading in dark the bottom a_i cells of the ith column for $1 \leqslant i \leqslant n$. We will prove that there are at most n^2 dark cells.

To do it, consider the $n \times n$ square S in the first quadrant with a vertex at the origin. Also consider the $n \times n$ square directly to the left of S. Starting from its lower left corner, shade in light the leftmost a_j cells of the jth row for $1 \leqslant j \leqslant n$. Equivalently, the light shading is obtained by reflecting the dark shading across the line $x = y$ and translating it n units to the left. The figure below illustrates this construction for the sequence $6, 6, 6, 7, 7, 7, 8, 12, 12, 14$.



We claim that there is no cell in S which is both dark and light. Assume, contrariwise, that there is such a cell in column i. Consider the highest dark cell in column i which is inside S. Since it is above a light cell and inside S, it must be light as well. There are two cases:

Case 1. $a_i \leqslant n$

If $a_i \leqslant n$ then this dark and light cell is (i, a_i), as highlighted in the figure. However, this is the $(n+i)$-th cell in row a_i, and we only shaded $a_{a_i} < n + i$ light cells in that row, a contradiction.

Case 2. $a_i \geqslant n + 1$

If $a_i \geqslant n + 1$, this dark and light cell is (i, n). This is the $(n + i)$-th cell in row n and we shaded $a_n \leqslant a_1 + n$ light cells in this row, so we must have $i \leqslant a_1$. But $a_1 \leqslant a_{a_1} \leqslant n$ by (1) and (2), so $i \leqslant a_1$ implies $a_i \leqslant a_{a_1} \leqslant n$, contradicting our assumption.

We conclude that there are no cells in S which are both dark and light. It follows that the number of shaded cells in S is at most n^2.

Finally, observe that if we had a light cell to the right of S, then by symmetry we would have a dark cell above S, and then the cell (n, n) would be dark and light. It follows that the number of light cells in S equals the number of dark cells outside of S, and therefore the number of shaded cells in S equals $a_1 + \cdots + a_n$. The desired result follows.

Solution 3. As in Solution 1, we first establish that $a_i \leqslant n + i - 1$ for $1 \leqslant i \leqslant n$. Now define $c_i = \max(a_i, i)$ for $1 \leqslant i \leqslant n$ and extend the sequence $c_1, c_2, \ldots$ periodically modulo n. We claim that this sequence also satisfies the conditions of the problem.

For $1 \leqslant i < j \leqslant n$ we have $a_i \leqslant a_j$ and $i < j$, so $c_i \leqslant c_j$. Also $a_n \leqslant a_1 + n$ and $n < 1 + n$ imply $c_n \leqslant c_1 + n$. Finally, the definitions imply that $c_{c_i} \in \{a_{a_i}, a_i, a_i - n, i\}$ so $c_{c_i} \leqslant n + i - 1$ by (2) and (3). This establishes (1) and (2) for $c_1, c_2, \ldots$.

Our new sequence has the additional property that

$$c_i \geqslant i \qquad \text{for } i = 1, 2, \ldots, n, \tag{6}$$

which allows us to construct the following visualization: Consider n equally spaced points on a circle, sequentially labelled 1, 2, ..., n (mod n), so point k is also labelled $n + k$. We draw arrows from vertex i to vertices $i + 1, \ldots, c_i$ for $1 \leqslant i \leqslant n$, keeping in mind that $c_i \geqslant i$ by (6). Since $c_i \leqslant n + i - 1$ by (3), no arrow will be drawn twice, and there is no arrow from a vertex to itself. The total number of arrows is

$$\text{number of arrows } = \sum_{i=1}^{n}(c_i - i) = \sum_{i=1}^{n} c_i - \binom{n+1}{2}$$

Now we show that we never draw both arrows $i \to j$ and $j \to i$ for $1 \leqslant i < j \leqslant n$. Assume contrariwise. This means, respectively, that

$$i < j \leqslant c_i \qquad \text{and} \qquad j < n + i \leqslant c_j.$$

We have $n + i \leqslant c_j \leqslant c_1 + n$ by (1), so $i \leqslant c_1$. Since $c_1 \leqslant n$ by (3), this implies that $c_i \leqslant c_{c_1} \leqslant n$ using (1) and (3). But then, using (1) again, $j \leqslant c_i \leqslant n$ implies $c_j \leqslant c_{c_i}$, which combined with $n + i \leqslant c_j$ gives us that $n + i \leqslant c_{c_i}$. This contradicts (2).

This means that the number of arrows is at most $\binom{n}{2}$, which implies that

$$\sum_{i=1}^{n} c_i \leqslant \binom{n}{2} + \binom{n+1}{2} = n^2.$$

Recalling that $a_i \leqslant c_i$ for $1 \leqslant i \leqslant n$, the desired inequality follows.

Comment 1. We sketch an alternative proof by induction. Begin by verifying the initial case $n = 1$ and the simple cases when $a_1 = 1$, $a_1 = n$, or $a_n \leqslant n$. Then, as in Solution 1, consider the index t such that $a_1 \leqslant \cdots \leqslant a_t \leqslant n < a_{t+1} \leqslant \cdots \leqslant a_n$. Observe again that $a_1 \leqslant t$. Define the sequence $d_1, \ldots, d_{n-1}$ by

$$d_i = \begin{cases} a_{i+1} - 1 & \text{if } i \leqslant t - 1 \\ a_{i+1} - 2 & \text{if } i \geqslant t \end{cases}$$

and extend it periodically modulo $n - 1$. One may verify that this sequence also satisfies the hypotheses of the problem. The induction hypothesis then gives $d_1 + \cdots + d_{n-1} \leqslant (n-1)^2$, which implies that

$$\sum_{i=1}^{n} a_i = a_1 + \sum_{i=2}^{t}(d_{i-1} + 1) + \sum_{i=t+1}^{n} (d_{i-1} + 2) \leqslant t + (t-1) + 2(n-t) + (n-1)^2 = n^2.$$

Comment 2. One unusual feature of this problem is that there are many different sequences for which equality holds. The discovery of such *optimal sequences* is not difficult, and it is useful in guiding the steps of a proof.

In fact, Solution 2 gives a complete description of the optimal sequences. Start with any lattice path P from the lower left to the upper right corner of the $n \times n$ square S using only steps up and right, such that the total number of steps along the left and top edges of S is at least n. Shade the cells of S below P dark, and the cells of S above P light. Now reflect the light shape across the line $x = y$ and shift it up n units, and shade it dark. As Solution 2 shows, the dark region will then correspond to an optimal sequence, and every optimal sequence arises in this way.

A5. Let $\mathbb{Z}_{\geqslant 0}$ be the set of all nonnegative integers. Find all the functions $f\colon \mathbb{Z}_{\geqslant 0} \to \mathbb{Z}_{\geqslant 0}$ satisfying the relation

$$f(f(f(n))) = f(n+1) + 1 \tag{$*$}$$

for all $n \in \mathbb{Z}_{\geqslant 0}$.

(Serbia)

Answer. There are two such functions: $f(n) = n + 1$ for all $n \in \mathbb{Z}_{\geqslant 0}$, and

$$f(n) = \begin{cases} n+1, & n \equiv 0 \pmod 4 \text{ or } n \equiv 2 \pmod 4, \\ n+5, & n \equiv 1 \pmod 4, \\ n-3, & n \equiv 3 \pmod 4 \end{cases} \qquad \text{for all } n \in \mathbb{Z}_{\geqslant 0}. \tag{1}$$

Throughout all the solutions, we write $h^k(x)$ to abbreviate the kth iteration of function h, so h^0 is the identity function, and $h^k(x) = \underbrace{h(\dots h}_{k \text{ times}}(x)\dots))$ for $k \geqslant 1$.

Solution 1. To start, we get from $(*)$ that

$$f^4(n) = f(f^3(n)) = f\big(f(n+1)+1\big) \quad \text{and} \quad f^4(n+1) = f^3(f(n+1)) = f\big(f(n+1)+1\big) + 1,$$

thus

$$f^4(n) + 1 = f^4(n+1). \tag{2}$$

I. Let us denote by R_i the range of f^i; note that $R_0 = \mathbb{Z}_{\geqslant 0}$ since f^0 is the identity function. Obviously, $R_0 \supseteq R_1 \supseteq \dots$. Next, from (2) we get that if $a \in R_4$ then also $a + 1 \in R_4$. This implies that $\mathbb{Z}_{\geqslant 0} \backslash R_4$ — and hence $\mathbb{Z}_{\geqslant 0} \backslash R_1$ — is finite. In particular, R_1 is unbounded.

Assume that $f(m) = f(n)$ for some distinct m and n. Then from $(*)$ we obtain $f(m+1) = f(n+1)$; by an easy induction we then get that $f(m+c) = f(n+c)$ for every $c \geqslant 0$. So the function $f(k)$ is periodic with period $|m-n|$ for $k \geqslant m$, and thus R_1 should be bounded, which is false. So, f is injective.

II. Denote now $S_i = R_{i-1} \backslash R_i$; all these sets are finite for $i \leqslant 4$. On the other hand, by the injectivity we have $n \in S_i \iff f(n) \in S_{i+1}$. By the injectivity again, f implements a bijection between S_i and S_{i+1}, thus $|S_1| = |S_2| = \dots$; denote this common cardinality by k. If $0 \in R_3$ then $0 = f(f(f(n)))$ for some n, thus from $(*)$ we get $f(n+1) = -1$ which is impossible. Therefore $0 \in R_0 \backslash R_3 = S_1 \cup S_2 \cup S_3$, thus $k \geqslant 1$.

Next, let us describe the elements b of $R_0 \backslash R_3 = S_1 \cup S_2 \cup S_3$. We claim that each such element satisfies at least one of three conditions (i) $b = 0$, (ii) $b = f(0) + 1$, and (iii) $b - 1 \in S_1$. Otherwise $b - 1 \in \mathbb{Z}_{\geqslant 0}$, and there exists some $n > 0$ such that $f(n) = b - 1$; but then $f^3(n-1) = f(n) + 1 = b$, so $b \in R_3$.

This yields

$$3k = |S_1 \cup S_2 \cup S_3| \leqslant 1 + 1 + |S_1| = k + 2,$$

or $k \leqslant 1$. Therefore $k = 1$, and the inequality above comes to equality. So we have $S_1 = \{a\}$, $S_2 = \{f(a)\}$, and $S_3 = \{f^2(a)\}$ for some $a \in \mathbb{Z}_{\geqslant 0}$, and each one of the three options (i), (ii), and (iii) should be realized exactly once, which means that

$$\{a, f(a), f^2(a)\} = \{0, a+1, f(0)+1\}. \tag{3}$$

III. From (3), we get $a+1 \in \{f(a), f^2(a)\}$ (the case $a+1=a$ is impossible). If $a+1=f^2(a)$ then we have $f(a+1)=f^3(a)=f(a+1)+1$ which is absurd. Therefore

$$f(a)=a+1. \tag{4}$$

Next, again from (3) we have $0 \in \{a, f^2(a)\}$. Let us consider these two cases separately.

Case 1. Assume that $a=0$, then $f(0)=f(a)=a+1=1$. Also from (3) we get $f(1)=f^2(a)=f(0)+1=2$. Now, let us show that $f(n)=n+1$ by induction on n; the base cases $n \leqslant 1$ are established. Next, if $n \geqslant 2$ then the induction hypothesis implies

$$n+1=f(n-1)+1=f^3(n-2)=f^2(n-1)=f(n),$$

establishing the step. In this case we have obtained the first of two answers; checking that is satisfies $(*)$ is straightforward.

Case 2. Assume now that $f^2(a)=0$; then by (3) we get $a=f(0)+1$. By (4) we get $f(a+1)=f^2(a)=0$, then $f(0)=f^3(a)=f(a+1)+1=1$, hence $a=f(0)+1=2$ and $f(2)=3$ by (4). To summarize,

$$f(0)=1, \quad f(2)=3, \quad f(3)=0.$$

Now let us prove by induction on m that (1) holds for all $n=4k, 4k+2, 4k+3$ with $k \leqslant m$ and for all $n=4k+1$ with $k<m$. The base case $m=0$ is established above. For the step, assume that $m \geqslant 1$. From $(*)$ we get $f^3(4m-3)=f(4m-2)+1=4m$. Next, by (2) we have

$$f(4m)=f^4(4m-3)=f^4(4m-4)+1=f^3(4m-3)+1=4m+1.$$

Then by the induction hypothesis together with $(*)$ we successively obtain

$$\begin{aligned} f(4m-3)&=f^3(4m-1)=f(4m)+1=4m+2,\\ f(4m+2)&=f^3(4m-4)=f(4m-3)+1=4m+3,\\ f(4m+3)&=f^3(4m-3)=f(4m-2)+1=4m, \end{aligned}$$

thus finishing the induction step.

Finally, it is straightforward to check that the constructed function works:

$$\begin{aligned} f^3(4k)&=4k+7=f(4k+1)+1, & f^3(4k+1)&=4k+4=f(4k+2)+1,\\ f^3(4k+2)&=4k+1=f(4k+3)+1, & f^3(4k+3)&=4k+6=f(4k+4)+1. \end{aligned}$$

Solution 2. I. For convenience, let us introduce the function $g(n)=f(n)+1$. Substituting $f(n)$ instead of n into $(*)$ we obtain

$$f^4(n)=f\big(f(n)+1\big)+1, \quad \text{or} \quad f^4(n)=g^2(n). \tag{5}$$

Applying f to both parts of $(*)$ and using (5) we get

$$f^4(n)+1=f\big(f(n+1)+1\big)+1=f^4(n+1). \tag{6}$$

Thus, if $g^2(0)=f^4(0)=c$ then an easy induction on n shows that

$$g^2(n)=f^4(n)=n+c, \qquad n \in \mathbb{Z}_{\geqslant 0}. \tag{7}$$

This relation implies that both f and g are injective: if, say, $f(m) = f(n)$ then $m + c = f^4(m) = f^4(n) = n + c$. Next, since $g(n) \geqslant 1$ for every n, we have $c = g^2(0) \geqslant 1$. Thus from (7) again we obtain $f(n) \neq n$ and $g(n) \neq n$ for all $n \in \mathbb{Z}_{\geqslant 0}$.

II. Next, application of f and g to (7) yields

$$f(n+c) = f^5(n) = f^4(f(n)) = f(n) + c \quad \text{and} \quad g(n+c) = g^3(n) = g(n) + c. \tag{8}$$

In particular, this means that if $m \equiv n \pmod c$ then $f(m) \equiv f(n) \pmod c$. Conversely, if $f(m) \equiv f(n) \pmod c$ then we get $m + c = f^4(m) \equiv f^4(n) = n + c \pmod c$. Thus,

$$m \equiv n \pmod c \iff f(m) \equiv f(n) \pmod c \iff g(m) \equiv g(n) \pmod c. \tag{9}$$

Now, let us introduce the function $\delta(n) = f(n) - n = g(n) - n - 1$. Set

$$S = \sum_{n=0}^{c-1} \delta(n).$$

Using (8), we get that for every complete residue system $n_1, \dots, n_c$ modulo c we also have

$$S = \sum_{i=1}^{c} \delta(n_i).$$

By (9), we get that $\{f^k(n) \colon n = 0, \dots, c-1\}$ and $\{g^k(n) \colon n = 0, \dots, c-1\}$ are complete residue systems modulo c for all k. Thus we have

$$c^2 = \sum_{n=0}^{c-1} \big(f^4(n) - n\big) = \sum_{k=0}^{3} \sum_{n=0}^{c-1} \big(f^{k+1}(n) - f^k(n)\big) = \sum_{k=0}^{3} \sum_{n=0}^{c-1} \delta(f^k(n)) = 4S$$

and similarly

$$c^2 = \sum_{n=0}^{c-1} \big(g^2(n) - n\big) = \sum_{k=0}^{1} \sum_{n=0}^{c-1} \big(g^{k+1}(n) - g^k(n)\big) = \sum_{k=0}^{1} \sum_{n=0}^{c-1} \big(\delta(g^k(n)) + 1\big) = 2S + 2c.$$

Therefore $c^2 = 4S = 2 \cdot 2S = 2(c^2 - 2c)$, or $c^2 = 4c$. Since $c \neq 0$, we get $c = 4$. Thus, in view of (8) it is sufficient to determine the values of f on the numbers $0, 1, 2, 3$.

III. Let $d = g(0) \geqslant 1$. Then $g(d) = g^2(0) = 0 + c = 4$. Now, if $d \geqslant 4$, then we would have $g(d-4) = g(d) - 4 = 0$ which is impossible. Thus $d \in \{1, 2, 3\}$. If $d = 1$ then we have $f(0) = g(0) - 1 = 0$ which is impossible since $f(n) \neq n$ for all n. If $d = 3$ then $g(3) = g^2(0) = 4$ and hence $f(3) = 3$ which is also impossible. Thus $g(0) = 2$ and hence $g(2) = g^2(0) = 4$.

Next, if $g(1) = 1 + 4k$ for some integer k, then $5 = g^2(1) = g(1 + 4k) = g(1) + 4k = 1 + 8k$ which is impossible. Thus, since $\{g(n) \colon n = 0, 1, 2, 3\}$ is a complete residue system modulo 4, we get $g(1) = 3 + 4k$ and hence $g(3) = g^2(1) - 4k = 5 - 4k$, leading to $k = 0$ or $k = 1$. So, we obtain iether

$$f(0) = 1,\ f(1) = 2,\ f(2) = 3,\ f(3) = 4, \quad \text{or} \quad f(0) = 1,\ f(1) = 6,\ f(2) = 3,\ f(3) = 0,$$

thus arriving to the two functions listed in the answer.

Finally, one can check that these two function work as in Solution 1. One may simplify the checking by noticing that (8) allows us to reduce it to $n = 0, 1, 2, 3$.

A6. Let $m \neq 0$ be an integer. Find all polynomials $P(x)$ with real coefficients such that

$$(x^3 - mx^2 + 1)P(x+1) + (x^3 + mx^2 + 1)P(x-1) = 2(x^3 - mx + 1)P(x) \tag{1}$$

for all real numbers x.

(Serbia)

Answer. $P(x) = tx$ for any real number t.

Solution. Let $P(x) = a_n x^n + \cdots + a_0 x^0$ with $a_n \neq 0$. Comparing the coefficients of x^{n+1} on both sides gives $a_n(n-2m)(n-1) = 0$, so $n = 1$ or $n = 2m$.

If $n = 1$, one easily verifies that $P(x) = x$ is a solution, while $P(x) = 1$ is not. Since the given condition is linear in P, this means that the linear solutions are precisely $P(x) = tx$ for $t \in \mathbb{R}$.

Now assume that $n = 2m$. The polynomial $xP(x+1) - (x+1)P(x) = (n-1)a_n x^n + \cdots$ has degree n, and therefore it has at least one (possibly complex) root r. If $r \notin \{0, -1\}$, define $k = P(r)/r = P(r+1)/(r+1)$. If $r = 0$, let $k = P(1)$. If $r = -1$, let $k = -P(-1)$. We now consider the polynomial $S(x) = P(x) - kx$. It also satisfies (1) because $P(x)$ and kx satisfy it. Additionally, it has the useful property that r and $r+1$ are roots.

Let $A(x) = x^3 - mx^2 + 1$ and $B(x) = x^3 + mx^2 + 1$. Plugging in $x = s$ into (1) implies that:

If $s-1$ and s are roots of S and s is not a root of A, then $s+1$ is a root of S.

If s and $s+1$ are roots of S and s is not a root of B, then $s-1$ is a root of S.

Let $a \geqslant 0$ and $b \geqslant 1$ be such that $r-a, r-a+1, \ldots, r, r+1, \ldots, r+b-1, r+b$ are roots of S, while $r-a-1$ and $r+b+1$ are not. The two statements above imply that $r-a$ is a root of B and $r+b$ is a root of A.

Since $r-a$ is a root of $B(x)$ and of $A(x+a+b)$, it is also a root of their greatest common divisor $C(x)$ as integer polynomials. If $C(x)$ was a non-trivial divisor of $B(x)$, then B would have a rational root α. Since the first and last coefficients of B are 1, α can only be 1 or -1; but $B(-1) = m > 0$ and $B(1) = m + 2 > 0$ since $n = 2m$.

Therefore $B(x) = A(x+a+b)$. Writing $c = a + b \geqslant 1$ we compute

$$0 = A(x+c) - B(x) = (3c - 2m)x^2 + c(3c - 2m)x + c^2(c - m).$$

Then we must have $3c - 2m = c - m = 0$, which gives $m = 0$, a contradiction. We conclude that $f(x) = tx$ is the only solution.

Solution 2. Multiplying (1) by x, we rewrite it as

$$x(x^3 - mx^2 + 1)P(x+1) + x(x^3 + mx^2 + 1)P(x-1) = [(x+1) + (x-1)]\,(x^3 - mx + 1)P(x).$$

After regrouping, it becomes

$$(x^3 - mx^2 + 1)Q(x) = (x^3 + mx^2 + 1)Q(x-1), \tag{2}$$

where $Q(x) = xP(x+1) - (x+1)P(x)$. If $\deg P \geqslant 2$ then $\deg Q = \deg P$, so $Q(x)$ has a finite multiset of complex roots, which we denote R_Q. Each root is taken with its multiplicity. Then the multiset of complex roots of $Q(x-1)$ is $R_Q + 1 = \{z + 1 : z \in R_Q\}$.

Let $\{x_1, x_2, x_3\}$ and $\{y_1, y_2, y_3\}$ be the multisets of roots of the polynomials $A(x) = x^3 - mx^2 + 1$ and $B(x) = x^3 + mx^2 + 1$, respectively. From (2) we get the equality of multisets

$$\{x_1, x_2, x_3\} \cup R_Q = \{y_1, y_2, y_3\} \cup (R_Q + 1).$$

For every $r \in R_Q$, since $r + 1$ is in the set of the right hand side, we must have $r + 1 \in R_Q$ or $r + 1 = x_i$ for some i. Similarly, since r is in the set of the left hand side, either $r - 1 \in R_Q$ or $r = y_i$ for some i. This implies that, possibly after relabelling y_1, y_2, y_3, all the roots of (2) may be partitioned into three chains of the form $\{y_i,\, y_i + 1, \ldots, y_i + k_i = x_i\}$ for $i = 1, 2, 3$ and some integers $k_1, k_2, k_3 \geqslant 0$.

Now we analyze the roots of the polynomial $A_a(x) = x^3 + ax^2 + 1$. Using calculus or elementary methods, we find that the local extrema of $A_a(x)$ occur at $x = 0$ and $x = -2a/3$; their values are $A_a(0) = 1 > 0$ and $A_a(-2a/3) = 1 + 4a^3/27$, which is positive for integers $a \geqslant -1$ and negative for integers $a \leqslant -2$. So when $a \in \mathbb{Z}$, A_a has three real roots if $a \leqslant -2$ and one if $a \geqslant -1$.

Now, since $y_i - x_i \in \mathbb{Z}$ for $i = 1, 2, 3$, the cubics A_m and A_{-m} must have the same number of real roots. The previous analysis then implies that $m = 1$ or $m = -1$. Therefore the real root α of $A_1(x) = x^3 + x^2 + 1$ and the real root β of $A_{-1}(x) = x^3 - x^2 + 1$ must differ by an integer. But this is impossible, because $A_1\left(-\frac{3}{2}\right) = -\frac{1}{8}$ and $A_1(-1) = 1$ so $-1.5 < \alpha < -1$, while $A_{-1}(-1) = -1$ and $A_{-1}\left(-\frac{1}{2}\right) = \frac{5}{8}$, so $-1 < \beta < -0.5$.

It follows that $\deg P \leqslant 1$. Then, as shown in Solution 1, we conclude that the solutions are $P(x) = tx$ for all real numbers t.

Combinatorics

C1. Let n be a positive integer. Find the smallest integer k with the following property: Given any real numbers $a_1, \ldots, a_d$ such that $a_1 + a_2 + \cdots + a_d = n$ and $0 \leqslant a_i \leqslant 1$ for $i = 1, 2, \ldots, d$, it is possible to partition these numbers into k groups (some of which may be empty) such that the sum of the numbers in each group is at most 1.

(Poland)

Answer. $k = 2n - 1$.

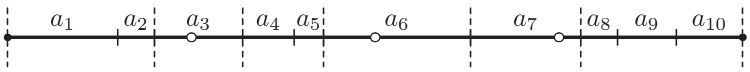
Solution 1. If $d = 2n-1$ and $a_1 = \cdots = a_{2n-1} = n/(2n-1)$, then each group in such a partition can contain at most one number, since $2n/(2n-1) > 1$. Therefore $k \geqslant 2n-1$. It remains to show that a suitable partition into $2n-1$ groups always exists.

We proceed by induction on d. For $d \leqslant 2n-1$ the result is trivial. If $d \geqslant 2n$, then since

$$(a_1 + a_2) + \ldots + (a_{2n-1} + a_{2n}) \leqslant n$$

we may find two numbers a_i, a_{i+1} such that $a_i + a_{i+1} \leqslant 1$. We "merge" these two numbers into one new number $a_i + a_{i+1}$. By the induction hypothesis, a suitable partition exists for the $d-1$ numbers $a_1, \ldots, a_{i-1}, a_i + a_{i+1}, a_{i+2}, \ldots, a_d$. This induces a suitable partition for $a_1, \ldots, a_d$.

Solution 2. We will show that it is even possible to split the sequence $a_1, \ldots, a_d$ into $2n-1$ *contiguous* groups so that the sum of the numbers in each groups does not exceed 1. Consider a segment S of length n, and partition it into segments $S_1, \ldots, S_d$ of lengths $a_1, \ldots, a_d$, respectively, as shown below. Consider a second partition of S into n equal parts by $n-1$ "empty dots".



Assume that the $n-1$ empty dots are in segments $S_{i_1}, \ldots, S_{i_{n-1}}$. (If a dot is on the boundary of two segments, we choose the right segment). These $n-1$ segments are distinct because they have length at most 1. Consider the partition:

$$\{a_1, \ldots, a_{i_1-1}\}, \{a_{i_1}\}, \{a_{i_1+1}, \ldots, a_{i_2-1}\}, \{a_{i_2}\}, \ldots \{a_{i_{n-1}}\}, \{a_{i_{n-1}+1}, \ldots, a_d\}.$$

In the example above, this partition is $\{a_1, a_2\}, \{a_3\}, \{a_4, a_5\}, \{a_6\}, \varnothing, \{a_7\}, \{a_8, a_9, a_{10}\}$. We claim that in this partition, the sum of the numbers in this group is at most 1.

For the sets $\{a_{i_t}\}$ this is obvious since $a_{i_t} \leqslant 1$. For the sets $\{a_{i_t} + 1, \ldots, a_{i_{t+1}-1}\}$ this follows from the fact that the corresponding segments lie between two neighboring empty dots, or between an endpoint of S and its nearest empty dot. Therefore the sum of their lengths cannot exceed 1.

Solution 3. First put all numbers greater than $\frac{1}{2}$ in their own groups. Then, form the remaining groups as follows: For each group, add new a_is one at a time until their sum exceeds $\frac{1}{2}$. Since the last summand is at most $\frac{1}{2}$, this group has sum at most 1. Continue this procedure until we have used all the a_is. Notice that the last group may have sum less than $\frac{1}{2}$. If the sum of the numbers in the last two groups is less than or equal to 1, we merge them into one group. In the end we are left with m groups. If $m = 1$ we are done. Otherwise the first $m-2$ have sums greater than $\frac{1}{2}$ and the last two have total sum greater than 1. Therefore $n > (m-2)/2 + 1$ so $m \leqslant 2n-1$ as desired.

Comment 1. The original proposal asked for the minimal value of k when $n = 2$.

Comment 2. More generally, one may ask the same question for real numbers between 0 and 1 whose sum is a real number r. In this case the smallest value of k is $k = \lceil 2r \rceil - 1$, as Solution 3 shows.

Solutions 1 and 2 lead to the slightly weaker bound $k \leqslant 2\lceil r \rceil - 1$. This is actually the optimal bound for partitions into *consecutive* groups, which are the ones contemplated in these two solutions. To see this, assume that r is not an integer and let $c = (r + 1 - \lceil r \rceil)/(1 + \lceil r \rceil)$. One easily checks that $0 < c < \frac{1}{2}$ and $\lceil r \rceil (2c) + (\lceil r \rceil - 1)(1 - c) = r$, so the sequence

$$2c,\ 1-c,\ 2c,\ 1-c,\ \ldots,\ 1-c,\ 2c$$

of $2\lceil r \rceil - 1$ numbers satisfies the given conditions. For this sequence, the only suitable partition into consecutive groups is the trivial partition, which requires $2\lceil r \rceil - 1$ groups.

C2. In the plane, 2013 red points and 2014 blue points are marked so that no three of the marked points are collinear. One needs to draw k lines not passing through the marked points and dividing the plane into several regions. The goal is to do it in such a way that no region contains points of both colors.

Find the minimal value of k such that the goal is attainable for every possible configuration of 4027 points.

(Australia)

Answer. $k = 2013$.

Solution 1. Firstly, let us present an example showing that $k \geqslant 2013$. Mark 2013 red and 2013 blue points on some circle alternately, and mark one more blue point somewhere in the plane. The circle is thus split into 4026 arcs, each arc having endpoints of different colors. Thus, if the goal is reached, then each arc should intersect some of the drawn lines. Since any line contains at most two points of the circle, one needs at least $4026/2 = 2013$ lines.

It remains to prove that one can reach the goal using 2013 lines. First of all, let us mention that for every two points A and B having the same color, one can draw two lines separating these points from all other ones. Namely, it suffices to take two lines parallel to AB and lying on different sides of AB sufficiently close to it: the only two points between these lines will be A and B.

Now, let P be the convex hull of all marked points. Two cases are possible.

Case 1. Assume that P has a red vertex A. Then one may draw a line separating A from all the other points, pair up the other 2012 red points into 1006 pairs, and separate each pair from the other points by two lines. Thus, 2013 lines will be used.

Case 2. Assume now that all the vertices of P are blue. Consider any two consecutive vertices of P, say A and B. One may separate these two points from the others by a line parallel to AB. Then, as in the previous case, one pairs up all the other 2012 blue points into 1006 pairs, and separates each pair from the other points by two lines. Again, 2013 lines will be used.

Comment 1. Instead of considering the convex hull, one may simply take a line containing two marked points A and B such that all the other marked points are on one side of this line. If one of A and B is red, then one may act as in Case 1; otherwise both are blue, and one may act as in Case 2.

Solution 2. Let us present a different proof of the fact that $k = 2013$ suffices. In fact, we will prove a more general statement:

If n points in the plane, no three of which are collinear, are colored in red and blue arbitrarily, then it suffices to draw $\lfloor n/2 \rfloor$ lines to reach the goal.

We proceed by induction on n. If $n \leqslant 2$ then the statement is obvious. Now assume that $n \geqslant 3$, and consider a line ℓ containing two marked points A and B such that all the other marked points are on one side of ℓ; for instance, any line containing a side of the convex hull works.

Remove for a moment the points A and B. By the induction hypothesis, for the remaining configuration it suffices to draw $\lfloor n/2 \rfloor - 1$ lines to reach the goal. Now return the points A and B back. Three cases are possible.

Case 1. If A and B have the same color, then one may draw a line parallel to ℓ and separating A and B from the other points. Obviously, the obtained configuration of $\lfloor n/2 \rfloor$ lines works.

Case 2. If A and B have different colors, but they are separated by some drawn line, then again the same line parallel to ℓ works.

Case 3. Finally, assume that A and B have different colors and lie in one of the regions defined by the drawn lines. By the induction assumption, this region contains no other points of one of the colors — without loss of generality, the only blue point it contains is A. Then it suffices to draw a line separating A from all other points.

Thus the step of the induction is proved.

Comment 2. One may ask a more general question, replacing the numbers 2013 and 2014 by any positive integers m and n, say with $m \leqslant n$. Denote the answer for this problem by $f(m, n)$.

One may show along the lines of Solution 1 that $m \leqslant f(m, n) \leqslant m + 1$; moreover, if m is even then $f(m, n) = m$. On the other hand, for every odd m there exists an N such that $f(m, n) = m$ for all $m \leqslant n \leqslant N$, and $f(m, n) = m + 1$ for all $n > N$.

C3. A crazy physicist discovered a new kind of particle which he called an *imon*, after some of them mysteriously appeared in his lab. Some pairs of imons in the lab can be *entangled*, and each imon can participate in many entanglement relations. The physicist has found a way to perform the following two kinds of operations with these particles, one operation at a time.

(i) If some imon is entangled with an odd number of other imons in the lab, then the physicist can destroy it.

(ii) At any moment, he may double the whole family of imons in his lab by creating a copy I' of each imon I. During this procedure, the two copies I' and J' become entangled if and only if the original imons I and J are entangled, and each copy I' becomes entangled with its original imon I; no other entanglements occur or disappear at this moment.

Prove that the physicist may apply a sequence of such operations resulting in a family of imons, no two of which are entangled.

(Japan)

Solution 1. Let us consider a graph with the imons as vertices, and two imons being connected if and only if they are entangled. Recall that a *proper coloring* of a graph G is a coloring of its vertices in several colors so that every two connected vertices have different colors.

Lemma. Assume that a graph G admits a proper coloring in n colors ($n > 1$). Then one may perform a sequence of operations resulting in a graph which admits a proper coloring in $n-1$ colors.

Proof. Let us apply repeatedly operation (i) to any appropriate vertices while it is possible. Since the number of vertices decreases, this process finally results in a graph where all the degrees are even. Surely this graph also admits a proper coloring in n colors $1, \ldots, n$; let us fix this coloring.

Now apply the operation (ii) to this graph. A proper coloring of the resulting graph in n colors still exists: one may preserve the colors of the original vertices and color the vertex I' in a color $k+1 \pmod{n}$ if the vertex I has color k. Then two connected original vertices still have different colors, and so do their two connected copies. On the other hand, the vertices I and I' have different colors since $n > 1$.

All the degrees of the vertices in the resulting graph are odd, so one may apply operation (i) to delete consecutively all the vertices of color n one by one; no two of them are connected by an edge, so their degrees do not change during the process. Thus, we obtain a graph admitting a proper coloring in $n-1$ colors, as required. The lemma is proved. $\square$

Now, assume that a graph G has n vertices; then it admits a proper coloring in n colors. Applying repeatedly the lemma we finally obtain a graph admitting a proper coloring in one color, that is — a graph with no edges, as required.

Solution 2. Again, we will use the graph language.

I. We start with the following observation.

Lemma. Assume that a graph G contains an isolated vertex A, and a graph G° is obtained from G by deleting this vertex. Then, if one can apply a sequence of operations which makes a graph with no edges from G°, then such a sequence also exists for G.

Proof. Consider any operation applicable to G° resulting in a graph G_1°; then there exists a sequence of operations applicable to G and resulting in a graph G_1 differing from G_1° by an addition of an isolated vertex A. Indeed, if this operation is of type (i), then one may simply repeat it in G.

Otherwise, the operation is of type (ii), and one may apply it to G and then delete the vertex A' (it will have degree 1).

Thus one may change the process for G° into a corresponding process for G step by step. $\square$

In view of this lemma, if at some moment a graph contains some isolated vertex, then we may simply delete it; let us call this operation (iii).

II. Let $V = \{A_1^0, \ldots, A_n^0\}$ be the vertices of the initial graph. Let us describe which graphs can appear during our operations. Assume that operation (ii) was applied m times. If these were the only operations applied, then the resulting graph G_n^m has the set of vertices which can be enumerated as

$$V_n^m = \{A_i^j \colon 1 \leqslant i \leqslant n,\ 0 \leqslant j \leqslant 2^m - 1\},$$

where A_i^0 is the common "ancestor" of all the vertices A_i^j, and the binary expansion of j (adjoined with some zeroes at the left to have m digits) "keeps the history" of this vertex: the dth digit from the right is 0 if at the dth doubling the ancestor of A_i^j was in the original part, and this digit is 1 if it was in the copy.

Next, the two vertices A_i^j and A_k^ℓ in G_n^m are connected with an edge exactly if either (1) $j = \ell$ and there was an edge between A_i^0 and A_k^0 (so these vertices appeared at the same application of operation (ii)); or (2) $i = k$ and the binary expansions of j and ℓ differ in exactly one digit (so their ancestors became connected as a copy and the original vertex at some application of (ii)).

Now, if some operations (i) were applied during the process, then simply some vertices in G_n^m disappeared. So, in any case the resulting graph is some induced subgraph of G_n^m.

III. Finally, we will show that from each (not necessarily induced) subgraph of G_n^m one can obtain a graph with no vertices by applying operations (i), (ii) and (iii). We proceed by induction on n; the base case $n = 0$ is trivial.

For the induction step, let us show how to apply several operations so as to obtain a graph containing no vertices of the form A_n^j for $j \in \mathbb{Z}$. We will do this in three steps.

Step 1. We apply repeatedly operation (i) to any appropriate vertices while it is possible. In the resulting graph, all vertices have even degrees.

Step 2. Apply operation (ii) obtaining a subgraph of G_n^{m+1} with all degrees being odd. In this graph, we delete one by one all the vertices A_n^j where the sum of the binary digits of j is even; it is possible since there are no edges between such vertices, so all their degrees remain odd. After that, we delete all isolated vertices.

Step 3. Finally, consider any remaining vertex A_n^j (then the sum of digits of j is odd). If its degree is odd, then we simply delete it. Otherwise, since A_n^j is not isolated, we consider any vertex adjacent to it. It has the form A_k^j for some $k < n$ (otherwise it would have the form A_n^ℓ, where ℓ has an even digit sum; but any such vertex has already been deleted at Step 2). No neighbor of A_k^j was deleted at Steps 2 and 3, so it has an odd degree. Then we successively delete A_k^j and A_n^j.

Notice that this deletion does not affect the applicability of this step to other vertices, since no two vertices A_i^j and A_k^ℓ for different j, ℓ with odd digit sum are connected with an edge. Thus we will delete all the remaining vertices of the form A_n^j, obtaining a subgraph of G_{n-1}^{m+1}. The application of the induction hypothesis finishes the proof.

Comment. In fact, the graph G_n^m is a Cartesian product of G and the graph of an m-dimensional hypercube.

C4. Let n be a positive integer, and let A be a subset of $\{1, \ldots, n\}$. An *A-partition* of n *into k parts* is a representation of n as a sum $n = a_1 + \cdots + a_k$, where the parts $a_1, \ldots, a_k$ belong to A and are not necessarily distinct. The *number of different parts* in such a partition is the number of (distinct) elements in the set $\{a_1, a_2, \ldots, a_k\}$.

We say that an A-partition of n into k parts is *optimal* if there is no A-partition of n into r parts with $r < k$. Prove that any optimal A-partition of n contains at most $\sqrt[3]{6n}$ different parts.

(Germany)

Solution 1. If there are no A-partitions of n, the result is vacuously true. Otherwise, let $k_{\min}$ be the minimum number of parts in an A-partition of n, and let $n = a_1 + \cdots + a_{k_{\min}}$ be an optimal partition. Denote by s the number of different parts in this partition, so we can write $S = \{a_1, \ldots, a_{k_{\min}}\} = \{b_1, \ldots, b_s\}$ for some pairwise different numbers $b_1 < \cdots < b_s$ in A.

If $s > \sqrt[3]{6n}$, we will prove that there exist subsets X and Y of S such that $|X| < |Y|$ and $\sum_{x \in X} x = \sum_{y \in Y} y$. Then, deleting the elements of Y from our partition and adding the elements of X to it, we obtain an A-partition of n into less than $k_{\min}$ parts, which is the desired contradiction.

For each positive integer $k \leqslant s$, we consider the k-element subset

$$S_{1,0}^k := \{b_1, \ldots, b_k\}$$

as well as the following k-element subsets $S_{i,j}^k$ of S:

$$S_{i,j}^k := \big\{b_1, \ldots, b_{k-i}, b_{k-i+j+1}, b_{s-i+2}, \ldots, b_s\big\}, \quad i = 1, \ldots, k, \quad j = 1, \ldots, s-k.$$

Pictorially, if we represent the elements of S by a sequence of dots in increasing order, and represent a subset of S by shading in the appropriate dots, we have:

$$S_{i,j}^k = \underbrace{\bullet\bullet\bullet\bullet\bullet\bullet\bullet}_{k-i}\underbrace{\circ\circ\circ\circ\circ}_{j}\bullet\underbrace{\circ\circ\circ\circ\circ\circ\circ}_{s-k-j}\underbrace{\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet}_{i-1}$$

Denote by $\Sigma_{i,j}^k$ the sum of elements in $S_{i,j}^k$. Clearly, $\Sigma_{1,0}^k$ is the minimum sum of a k-element subset of S. Next, for all appropriate indices i and j we have

$$\Sigma_{i,j}^k = \Sigma_{i,j+1}^k + b_{k-i+j+1} - b_{k-i+j+2} < \Sigma_{i,j+1}^k \quad \text{and} \quad \Sigma_{i,s-k}^k = \Sigma_{i+1,1}^k + b_{k-i} - b_{k-i+1} < \Sigma_{i+1,1}^k.$$

Therefore

$$1 \leqslant \Sigma_{1,0}^k < \Sigma_{1,1}^k < \Sigma_{1,2}^k < \cdots < \Sigma_{1,s-k}^k < \Sigma_{2,1}^k < \cdots < \Sigma_{2,s-k}^k < \Sigma_{3,1}^k < \cdots < \Sigma_{k,s-k}^k \leqslant n.$$

To see this in the picture, we start with the k leftmost points marked. At each step, we look for the rightmost point which can move to the right, and move it one unit to the right. We continue until the k rightmost points are marked. As we do this, the corresponding sums clearly increase.

For each k we have found $k(s-k)+1$ different integers of the form $\Sigma_{i,j}^k$ between 1 and n. As we vary k, the total number of integers we are considering is

$$\sum_{k=1}^{s} \big(k(s-k)+1\big) = s \cdot \frac{s(s+1)}{2} - \frac{s(s+1)(2s+1)}{6} + s = \frac{s(s^2+5)}{6} > \frac{s^3}{6} > n.$$

Since they are between 1 and n, at least two of these integers are equal. Consequently, there exist $1 \leqslant k < k' \leqslant s$ and $X = S_{i,j}^k$ as well as $Y = S_{i',j'}^{k'}$ such that

$$\sum_{x \in X} x = \sum_{y \in Y} y, \quad \text{but} \quad |X| = k < k' = |Y|,$$

as required. The result follows.

Solution 2. Assume, to the contrary, that the statement is false, and choose the minimum number n for which it fails. So there exists a set $A \subseteq \{1, \ldots, n\}$ together with an optimal A-partition $n = a_1 + \cdots + a_{k_{\min}}$ of n refuting our statement, where, of course, $k_{\min}$ is the minimum number of parts in an A-partition of n. Again, we define $S = \{a_1, \ldots, a_{k_{\min}}\} = \{b_1, \ldots, b_s\}$ with $b_1 < \cdots < b_s$; by our assumption we have $s > \sqrt[3]{6n} > 1$. Without loss of generality we assume that $a_{k_{\min}} = b_s$. Let us distinguish two cases.

Case 1. $b_s \geqslant \frac{s(s-1)}{2} + 1$.

Consider the partition $n - b_s = a_1 + \cdots + a_{k_{\min}-1}$, which is clearly a minimum A-partition of $n - b_s$ with at least $s - 1 \geqslant 1$ different parts. Now, from $n < \frac{s^3}{6}$ we obtain

$$n - b_s \leqslant n - \frac{s(s-1)}{2} - 1 < \frac{s^3}{6} - \frac{s(s-1)}{2} - 1 < \frac{(s-1)^3}{6},$$

so $s - 1 > \sqrt[3]{6(n - b_s)}$, which contradicts the choice of n.

Case 2. $b_s \leqslant \frac{s(s-1)}{2}$.

Set $b_0 = 0$, $\Sigma_{0,0} = 0$, and $\Sigma_{i,j} = b_1 + \cdots + b_{i-1} + b_j$ for $1 \leqslant i \leqslant j < s$. There are $\frac{s(s-1)}{2} + 1 > b_s$ such sums; so at least two of them, say $\Sigma_{i,j}$ and $\Sigma_{i',j'}$, are congruent modulo b_s (where $(i,j) \neq (i',j')$). This means that $\Sigma_{i,j} - \Sigma_{i',j'} = rb_s$ for some integer r. Notice that for $i \leqslant j < k < s$ we have

$$0 < \Sigma_{i,k} - \Sigma_{i,j} = b_k - b_j < b_s,$$

so the indices i and i' are distinct, and we may assume that $i > i'$. Next, we observe that $\Sigma_{i,j} - \Sigma_{i',j'} = (b_{i'} - b_{j'}) + b_j + b_{i'+1} + \cdots + b_{i-1}$ and $b_{i'} \leqslant b_{j'}$ imply

$$-b_s < -b_{j'} < \Sigma_{i,j} - \Sigma_{i',j'} < (i - i')b_s,$$

so $0 \leqslant r \leqslant i - i' - 1$.

Thus, we may remove the i terms of $\Sigma_{i,j}$ in our A-partition, and replace them by the i' terms of $\Sigma_{i',j'}$ and r terms equal to b_s, for a total of $r + i' < i$ terms. The result is an A-partition of n into a smaller number of parts, a contradiction.

Comment. The original proposal also contained a second part, showing that the estimate appearing in the problem has the correct order of magnitude:

For every positive integer n, there exist a set A and an optimal A-partition of n that contains $\lfloor \sqrt[3]{2n} \rfloor$ different parts.

The Problem Selection Committee removed this statement from the problem, since it seems to be less suitable for the competiton; but for completeness we provide an outline of its proof here.

Let $k = \lfloor \sqrt[3]{2n} \rfloor - 1$. The statement is trivial for $n < 4$, so we assume $n \geqslant 4$ and hence $k \geqslant 1$. Let $h = \lfloor \frac{n-1}{k} \rfloor$. Notice that $h \geqslant \frac{n}{k} - 1$.

Now let $A = \{1, \ldots, h\}$, and set $a_1 = h$, $a_2 = h - 1$, …, $a_k = h - k + 1$, and $a_{k+1} = n - (a_1 + \cdots + a_k)$. It is not difficult to prove that $a_k > a_{k+1} \geqslant 1$, which shows that

$$n = a_1 + \ldots + a_{k+1}$$

is an A-partition of n into $k + 1$ different parts. Since $kh < n$, any A-partition of n has at least $k + 1$ parts. Therefore our A-partition is optimal, and it has $\lfloor \sqrt[3]{2n} \rfloor$ distinct parts, as desired.

C5. Let r be a positive integer, and let $a_0, a_1, \ldots$ be an infinite sequence of real numbers. Assume that for all nonnegative integers m and s there exists a positive integer $n \in [m+1, m+r]$ such that

$$a_m + a_{m+1} + \cdots + a_{m+s} = a_n + a_{n+1} + \cdots + a_{n+s}.$$

Prove that the sequence is periodic, i. e. there exists some $p \geqslant 1$ such that $a_{n+p} = a_n$ for all $n \geqslant 0$.

(India)

Solution. For every indices $m \leqslant n$ we will denote $S(m,n) = a_m + a_{m+1} + \cdots + a_{n-1}$; thus $S(n,n) = 0$. Let us start with the following lemma.

Lemma. Let $b_0, b_1, \ldots$ be an infinite sequence. Assume that for every nonnegative integer m there exists a nonnegative integer $n \in [m+1, m+r]$ such that $b_m = b_n$. Then for every indices $k \leqslant \ell$ there exists an index $t \in [\ell, \ell+r-1]$ such that $b_t = b_k$. Moreover, there are at most r distinct numbers among the terms of (b_i).

Proof. To prove the first claim, let us notice that there exists an infinite sequence of indices $k_1 = k, k_2, k_3, \ldots$ such that $b_{k_1} = b_{k_2} = \cdots = b_k$ and $k_i < k_{i+1} \leqslant k_i + r$ for all $i \geqslant 1$. This sequence is unbounded from above, thus it hits each segment of the form $[\ell, \ell+r-1]$ with $\ell \geqslant k$, as required.

To prove the second claim, assume, to the contrary, that there exist $r+1$ distinct numbers $b_{i_1}, \ldots, b_{i_{r+1}}$. Let us apply the first claim to $k = i_1, \ldots, i_{r+1}$ and $\ell = \max\{i_1, \ldots, i_{r+1}\}$; we obtain that for every $j \in \{1, \ldots, r+1\}$ there exists $t_j \in [s, s+r-1]$ such that $b_{t_j} = b_{i_j}$. Thus the segment $[s, s+r-1]$ should contain $r+1$ distinct integers, which is absurd. □

Setting $s = 0$ in the problem condition, we see that the sequence (a_i) satisfies the condition of the lemma, thus it attains at most r distinct values. Denote by A_i the ordered r-tuple $(a_i, \ldots, a_{i+r-1})$; then among A_i's there are at most r^r distinct tuples, so for every $k \geqslant 0$ two of the tuples $A_k, A_{k+1}, \ldots, A_{k+r^r}$ are identical. This means that there exists a positive integer $p \leqslant r^r$ such that the equality $A_d = A_{d+p}$ holds infinitely many times. Let D be the set of indices d satisfying this relation.

Now we claim that D coincides with the set of all nonnegative integers. Since D is unbounded, it suffices to show that $d \in D$ whenever $d+1 \in D$. For that, denote $b_k = S(k, p+k)$. The sequence $b_0, b_1, \ldots$ satisfies the lemma conditions, so there exists an index $t \in [d+1, d+r]$ such that $S(t, t+p) = S(d, d+p)$. This last relation rewrites as $S(d,t) = S(d+p, t+p)$. Since $A_{d+1} = A_{d+p+1}$, we have $S(d+1, t) = S(d+p+1, t+p)$, therefore we obtain

$$a_d = S(d,t) - S(d+1,t) = S(d+p, t+p) - S(d+p+1, t+p) = a_{d+p}$$

and thus $A_d = A_{d+p}$, as required.

Finally, we get $A_d = A_{d+p}$ for all d, so in particular $a_d = a_{d+p}$ for all d, QED.

Comment 1. In the present proof, the upper bound for the minimal period length is r^r. This bound is not sharp; for instance, one may improve it to $(r-1)^r$ for $r \geqslant 3$..

On the other hand, this minimal length may happen to be greater than r. For instance, it is easy to check that the sequence with period $(3, -3, 3, -3, 3, -1, -1, -1)$ satisfies the problem condition for $r = 7$.

Comment 2. The conclusion remains true even if the problem condition only holds for every $s \geqslant N$ for some positive integer N. To show that, one can act as follows. Firstly, the sums of the form $S(i, i+N)$ attain at most r values, as well as the sums of the form $S(i, i+N+1)$. Thus the terms $a_i = S(i, i+N+1) - S(i+1, i+N+1)$ attain at most r^2 distinct values. Then, among the tuples $A_k, A_{k+N}, \ldots, A_{k+r^{2r}N}$ two are identical, so for some $p \leqslant r^{2r}$ the set $D = \{d\colon A_d = A_{d+Np}\}$ is infinite. The further arguments apply almost literally, with p being replaced by Np.

After having proved that such a sequence is also necessarily periodic, one may reduce the bound for the minimal period length to r^r — essentially by verifying that the sequence satisfies the original version of the condition.

C6. In some country several pairs of cities are connected by direct two-way flights. It is possible to go from any city to any other by a sequence of flights. The *distance* between two cities is defined to be the least possible number of flights required to go from one of them to the other. It is known that for any city there are at most 100 cities at distance exactly three from it. Prove that there is no city such that more than 2550 other cities have distance exactly four from it.

(Russia)

Solution. Let us denote by $d(a, b)$ the distance between the cities a and b, and by

$$S_i(a) = \{c : d(a, c) = i\}$$

the set of cities at distance exactly i from city a.

Assume that for some city x the set $D = S_4(x)$ has size at least 2551. Let $A = S_1(x)$. A subset A' of A is said to be *substantial*, if every city in D can be reached from x with four flights while passing through some member of A'; in other terms, every city in D has distance 3 from some member of A', or $D \subseteq \bigcup_{a \in A'} S_3(a)$. For instance, A itself is substantial. Now let us fix some substantial subset A^* of A having the minimal cardinality $m = |A^*|$.

Since

$$m(101 - m) \leqslant 50 \cdot 51 = 2550,$$

there has to be a city $a \in A^*$ such that $|S_3(a) \cap D| \geqslant 102 - m$. As $|S_3(a)| \leqslant 100$, we obtain that $S_3(a)$ may contain at most $100 - (102 - m) = m - 2$ cities c with $d(c, x) \leqslant 3$. Let us denote by $T = \{c \in S_3(a) : d(x, c) \leqslant 3\}$ the set of all such cities, so $|T| \leqslant m - 2$. Now, to get a contradiction, we will construct $m - 1$ distinct elements in T, corresponding to $m - 1$ elements of the set $A_a = A^* \backslash \{a\}$.

Firstly, due to the minimality of A^*, for each $y \in A_a$ there exists some city $d_y \in D$ which can only be reached with four flights from x by passing through y. So, there is a way to get from x to d_y along x–y–b_y–c_y–d_y for some cities b_y and c_y; notice that $d(x, b_y) = 2$ and $d(x, c_y) = 3$ since this path has the minimal possible length.

Now we claim that all $2(m - 1)$ cities of the form b_y, c_y with $y \in A_a$ are distinct. Indeed, no b_y may coincide with any c_z since their distances from x are different. On the other hand, if one had $b_y = b_z$ for $y \neq z$, then there would exist a path of length 4 from x to d_z via y, namely x–y–b_z–c_z–d_z; this is impossible by the choice of d_z. Similarly, $c_y \neq c_z$ for $y \neq z$.

So, it suffices to prove that for every $y \in A_a$, one of the cities b_y and c_y has distance 3 from a (and thus belongs to T). For that, notice that $d(a, y) \leqslant 2$ due to the path a–x–y, while $d(a, d_y) \geqslant d(x, d_y) - d(x, a) = 3$. Moreover, $d(a, d_y) \neq 3$ by the choice of d_y; thus $d(a, d_y) > 3$. Finally, in the sequence $d(a, y)$, $d(a, b_y)$, $d(a, c_y)$, $d(a, d_y)$ the neighboring terms differ by at most 1, the first term is less than 3, and the last one is greater than 3; thus there exists one which is equal to 3, as required.

Comment 1. The upper bound 2550 is sharp. This can be seen by means of various examples; one of them is the "Roman Empire": it has one capital, called "Rome", that is connected to 51 semicapitals by internally disjoint paths of length 3. Moreover, each of these semicapitals is connected to 50 rural cities by direct flights.

Comment 2. Observe that, under the conditions of the problem, there exists no bound for the size of $S_1(x)$ or $S_2(x)$.

Comment 3. The numbers 100 and 2550 appearing in the statement of the problem may be replaced by n and $\left\lfloor \frac{(n+1)^2}{4} \right\rfloor$ for any positive integer n. Still more generally, one can also replace the pair $(3, 4)$ of distances under consideration by any pair (r, s) of positive integers satisfying $r < s \leqslant \frac{3}{2}r$.

To adapt the above proof to this situation, one takes $A = S_{s-r}(x)$ and defines the concept of substantiality as before. Then one takes A^* to be a minimal substantial subset of A, and for each $y \in A^*$ one fixes an element $d_y \in S_s(x)$ which is only reachable from x by a path of length s by passing through y. As before, it suffices to show that for distinct $a, y \in A^*$ and a path $y = y_0 - y_1 - \ldots - y_r = d_y$, at least one of the cities $y_0, \ldots, y_{r-1}$ has distance r from a. This can be done as above; the relation $s \leqslant \frac{3}{2}r$ is used here to show that $d(a, y_0) \leqslant r$.

Moreover, the estimate $\left\lfloor \frac{(n+1)^2}{4} \right\rfloor$ is also sharp for every positive integer n and every positive integers r, s with $r < s \leqslant \frac{3}{2}r$. This may be shown by an example similar to that in the previous comment.

C7. Let $n \geqslant 2$ be an integer. Consider all circular arrangements of the numbers $0, 1, \ldots, n$; the $n+1$ rotations of an arrangement are considered to be equal. A circular arrangement is called *beautiful* if, for any four distinct numbers $0 \leqslant a, b, c, d \leqslant n$ with $a + c = b + d$, the chord joining numbers a and c does not intersect the chord joining numbers b and d.

Let M be the number of beautiful arrangements of $0, 1, \ldots, n$. Let N be the number of pairs (x, y) of positive integers such that $x + y \leqslant n$ and $\gcd(x, y) = 1$. Prove that

$$M = N + 1.$$

(Russia)

Solution 1. Given a circular arrangement of $[0, n] = \{0, 1, \ldots, n\}$, we define a k-*chord* to be a (possibly degenerate) chord whose (possibly equal) endpoints add up to k. We say that three chords of a circle are *aligned* if one of them separates the other two. Say that $m \geqslant 3$ chords are aligned if any three of them are aligned. For instance, in Figure 1, A, B, and C are aligned, while B, C, and D are not.

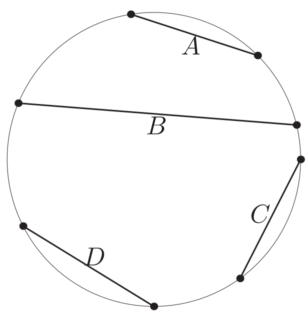


Figure 1

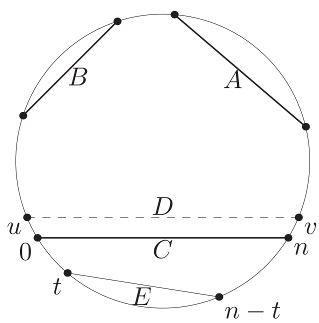


Figure 2

Claim. In a beautiful arrangement, the k–chords are aligned for any integer k.

Proof. We proceed by induction. For $n \leqslant 3$ the statement is trivial. Now let $n \geqslant 4$, and proceed by contradiction. Consider a beautiful arrangement S where the three k–chords A, B, C are not aligned. If n is not among the endpoints of A, B, and C, then by deleting n from S we obtain a beautiful arrangement $S \backslash \{n\}$ of $[0, n-1]$, where A, B, and C are aligned by the induction hypothesis. Similarly, if 0 is not among these endpoints, then deleting 0 and decreasing all the numbers by 1 gives a beautiful arrangement $S \backslash \{0\}$ where A, B, and C are aligned. Therefore both 0 and n are among the endpoints of these segments. If x and y are their respective partners, we have $n \geqslant 0 + x = k = n + y \geqslant n$. Thus 0 and n are the endpoints of one of the chords; say it is C.

Let D be the chord formed by the numbers u and v which are adjacent to 0 and n and on the same side of C as A and B, as shown in Figure 2. Set $t = u + v$. If we had $t = n$, the n–chords A, B, and D would not be aligned in the beautiful arrangement $S \backslash \{0, n\}$, contradicting the induction hypothesis. If $t < n$, then the t-chord from 0 to t cannot intersect D, so the chord C separates t and D. The chord E from t to $n - t$ does not intersect C, so t and $n - t$ are on the same side of C. But then the chords A, B, and E are not aligned in $S \backslash \{0, n\}$, a contradiction. Finally, the case $t > n$ is equivalent to the case $t < n$ via the beauty-preserving relabelling $x \mapsto n - x$ for $0 \leqslant x \leqslant n$, which sends t-chords to $(2n - t)$–chords. This proves the Claim.

Having established the Claim, we prove the desired result by induction. The case $n = 2$ is trivial. Now assume that $n \geqslant 3$. Let S be a beautiful arrangement of $[0, n]$ and delete n to obtain the beautiful arrangement T of $[0, n-1]$. The n–chords of T are aligned, and they contain every point except 0. Say T is of *Type 1* if 0 lies between two of these n–chords, and it is of *Type 2* otherwise; *i.e.*, if 0 is aligned with these n–chords. We will show that each Type 1 arrangement of $[0, n-1]$ arises from a unique arrangement of $[0, n]$, and each Type 2 arrangement of $[0, n-1]$ arises from exactly two beautiful arrangements of $[0, n]$.

If T is of Type 1, let 0 lie between chords A and B. Since the chord from 0 to n must be aligned with A and B in S, n must be on the other arc between A and B. Therefore S can be recovered uniquely from T. In the other direction, if T is of Type 1 and we insert n as above, then we claim the resulting arrangement S is beautiful. For $0 < k < n$, the k–chords of S are also k–chords of T, so they are aligned. Finally, for $n < k < 2n$, notice that the n–chords of S are parallel by construction, so there is an antisymmetry axis ℓ such that x is symmetric to $n - x$ with respect to ℓ for all x. If we had two k–chords which intersect, then their reflections across ℓ would be two $(2n-k)$-chords which intersect, where $0 < 2n - k < n$, a contradiction.

If T is of Type 2, there are two possible positions for n in S, on either side of 0. As above, we check that both positions lead to beautiful arrangements of $[0, n]$.

Hence if we let M_n be the number of beautiful arrangements of $[0, n]$, and let L_n be the number of beautiful arrangements of $[0, n-1]$ of Type 2, we have

$$M_n = (M_{n-1} - L_{n-1}) + 2L_{n-1} = M_{n-1} + L_{n-1}.$$

It then remains to show that L_{n-1} is the number of pairs (x, y) of positive integers with $x + y = n$ and $\gcd(x, y) = 1$. Since $n \geqslant 3$, this number equals $\varphi(n) = \#\{x : 1 \leqslant x \leqslant n, \gcd(x, n) = 1\}$.

To prove this, consider a Type 2 beautiful arrangement of $[0, n-1]$. Label the positions $0, \ldots, n-1 \pmod{n}$ clockwise around the circle, so that number 0 is in position 0. Let $f(i)$ be the number in position i; note that f is a permutation of $[0, n-1]$. Let a be the position such that $f(a) = n - 1$.

Since the n–chords are aligned with 0, and every point is in an n–chord, these chords are all parallel and

$$f(i) + f(-i) = n \qquad \text{for all } i.$$

Similarly, since the $(n-1)$–chords are aligned and every point is in an $(n-1)$–chord, these chords are also parallel and

$$f(i) + f(a - i) = n - 1 \qquad \text{for all } i.$$

Therefore $f(a - i) = f(-i) - 1$ for all i; and since $f(0) = 0$, we get

$$f(-ak) = k \qquad \text{for all } k. \tag{1}$$

Recall that this is an equality modulo n. Since f is a permutation, we must have $(a, n) = 1$. Hence $L_{n-1} \leqslant \varphi(n)$.

To prove equality, it remains to observe that the labeling (1) is beautiful. To see this, consider four numbers w, x, y, z on the circle with $w + y = x + z$. Their positions around the circle satisfy $(-aw) + (-ay) = (-ax) + (-az)$, which means that the chord from w to y and the chord from x to z are parallel. Thus (1) is beautiful, and by construction it has Type 2. The desired result follows.

Solution 2. Notice that there are exactly N irreducible fractions $f_1 < \cdots < f_N$ in $(0,1)$ whose denominator is at most n, since the pair (x,y) with $x+y \leqslant n$ and $(x,y)=1$ corresponds to the fraction $x/(x+y)$. Write $f_i = \frac{a_i}{b_i}$ for $1 \leqslant i \leqslant N$.

We begin by constructing $N+1$ beautiful arrangements. Take any $\alpha \in (0,1)$ which is not one of the above N fractions. Consider a circle of perimeter 1. Successively mark points $0, 1, 2, \ldots, n$ where 0 is arbitrary, and the clockwise distance from i to $i+1$ is α. The point k will be at clockwise distance $\{k\alpha\}$ from 0, where $\{r\}$ denotes the fractional part of r. Call such a circular arrangement *cyclic* and denote it by $A(\alpha)$. If the clockwise order of the points is the same in $A(\alpha_1)$ and $A(\alpha_2)$, we regard them as the same circular arrangement. Figure 3 shows the cyclic arrangement $A(3/5+\epsilon)$ of $[0,13]$ where $\epsilon > 0$ is very small.

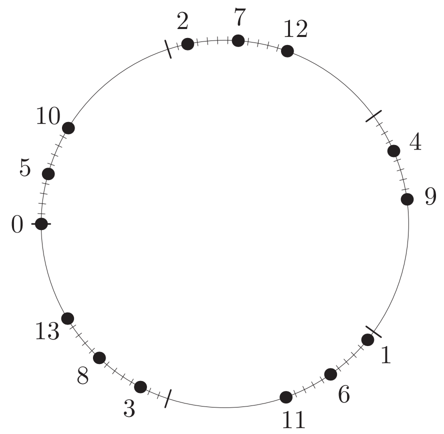


Figure 3

If $0 \leqslant a,\, b,\, c,\, d \leqslant n$ satisfy $a+c = b+d$, then $a\alpha + c\alpha = b\alpha + d\alpha$, so the chord from a to c is parallel to the chord from b to d in $A(\alpha)$. Hence in a cyclic arrangement all k—chords are parallel. In particular every cyclic arrangement is beautiful.

Next we show that there are exactly $N+1$ distinct cyclic arrangements. To see this, let us see how $A(\alpha)$ changes as we increase α from 0 to 1. The order of points p and q changes precisely when we cross a value $\alpha = f$ such that $\{pf\} = \{qf\}$; this can only happen if f is one of the N fractions $f_1, \ldots, f_N$. Therefore there are at most $N+1$ different cyclic arrangements.

To show they are all distinct, recall that $f_i = a_i/b_i$ and let $\epsilon > 0$ be a very small number. In the arrangement $A(f_i + \epsilon)$, point k lands at $\frac{ka_i \pmod{b_i}}{b_i} + k\epsilon$. Therefore the points are grouped into b_i clusters next to the points $0, \frac{1}{b_i}, \ldots, \frac{b_i - 1}{b_i}$ of the circle. The cluster following $\frac{k}{b_i}$ contains the numbers congruent to ka_i^{-1} modulo b_i, listed clockwise in increasing order. It follows that the first number after 0 in $A(f_i+\epsilon)$ is b_i, and the first number after 0 which is less than b_i is $a_i^{-1} \pmod{b_i}$, which uniquely determines a_i. In this way we can recover f_i from the cyclic arrangement. Note also that $A(f_i+\epsilon)$ is not the trivial arrangement where we list $0, 1, \ldots, n$ in order clockwise. It follows that the $N+1$ cyclic arrangements $A(\epsilon), A(f_1+\epsilon), \ldots, A(f_N+\epsilon)$ are distinct.

Let us record an observation which will be useful later:

$$\text{if } f_i < \alpha < f_{i+1} \text{ then 0 is immediately after } b_{i+1} \text{ and before } b_i \text{ in } A(\alpha). \tag{2}$$

Indeed, we already observed that b_i is the first number after 0 in $A(f_i+\epsilon) = A(\alpha)$. Similarly we see that b_{i+1} is the last number before 0 in $A(f_{i+1}-\epsilon) = A(\alpha)$.

Finally, we show that any beautiful arrangement of $[0, n]$ is cyclic by induction on n. For $n \leqslant 2$ the result is clear. Now assume that all beautiful arrangements of $[0, n-1]$ are cyclic, and consider a beautiful arrangement A of $[0, n]$. The subarrangement $A_{n-1} = A \backslash \{n\}$ of $[0, n-1]$ obtained by deleting n is cyclic; say $A_{n-1} = A_{n-1}(\alpha)$.

Let α be between the consecutive fractions $\frac{p_1}{q_1} < \frac{p_2}{q_2}$ among the irreducible fractions of denominator at most $n-1$. There is at most one fraction $\frac{i}{n}$ in $(\frac{p_1}{q_1}, \frac{p_2}{q_2})$, since $\frac{i}{n} < \frac{i}{n-1} \leqslant \frac{i+1}{n}$ for $0 < i \leqslant n-1$.

Case 1. There is no fraction with denominator n between $\frac{p_1}{q_1}$ and $\frac{p_2}{q_2}$.

In this case the only cyclic arrangement extending $A_{n-1}(\alpha)$ is $A_n(\alpha)$. We know that A and $A_n(\alpha)$ can only differ in the position of n. Assume n is immediately after x and before y in $A_n(\alpha)$. Since the neighbors of 0 are q_1 and q_2 by (2), we have $x, y \geqslant 1$.

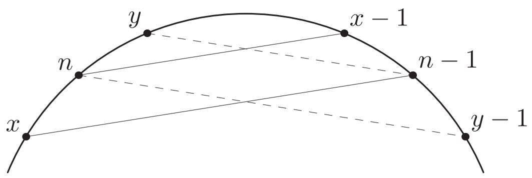


Figure 4

In $A_n(\alpha)$ the chord from $n-1$ to x is parallel and adjacent to the chord from n to $x-1$, so $n-1$ is between $x-1$ and x in clockwise order, as shown in Figure 4. Similarly, $n-1$ is between y and $y-1$. Therefore x, y, $x-1$, $n-1$, and $y-1$ occur in this order in $A_n(\alpha)$ and hence in A (possibly with $y = x-1$ or $x = y-1$).

Now, A may only differ from $A_n(\alpha)$ in the location of n. In A, since the chord from $n-1$ to x and the chord from n to $x-1$ do not intersect, n is between x and $n-1$. Similarly, n is between $n-1$ and y. Then n must be between x and y and $A = A_n(\alpha)$. Therefore A is cyclic as desired.

Case 2. There is exactly one i with $\frac{p_1}{q_1} < \frac{i}{n} < \frac{p_2}{q_2}$.

In this case there exist two cyclic arrangements $A_n(\alpha_1)$ and $A_n(\alpha_2)$ of the numbers $0, \ldots, n$ extending $A_{n-1}(\alpha)$, where $\frac{p_1}{q_1} < \alpha_1 < \frac{i}{n}$ and $\frac{i}{n} < \alpha_2 < \frac{p_2}{q_2}$. In $A_{n-1}(\alpha)$, 0 is the only number between q_2 and q_1 by (2). For the same reason, n is between q_2 and 0 in $A_n(\alpha_1)$, and between 0 and q_1 in $A_n(\alpha_2)$.

Letting $x = q_2$ and $y = q_1$, the argument of Case 1 tells us that n must be between x and y in A. Therefore A must equal $A_n(\alpha_1)$ or $A_n(\alpha_2)$, and therefore it is cyclic.

This concludes the proof that every beautiful arrangement is cyclic. It follows that there are exactly $N+1$ beautiful arrangements of $[0, n]$ as we wished to show.

C8. Players A and B play a paintful game on the real line. Player A has a pot of paint with four units of black ink. A quantity p of this ink suffices to blacken a (closed) real interval of length p. In every round, player A picks some positive integer m and provides $1/2^m$ units of ink from the pot. Player B then picks an integer k and blackens the interval from $k/2^m$ to $(k+1)/2^m$ (some parts of this interval may have been blackened before). The goal of player A is to reach a situation where the pot is empty and the interval $[0,1]$ is not completely blackened.

Decide whether there exists a strategy for player A to win in a finite number of moves.

(Austria)

Answer. No. Such a strategy for player A does not exist.

Solution. We will present a strategy for player B that guarantees that the interval $[0,1]$ is completely blackened, once the paint pot has become empty.

At the beginning of round r, let x_r denote the largest real number for which the interval between 0 and x_r has already been blackened; for completeness we define $x_1 = 0$. Let m be the integer picked by player A in this round; we define an integer y_r by

$$\frac{y_r}{2^m} \leqslant x_r < \frac{y_r+1}{2^m}.$$

Note that $I_0^r = [y_r/2^m, (y_r+1)/2^m]$ is the leftmost interval that may be painted in round r and that still contains some uncolored point.

Player B now looks at the *next* interval $I_1^r = [(y_r+1)/2^m, (y_r+2)/2^m]$. If I_1^r still contains an uncolored point, then player B blackens the interval I_1^r; otherwise he blackens the interval I_0^r. We make the convention that, at the beginning of the game, the interval $[1,2]$ is already blackened; thus, if $y_r + 1 = 2^m$, then B blackens I_0^r.

Our aim is to estimate the amount of ink used after each round. Firstly, we will prove by induction that, if before rth round the segment $[0,1]$ is not completely colored, then, before this move,

(i) the amount of ink used for the segment $[0, x_r]$ is at most $3x_r$; and

(ii) for every m, B has blackened at most one interval of length $1/2^m$ to the right of x_r.

Obviously, these conditions are satisfied for $r = 0$. Now assume that they were satisfied before the rth move, and consider the situation after this move; let m be the number A has picked at this move.

If B has blackened the interval I_1^r at this move, then $x_{r+1} = x_r$, and (i) holds by the induction hypothesis. Next, had B blackened before the rth move any interval of length $1/2^m$ to the right of x_r, this interval would necessarily coincide with I_1^r. By our strategy, this cannot happen. So, condition (ii) also remains valid.

Assume now that B has blackened the interval I_0^r at the rth move, but the interval $[0,1]$ still contains uncolored parts (which means that I_1^r is contained in $[0,1]$). Then condition (ii) clearly remains true, and we need to check (i) only. In our case, the intervals I_0^r and I_1^r are completely colored after the rth move, so x_{r+1} either reaches the right endpoint of I_1 or moves even further to the right. So, $x_{r+1} = x_r + \alpha$ for some $\alpha > 1/2^m$.

Next, any interval blackened by B before the rth move which intersects (x_r, x_{r+1}) should be contained in $[x_r, x_{r+1}]$; by (ii), all such intervals have different lengths not exceeding $1/2^m$, so the total amount of ink used for them is less than $2/2^m$. Thus, the amount of ink used for the segment $[0, x_{r+1}]$ does not exceed the sum of $2/2^m$, $3x_r$ (used for $[0, x_r]$), and $1/2^m$ used for the segment I_0^r. In total it gives at most $3(x_r + 1/2^m) < 3(x_r + \alpha) = 3x_{r+1}$. Thus condition (i) is also verified in this case. The claim is proved.

Finally, we can perform the desired estimation. Consider any situation in the game, say after the $(r-1)$st move; assume that the segment $[0,1]$ is not completely black. By (ii), in the segment $[x_r, 1]$ player B has colored several segments of different lengths; all these lengths are negative powers of 2 not exceeding $1 - x_r$; thus the total amount of ink used for this interval is at most $2(1 - x_r)$. Using (i), we obtain that the total amount of ink used is at most $3x_r + 2(1 - x_r) < 3$. Thus the pot is not empty, and therefore A never wins.

Comment 1. Notice that this strategy works even if the pot contains initially only 3 units of ink.

Comment 2. There exist other strategies for B allowing him to prevent emptying the pot before the whole interval is colored. On the other hand, let us mention some idea which *does not* work.

Player B could try a strategy in which the set of blackened points in each round is an interval of the type $[0, x]$. Such a strategy cannot work (even if there is more ink available). Indeed, under the assumption that B uses such a strategy, let us prove by induction on s the following statement:

For any positive integer s, player A has a strategy picking only positive integers $m \leqslant s$ in which, if player B ever paints a point $x \geqslant 1 - 1/2^s$ then after some move, exactly the interval $[0, 1 - 1/2^s]$ is blackened, and the amount of ink used up to this moment is at least $s/2$.

For the base case $s = 1$, player A just picks $m = 1$ in the first round. If for some positive integer k player A has such a strategy, for $s + 1$ he can first rescale his strategy to the interval $[0, 1/2]$ (sending in each round half of the amount of ink he would give by the original strategy). Thus, after some round, the interval $[0, 1/2 - 1/2^{s+1}]$ becomes blackened, and the amount of ink used is at least $s/4$. Now player A picks $m = 1/2$, and player B spends $1/2$ unit of ink to blacken the interval $[0, 1/2]$. After that, player A again rescales his strategy to the interval $[1/2, 1]$, and player B spends at least $s/4$ units of ink to blacken the interval $[1/2, 1 - 1/2^{s+1}]$, so he spends in total at least $s/4 + 1/2 + s/4 = (s + 1)/2$ units of ink.

Comment 3. In order to avoid finiteness issues, the statement could be replaced by the following one:

> *Players A and B play a paintful game on the real numbers. Player A has a paint pot with four units of black ink. A quantity p of this ink suffices to blacken a (closed) real interval of length p. In the beginning of the game, player A chooses (and announces) a positive integer N. In every round, player A picks some positive integer $m \leqslant N$ and provides $1/2^m$ units of ink from the pot. The player B picks an integer k and blackens the interval from $k/2^m$ to $(k + 1)/2^m$ (some parts of this interval may happen to be blackened before). The goal of player A is to reach a situation where the pot is empty and the interval $[0, 1]$ is not completely blackened.*
>
> *Decide whether there exists a strategy for player A to win.*

However, the Problem Selection Committee believes that this version may turn out to be harder than the original one.

Geometry

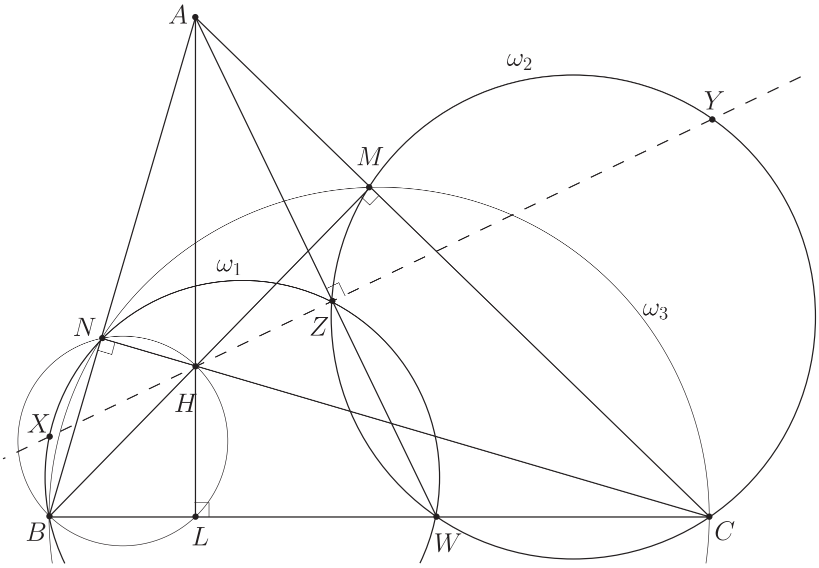
G1. Let ABC be an acute-angled triangle with orthocenter H, and let W be a point on side BC. Denote by M and N the feet of the altitudes from B and C, respectively. Denote by ω_1 the circumcircle of BWN, and let X be the point on ω_1 which is diametrically opposite to W. Analogously, denote by ω_2 the circumcircle of CWM, and let Y be the point on ω_2 which is diametrically opposite to W. Prove that X, Y and H are collinear.

(Thaliand)

Solution. Let L be the foot of the altitude from A, and let Z be the second intersection point of circles ω_1 and ω_2, other than W. We show that X, Y, Z and H lie on the same line.

Due to $\angle BNC = \angle BMC = 90^\circ$, the points B, C, N and M are concyclic; denote their circle by ω_3. Observe that the line WZ is the radical axis of ω_1 and ω_2; similarly, BN is the radical axis of ω_1 and ω_3, and CM is the radical axis of ω_2 and ω_3. Hence $A = BN \cap CM$ is the radical center of the three circles, and therefore WZ passes through A.

Since WX and WY are diameters in ω_1 and ω_2, respectively, we have $\angle WZX = \angle WZY = 90^\circ$, so the points X and Y lie on the line through Z, perpendicular to WZ.



The quadrilateral $BLHN$ is cyclic, because it has two opposite right angles. From the power of A with respect to the circles ω_1 and $BLHN$ we find $AL \cdot AH = AB \cdot AN = AW \cdot AZ$. If H lies on the line AW then this implies $H = Z$ immediately. Otherwise, by $\frac{AZ}{AH} = \frac{AL}{AW}$ the triangles AHZ and AWL are similar. Then $\angle HZA = \angle WLA = 90^\circ$, so the point H also lies on the line XYZ.

Comment. The original proposal also included a second statement:

> *Let P be the point on ω_1 such that WP is parallel to CN, and let Q be the point on ω_2 such that WQ is parallel to BM. Prove that P, Q and H are collinear if and only if $BW = CW$ or $AW \perp BC$.*

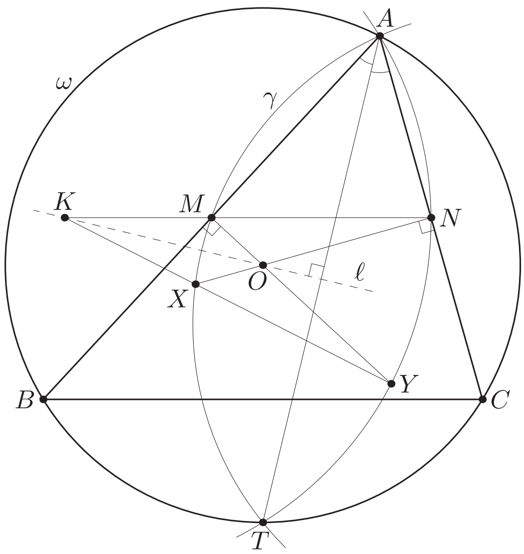
The Problem Selection Committee considered the first part more suitable for the competition.

G2. Let ω be the circumcircle of a triangle ABC. Denote by M and N the midpoints of the sides AB and AC, respectively, and denote by T the midpoint of the arc BC of ω not containing A. The circumcircles of the triangles AMT and ANT intersect the perpendicular bisectors of AC and AB at points X and Y, respectively; assume that X and Y lie inside the triangle ABC. The lines MN and XY intersect at K. Prove that $KA = KT$.

(Iran)

Solution 1. Let O be the center of ω, thus $O = MY \cap NX$. Let ℓ be the perpendicular bisector of AT (it also passes through O). Denote by r the operation of reflection about ℓ. Since AT is the angle bisector of $\angle BAC$, the line $r(AB)$ is parallel to AC. Since $OM \perp AB$ and $ON \perp AC$, this means that the line $r(OM)$ is parallel to the line ON and passes through O, so $r(OM) = ON$. Finally, the circumcircle γ of the triangle AMT is symmetric about ℓ, so $r(\gamma) = \gamma$. Thus the point M maps to the common point of ON with the arc AMT of γ — that is, $r(M) = X$.

Similarly, $r(N) = Y$. Thus, we get $r(MN) = XY$, and the common point K of MN nd XY lies on ℓ. This means exactly that $KA = KT$.



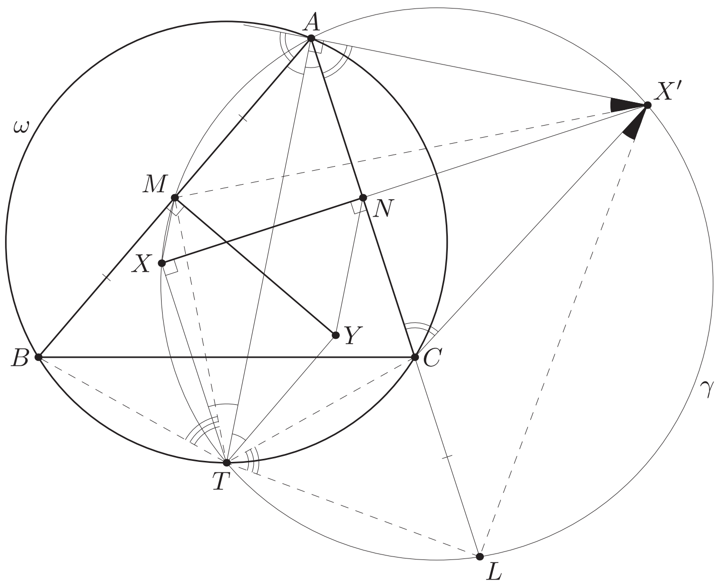
Solution 2. Let L be the second common point of the line AC with the circumcircle γ of the triangle AMT. From the cyclic quadrilaterals $ABTC$ and $AMTL$ we get $\angle BTC = 180^\circ - \angle BAC = \angle MTL$, which implies $\angle BTM = \angle CTL$. Since AT is an angle bisector in these quadrilaterals, we have $BT = TC$ and $MT = TL$. Thus the triangles BTM and CTL are congruent, so $CL = BM = AM$.

Let X' be the common point of the line NX with the external bisector of $\angle BAC$; notice that it lies outside the triangle ABC. Then we have $\angle TAX' = 90^\circ$ and $X'A = X'C$, so we get $\angle X'AM = 90^\circ + \angle BAC/2 = 180^\circ - \angle X'AC = 180^\circ - \angle X'CA = \angle X'CL$. Thus the triangles $X'AM$ and $X'CL$ are congruent, and therefore

$$\angle MX'L = \angle AX'C + (\angle CX'L - \angle AX'M) = \angle AX'C = 180^\circ - 2\angle X'AC = \angle BAC = \angle MAL.$$

This means that X' lies on γ.

Thus we have $\angle TXN = \angle TXX' = \angle TAX' = 90^\circ$, so $TX \parallel AC$. Then $\angle XTA = \angle TAC = \angle TAM$, so the cyclic quadrilateral $MATX$ is an isosceles trapezoid. Similarly, $NATY$ is an isosceles trapezoid, so again the lines MN and XY are the reflections of each other about the perpendicular bisector of AT. Thus K belongs to this perpendicular bisector.



Comment. There are several different ways of showing that the points X and M are symmetrical with respect to ℓ. For instance, one can show that the quadrilaterals $AMON$ and $TXOY$ are congruent. We chose Solution 1 as a simple way of doing it. On the other hand, Solution 2 shows some other interesting properties of the configuration.

Let us define Y', analogously to X', as the common point of MY and the external bisector of $\angle BAC$. One may easily see that in general the lines MN and $X'Y'$ (which is the external bisector of $\angle BAC$) do not intersect on the perpendicular bisector of AT. Thus, any solution should involve some argument using the choice of the intersection points X and Y.

G3. In a triangle ABC, let D and E be the feet of the angle bisectors of angles A and B, respectively. A rhombus is inscribed into the quadrilateral $AEDB$ (all vertices of the rhombus lie on different sides of $AEDB$). Let φ be the non-obtuse angle of the rhombus. Prove that $\varphi \leqslant \max\{\angle BAC, \angle ABC\}$.

(Serbia)

Solution 1. Let K, L, M, and N be the vertices of the rhombus lying on the sides AE, ED, DB, and BA, respectively. Denote by $d(X, YZ)$ the distance from a point X to a line YZ. Since D and E are the feet of the bisectors, we have $d(D, AB) = d(D, AC)$, $d(E, AB) = d(E, BC)$, and $d(D, BC) = d(E, AC) = 0$, which implies

$$d(D, AC) + d(D, BC) = d(D, AB) \quad \text{and} \quad d(E, AC) + d(E, BC) = d(E, AB).$$

Since L lies on the segment DE and the relation $d(X, AC) + d(X, BC) = d(X, AB)$ is linear in X inside the triangle, these two relations imply

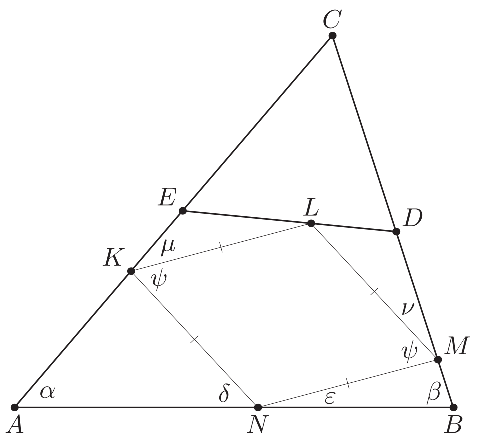
$$d(L, AC) + d(L, BC) = d(L, AB). \tag{1}$$

Denote the angles as in the figure below, and denote $a = KL$. Then we have $d(L, AC) = a \sin \mu$ and $d(L, BC) = a \sin \nu$. Since $KLMN$ is a parallelogram lying on one side of AB, we get

$$d(L, AB) = d(L, AB) + d(N, AB) = d(K, AB) + d(M, AB) = a(\sin \delta + \sin \varepsilon).$$

Thus the condition (1) reads

$$\sin \mu + \sin \nu = \sin \delta + \sin \varepsilon. \tag{2}$$



If one of the angles α and β is non-acute, then the desired inequality is trivial. So we assume that $\alpha, \beta < \pi/2$. It suffices to show then that $\psi = \angle NKL \leqslant \max\{\alpha, \beta\}$.

Assume, to the contrary, that $\psi > \max\{\alpha, \beta\}$. Since $\mu + \psi = \angle CKN = \alpha + \delta$, by our assumption we obtain $\mu = (\alpha - \psi) + \delta < \delta$. Similarly, $\nu < \varepsilon$. Next, since $KN \parallel ML$, we have $\beta = \delta + \nu$, so $\delta < \beta < \pi/2$. Similarly, $\varepsilon < \pi/2$. Finally, by $\mu < \delta < \pi/2$ and $\nu < \varepsilon < \pi/2$, we obtain

$$\sin \mu < \sin \delta \quad \text{and} \quad \sin \nu < \sin \varepsilon.$$

This contradicts (2).

Comment. One can see that the equality is achieved if $\alpha = \beta$ for every rhombus inscribed into the quadrilateral $AEDB$.

G4. Let ABC be a triangle with $\angle B > \angle C$. Let P and Q be two different points on line AC such that $\angle PBA = \angle QBA = \angle ACB$ and A is located between P and C. Suppose that there exists an interior point D of segment BQ for which $PD = PB$. Let the ray AD intersect the circle ABC at $R \neq A$. Prove that $QB = QR$.

(Georgia)

Solution 1. Denote by ω the circumcircle of the triangle ABC, and let $\angle ACB = \gamma$. Note that the condition $\gamma < \angle CBA$ implies $\gamma < 90^\circ$. Since $\angle PBA = \gamma$, the line PB is tangent to ω, so $PA \cdot PC = PB^2 = PD^2$. By $\frac{PA}{PD} = \frac{PD}{PC}$ the triangles PAD and PDC are similar, and $\angle ADP = \angle DCP$.

Next, since $\angle ABQ = \angle ACB$, the triangles ABC and AQB are also similar. Then $\angle AQB = \angle ABC = \angle ARC$, which means that the points D, R, C, and Q are concyclic. Therefore $\angle DRQ = \angle DCQ = \angle ADP$.

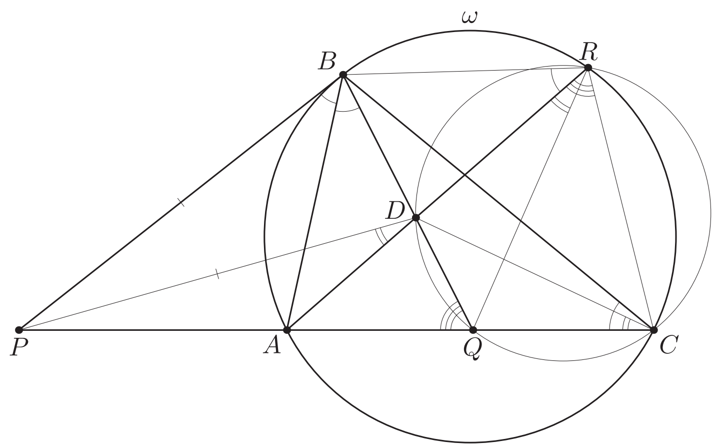


Figure 1

Now from $\angle ARB = \angle ACB = \gamma$ and $\angle PDB = \angle PBD = 2\gamma$ we get

$$\angle QBR = \angle ADB - \angle ARB = \angle ADP + \angle PDB - \angle ARB = \angle DRQ + \gamma = \angle QRB,$$

so the triangle QRB is isosceles, which yields $QB = QR$.

Solution 2. Again, denote by ω the circumcircle of the triangle ABC. Denote $\angle ACB = \gamma$. Since $\angle PBA = \gamma$, the line PB is tangent to ω.

Let E be the second intersection point of BQ with ω. If V' is any point on the ray CE beyond E, then $\angle BEV' = 180^\circ - \angle BEC = 180^\circ - \angle BAC = \angle PAB$; together with $\angle ABQ = \angle PBA$ this shows firstly, that the rays BA and CE intersect at some point V, and secondly that the triangle VEB is similar to the triangle PAB. Thus we have $\angle BVE = \angle BPA$. Next, $\angle AEV = \angle BEV - \gamma = \angle PAB - \angle ABQ = \angle AQB$; so the triangles PBQ and VAE are also similar.

Let PH be an altitude in the isosceles triangle PBD; then $BH = HD$. Let G be the intersection point of PH and AB. By the symmetry with respect to PH, we have $\angle BDG = \angle DBG = \gamma = \angle BEA$; thus $DG \parallel AE$ and hence $\frac{BG}{GA} = \frac{BD}{DE}$. Thus the points G and D correspond to each other in the similar triangles PAB and VEB, so $\angle DVB = \angle GPB = 90^\circ - \angle PBQ = 90^\circ - \angle VAE$. Thus $VD \perp AE$.

Let T be the common point of VD and AE, and let DS be an altitude in the triangle BDR. The points S and T are the feet of corresponding altitudes in the similar triangles ADE and BDR, so $\frac{BS}{SR} = \frac{AT}{TE}$. On the other hand, the points T and H are feet of corresponding altitudes in the similar triangles VAE and PBQ, so $\frac{AT}{TE} = \frac{BH}{HQ}$. Thus $\frac{BS}{SR} = \frac{AT}{TE} = \frac{BH}{HQ}$, and the triangles BHS and BQR are similar.

Finally, SH is a median in the right-angled triangle SBD; so $BH = HS$, and hence $BQ = QR$.

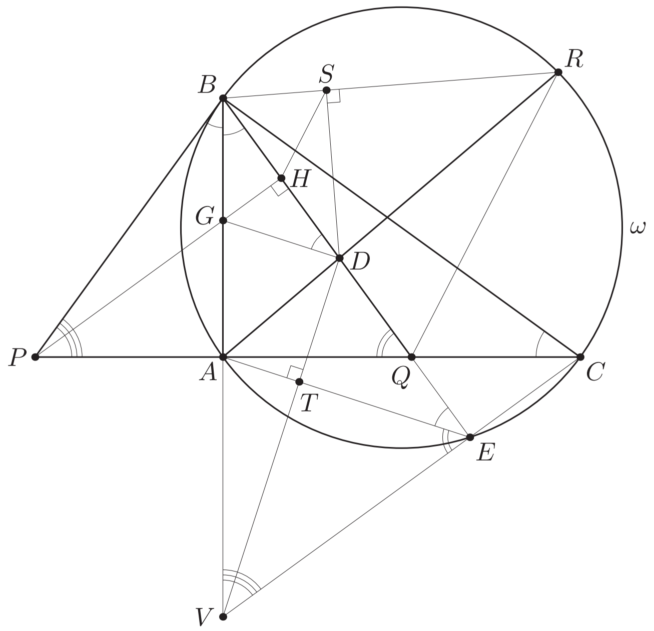


Figure 2

Solution 3. Denote by ω and O the circumcircle of the triangle ABC and its center, respectively. From the condition $\angle PBA = \angle BCA$ we know that BP is tangent to ω.

Let E be the second point of intersection of ω and BD. Due to the isosceles triangle BDP, the tangent of ω at E is parallel to DP and consequently it intersects BP at some point L. Of course, $PD \parallel LE$. Let M be the midpoint of BE, and let H be the midpoint of BR. Notice that $\angle AEB = \angle ACB = \angle ABQ = \angle ABE$, so A lies on the perpendicular bisector of BE; thus the points L, A, M, and O are collinear. Let ω_1 be the circle with diameter BO. Let $Q' = HO \cap BE$; since HO is the perpendicular bisector of BR, the statement of the problem is equivalent to $Q' = Q$.

Consider the following sequence of projections (see Fig. 3).

1. Project the line BE to the line LB through the center A. (This maps Q to P.)
2. Project the line LB to BE in parallel direction with LE. ($P \mapsto D$.)
3. Project the line BE to the circle ω through its point A. ($D \mapsto R$.)
4. Scale ω by the ratio $\frac{1}{2}$ from the point B to the circle ω_1. ($R \mapsto H$.)
5. Project ω_1 to the line BE through its point O. ($H \mapsto Q'$.)

We prove that the composition of these transforms, which maps the line BE to itself, is the identity. To achieve this, it suffices to show three fixed points. An obvious fixed point is B which is fixed by all the transformations above. Another fixed point is M, its path being $M \mapsto L \mapsto E \mapsto E \mapsto M \mapsto M$.

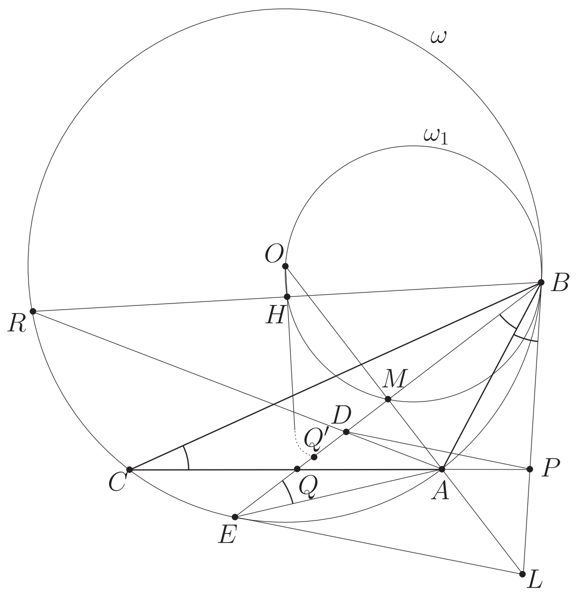


Figure 3

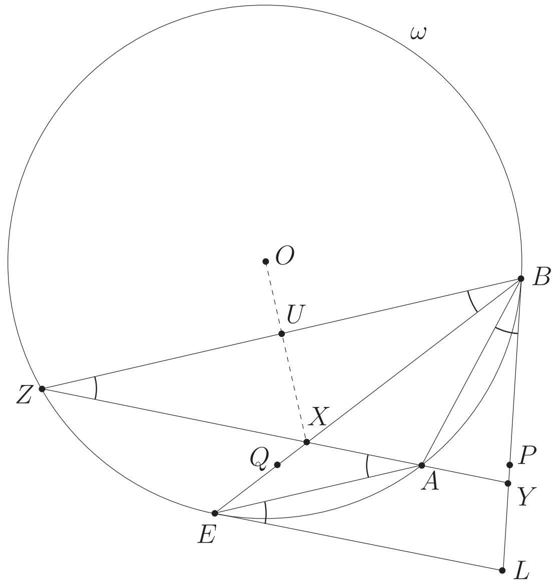


Figure 4

In order to show a third fixed point, draw a line parallel with LE through A; let that line intersect BE, LB and ω at X, Y and $Z \neq A$, respectively (see Fig. 4). We show that X is a fixed point. The images of X at the first three transformations are $X \mapsto Y \mapsto X \mapsto Z$. From $\angle XBZ = \angle EAZ = \angle AEL = \angle LBA = \angle BZX$ we can see that the triangle XBZ is isosceles. Let U be the midpoint of BZ; then the last two transformations do $Z \mapsto U \mapsto X$, and the point X is fixed.

Comment. Verifying that the point E is fixed seems more natural at first, but it appears to be less straightforward. Here we outline a possible proof.

Let the images of E at the first three transforms above be F, G and I. After comparing the angles depicted in Fig. 5 (noticing that the quadrilateral $AFBG$ is cyclic) we can observe that the tangent LE of ω is parallel to BI. Then, similarly to the above reasons, the point E is also fixed.

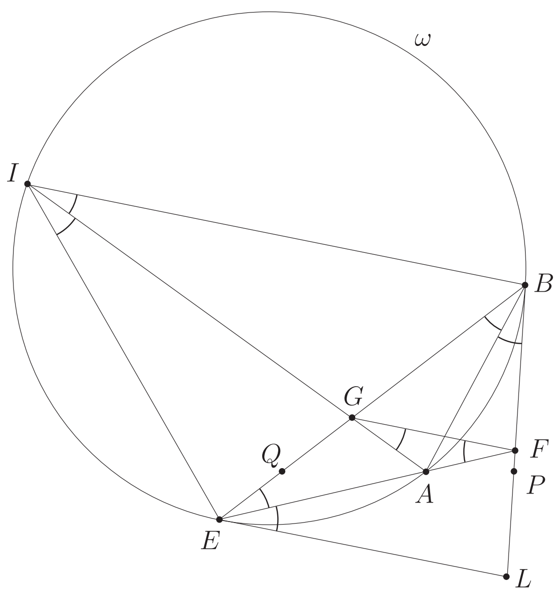


Figure 5

G5. Let $ABCDEF$ be a convex hexagon with $AB = DE$, $BC = EF$, $CD = FA$, and $\angle A - \angle D = \angle C - \angle F = \angle E - \angle B$. Prove that the diagonals AD, BE, and CF are concurrent.

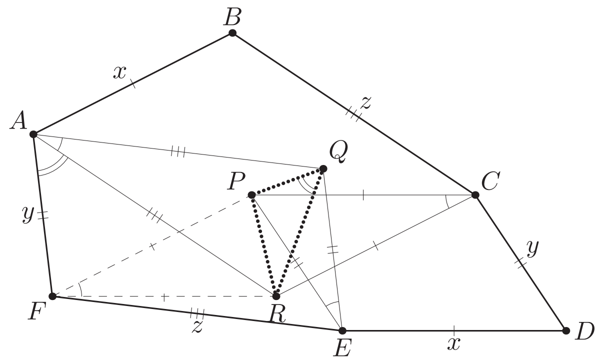
(Ukraine)

In all three solutions, we denote $\theta = \angle A - \angle D = \angle C - \angle F = \angle E - \angle B$ and assume without loss of generality that $\theta \geqslant 0$.

Solution 1. Let $x = AB = DE$, $y = CD = FA$, $z = EF = BC$. Consider the points P, Q, and R such that the quadrilaterals $CDEP$, $EFAQ$, and $ABCR$ are parallelograms. We compute

$$\begin{aligned}\angle PEQ &= \angle FEQ + \angle DEP - \angle E = (180^\circ - \angle F) + (180^\circ - \angle D) - \angle E \\ &= 360^\circ - \angle D - \angle E - \angle F = \tfrac{1}{2}\big(\angle A + \angle B + \angle C - \angle D - \angle E - \angle F\big) = \theta/2.\end{aligned}$$

Similarly, $\angle QAR = \angle RCP = \theta/2$.



If $\theta = 0$, since $\triangle RCP$ is isosceles, $R = P$. Therefore $AB \parallel RC = PC \parallel ED$, so $ABDE$ is a parallelogram. Similarly, $BCEF$ and $CDFA$ are parallelograms. It follows that AD, BE and CF meet at their common midpoint.

Now assume $\theta > 0$. Since $\triangle PEQ$, $\triangle QAR$, and $\triangle RCP$ are isosceles and have the same angle at the apex, we have $\triangle PEQ \sim \triangle QAR \sim \triangle RCP$ with ratios of similarity $y : z : x$. Thus

$$\triangle PQR \text{ is similar to the triangle with sidelengths } y,\ z, \text{ and } x. \tag{1}$$

Next, notice that

$$\frac{RQ}{QP} = \frac{z}{y} = \frac{RA}{AF}$$

and, using directed angles between rays,

$$\begin{aligned}\measuredangle(RQ, QP) &= \measuredangle(RQ, QE) + \measuredangle(QE, QP) \\ &= \measuredangle(RQ, QE) + \measuredangle(RA, RQ) = \measuredangle(RA, QE) = \measuredangle(RA, AF).\end{aligned}$$

Thus $\triangle PQR \sim \triangle FAR$. Since $FA = y$ and $AR = z$, (1) then implies that $FR = x$. Similarly $FP = x$. Therefore $CRFP$ is a rhombus.

We conclude that CF is the perpendicular bisector of PR. Similarly, BE is the perpendicular bisector of PQ and AD is the perpendicular bisector of QR. It follows that AD, BE, and CF are concurrent at the circumcenter of PQR.

Solution 2. Let $X = CD \cap EF$, $Y = EF \cap AB$, $Z = AB \cap CD$, $X' = FA \cap BC$, $Y' = BC \cap DE$, and $Z' = DE \cap FA$. From $\angle A + \angle B + \angle C = 360^\circ + \theta/2$ we get $\angle A + \angle B > 180^\circ$ and $\angle B + \angle C > 180^\circ$, so Z and X' are respectively on the opposite sides of BC and AB from the hexagon. Similar conclusions hold for X, Y, Y', and Z'. Then

$$\angle YZX = \angle B + \angle C - 180^\circ = \angle E + \angle F - 180^\circ = \angle Y'Z'X',$$

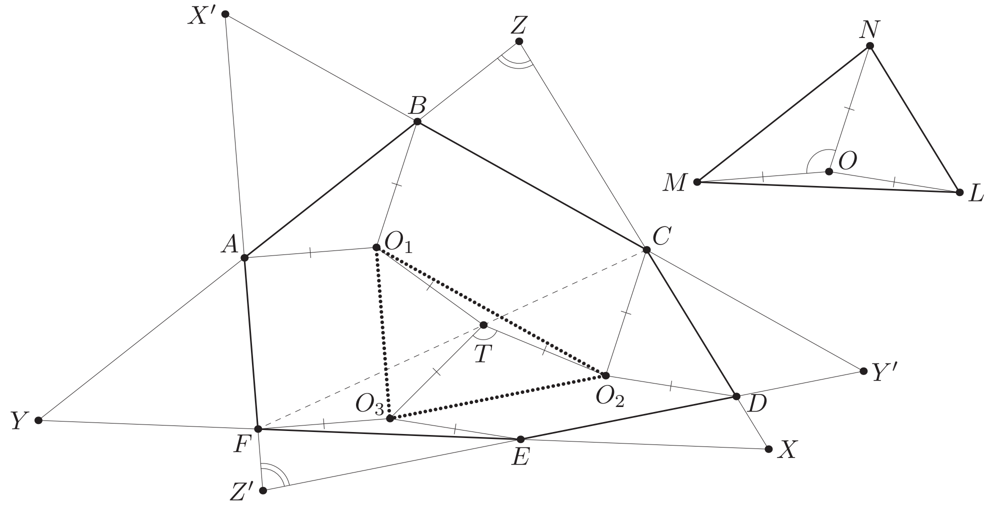
and similarly $\angle ZXY = \angle Z'X'Y'$ and $\angle XYZ = \angle X'Y'Z'$, so $\triangle XYZ \sim \triangle X'Y'Z'$. Thus there is a rotation R which sends $\triangle XYZ$ to a triangle with sides parallel to $\triangle X'Y'Z'$. Since $AB = DE$ we have $R(\overrightarrow{AB}) = \overrightarrow{DE}$. Similarly, $R(\overrightarrow{CD}) = \overrightarrow{FA}$ and $R(\overrightarrow{EF}) = \overrightarrow{BC}$. Therefore

$$\overrightarrow{0} = \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DE} + \overrightarrow{EF} + \overrightarrow{FA} = \left(\overrightarrow{AB} + \overrightarrow{CD} + \overrightarrow{EF}\right) + R\left(\overrightarrow{AB} + \overrightarrow{CD} + \overrightarrow{EF}\right).$$

If R is a rotation by 180°, then any two opposite sides of our hexagon are equal and parallel, so the three diagonals meet at their common midpoint. Otherwise, we must have

$$\overrightarrow{AB} + \overrightarrow{CD} + \overrightarrow{EF} = \overrightarrow{0},$$

or else we would have two vectors with different directions whose sum is $\overrightarrow{0}$.



This allows us to consider a triangle LMN with $\overrightarrow{LM} = \overrightarrow{EF}$, $\overrightarrow{MN} = \overrightarrow{AB}$, and $\overrightarrow{NL} = \overrightarrow{CD}$. Let O be the circumcenter of $\triangle LMN$ and consider the points O_1, O_2, O_3 such that $\triangle AO_1B$, $\triangle CO_2D$, and $\triangle EO_3F$ are translations of $\triangle MON$, $\triangle NOL$, and $\triangle LOM$, respectively. Since FO_3 and AO_1 are translations of MO, quadrilateral AFO_3O_1 is a parallelogram and $O_3O_1 = FA = CD = NL$. Similarly, $O_1O_2 = LM$ and $O_2O_3 = MN$. Therefore $\triangle O_1O_2O_3 \cong \triangle LMN$. Moreover, by means of the rotation R one may check that these triangles have the same orientation.

Let T be the circumcenter of $\triangle O_1O_2O_3$. We claim that AD, BE, and CF meet at T. Let us show that C, T, and F are collinear. Notice that $CO_2 = O_2T = TO_3 = O_3F$ since they are all equal to the circumradius of $\triangle LMN$. Therefore $\triangle TO_3F$ and $\triangle CO_2T$ are isosceles. Using directed angles between rays again, we get

$$\measuredangle(TF, TO_3) = \measuredangle(FO_3, FT) \qquad \text{and} \qquad \measuredangle(TO_2, TC) = \measuredangle(CT, CO_2). \tag{2}$$

Also, T and O are the circumcenters of the congruent triangles $\triangle O_1O_2O_3$ and $\triangle LMN$ so we have $\measuredangle(TO_3, TO_2) = \measuredangle(ON, OM)$. Since CO_2 and FO_3 are translations of NO and MO respectively, this implies

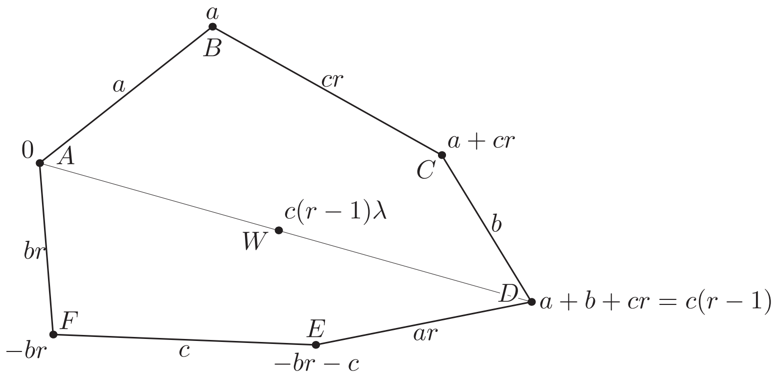
$$\measuredangle(TO_3, TO_2) = \measuredangle(CO_2, FO_3). \tag{3}$$

Adding the three equations in (2) and (3) gives

$$\measuredangle(TF, TC) = \measuredangle(CT, FT) = -\measuredangle(TF, TC)$$

which implies that T is on CF. Analogous arguments show that it is on AD and BE also. The desired result follows.

Solution 3. Place the hexagon on the complex plane, with A at the origin and vertices labelled clockwise. Now A, B, C, D, E, F represent the corresponding complex numbers. Also consider the complex numbers a, b, c, a', b', c' given by $B - A = a$, $D - C = b$, $F - E = c$, $E - D = a'$, $A - F = b'$, and $C - B = c'$. Let $k = |a|/|b|$. From $a/b' = -ke^{i\angle A}$ and $a'/b = -ke^{i\angle D}$ we get that $(a'/a)(b'/b) = e^{-i\theta}$ and similarly $(b'/b)(c'/c) = e^{-i\theta}$ and $(c'/c)(a'/a) = e^{-i\theta}$. It follows that $a' = ar$, $b' = br$, and $c' = cr$ for a complex number r with $|r| = 1$, as shown below.



We have

$$0 = a + cr + b + ar + c + br = (a + b + c)(1 + r).$$

If $r = -1$, then the hexagon is centrally symmetric and its diagonals intersect at its center of symmetry. Otherwise

$$a + b + c = 0.$$

Therefore

$$A = 0, \quad B = a, \quad C = a + cr, \quad D = c(r-1), \quad E = -br - c, \quad F = -br.$$

Now consider a point W on AD given by the complex number $c(r-1)\lambda$, where λ is a real number with $0 < \lambda < 1$. Since $D \neq A$, we have $r \neq 1$, so we can define $s = 1/(r-1)$. From $r\overline{r} = |r|^2 = 1$ we get

$$1 + s = \frac{r}{r-1} = \frac{r}{r - r\overline{r}} = \frac{1}{1 - \overline{r}} = -\overline{s}.$$

Now,

$$\begin{aligned} W \text{ is on } BE &\iff c(r-1)\lambda - a \parallel a - (-br - c) = b(r-1) \iff c\lambda - as \parallel b \\ &\iff -a\lambda - b\lambda - as \parallel b \iff a(\lambda + s) \parallel b. \end{aligned}$$

One easily checks that $r \neq \pm 1$ implies that $\lambda + s \neq 0$ since s is not real. On the other hand,

$$\begin{aligned} W \text{ on } CF &\iff c(r-1)\lambda + br \parallel -br - (a + cr) = a(r-1) \iff c\lambda + b(1+s) \parallel a \\ &\iff -a\lambda - b\lambda - b\overline{s} \parallel a \iff b(\lambda + \overline{s}) \parallel a \iff b \parallel a(\lambda + s), \end{aligned}$$

where in the last step we use that $(\lambda + s)(\lambda + \overline{s}) = |\lambda + s|^2 \in \mathbb{R}_{>0}$. We conclude that $AD \cap BE = CF \cap BE$, and the desired result follows.

G6. Let the excircle of the triangle ABC lying opposite to A touch its side BC at the point A_1. Define the points B_1 and C_1 analogously. Suppose that the circumcentre of the triangle $A_1B_1C_1$ lies on the circumcircle of the triangle ABC. Prove that the triangle ABC is right-angled.

(Russia)

Solution 1. Denote the circumcircles of the triangles ABC and $A_1B_1C_1$ by Ω and Γ, respectively. Denote the midpoint of the arc CB of Ω containing A by A_0, and define B_0 as well as C_0 analogously. By our hypothesis the centre Q of Γ lies on Ω.

Lemma. One has $A_0B_1 = A_0C_1$. Moreover, the points A, A_0, B_1, and C_1 are concyclic. Finally, the points A and A_0 lie on the same side of B_1C_1. Similar statements hold for B and C.

Proof. Let us consider the case $A = A_0$ first. Then the triangle ABC is isosceles at A, which implies $AB_1 = AC_1$ while the remaining assertions of the Lemma are obvious. So let us suppose $A \neq A_0$ from now on.

By the definition of A_0, we have $A_0B = A_0C$. It is also well known and easy to show that $BC_1 = CB_1$. Next, we have $\angle C_1BA_0 = \angle ABA_0 = \angle ACA_0 = \angle B_1CA_0$. Hence the triangles A_0BC_1 and A_0CB_1 are congruent. This implies $A_0C_1 = A_0B_1$, establishing the first part of the Lemma. It also follows that $\angle A_0C_1A = \angle A_0B_1A$, as these are exterior angles at the corresponding vertices C_1 and B_1 of the congruent triangles A_0BC_1 and A_0CB_1. For that reason the points A, A_0, B_1, and C_1 are indeed the vertices of some cyclic quadrilateral two opposite sides of which are AA_0 and B_1C_1. □

Now we turn to the solution. Evidently the points A_1, B_1, and C_1 lie interior to some semicircle arc of Γ, so the triangle $A_1B_1C_1$ is obtuse-angled. Without loss of generality, we will assume that its angle at B_1 is obtuse. Thus Q and B_1 lie on different sides of A_1C_1; obviously, the same holds for the points B and B_1. So, the points Q and B are on the same side of A_1C_1.

Notice that the perpendicular bisector of A_1C_1 intersects Ω at two points lying on different sides of A_1C_1. By the first statement from the Lemma, both points B_0 and Q are among these points of intersection; since they share the same side of A_1C_1, they coincide (see Figure 1).

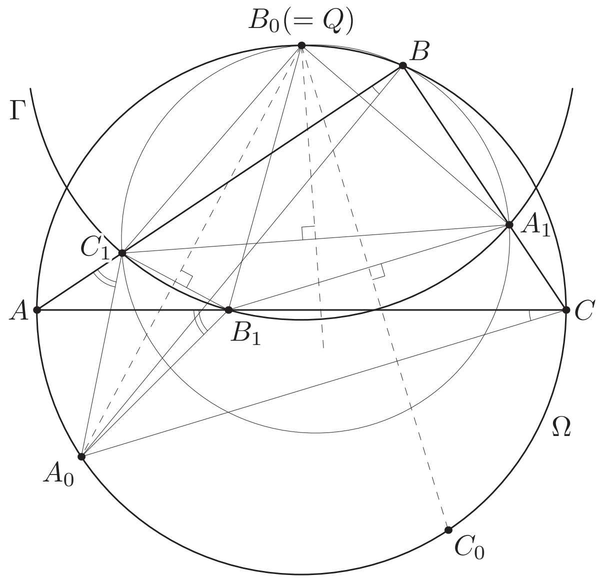


Figure 1

Now, by the first part of the Lemma again, the lines QA_0 and QC_0 are the perpendicular bisectors of B_1C_1 and A_1B_1, respectively. Thus

$$\angle C_1B_0A_1 = \angle C_1B_0B_1 + \angle B_1B_0A_1 = 2\angle A_0B_0B_1 + 2\angle B_1B_0C_0 = 2\angle A_0B_0C_0 = 180^\circ - \angle ABC,$$

recalling that A_0 and C_0 are the midpoints of the arcs CB and BA, respectively.

On the other hand, by the second part of the Lemma we have

$$\angle C_1B_0A_1 = \angle C_1BA_1 = \angle ABC.$$

From the last two equalities, we get $\angle ABC = 90^\circ$, whereby the problem is solved.

Solution 2. Let Q again denote the centre of the circumcircle of the triangle $A_1B_1C_1$, that lies on the circumcircle Ω of the triangle ABC. We first consider the case where Q coincides with one of the vertices of ABC, say $Q = B$. Then $BC_1 = BA_1$ and consequently the triangle ABC is isosceles at B. Moreover we have $BC_1 = B_1C$ in any triangle, and hence $BB_1 = BC_1 = B_1C$; similarly, $BB_1 = B_1A$. It follows that B_1 is the centre of Ω and that the triangle ABC has a right angle at B.

So from now on we may suppose $Q \notin \{A, B, C\}$. We start with the following well known fact.

Lemma. Let XYZ and $X'Y'Z'$ be two triangles with $XY = X'Y'$ and $YZ = Y'Z'$.

(i) If $XZ \neq X'Z'$ and $\angle YZX = \angle Y'Z'X'$, then $\angle ZXY + \angle Z'X'Y' = 180^\circ$.

(ii) If $\angle YZX + \angle X'Z'Y' = 180^\circ$, then $\angle ZXY = \angle Y'X'Z'$.

Proof. For both parts, we may move the triangle XYZ through the plane until $Y = Y'$ and $Z = Z'$. Possibly after reflecting one of the two triangles about YZ, we may also suppose that X and X' lie on the same side of YZ if we are in case (i) and on different sides if we are in case (ii). In both cases, the points X, Z, and X' are collinear due to the angle condition (see Fig. 2). Moreover we have $X \neq X'$, because in case (i) we assumed $XZ \neq X'Z'$ and in case (ii) these points even lie on different sides of YZ. Thus the triangle $XX'Y$ is isosceles at Y. The claim now follows by considering the equal angles at its base. □

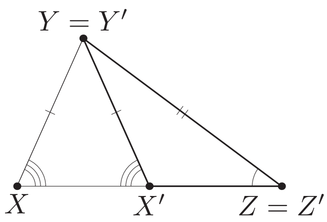


Figure 2(i)

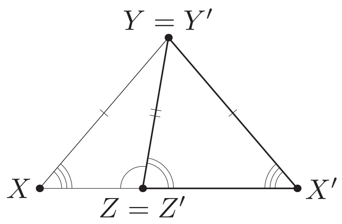


Figure 2(ii)

Relabeling the vertices of the triangle ABC if necessary we may suppose that Q lies in the interior of the arc AB of Ω not containing C. We will sometimes use tacitly that the six triangles QBA_1, QA_1C, QCB_1, QB_1A, QC_1A, and QBC_1 have the same orientation.

As Q cannot be the circumcentre of the triangle ABC, it is impossible that $QA = QB = QC$ and thus we may also suppose that $QC \neq QB$. Now the above Lemma (i) is applicable to the triangles QB_1C and QC_1B, since $QB_1 = QC_1$ and $B_1C = C_1B$, while $\angle B_1CQ = \angle C_1BQ$ holds as both angles appear over the same side of the chord QA in Ω (see Fig. 3). So we get

$$\angle CQB_1 + \angle BQC_1 = 180^\circ. \tag{1}$$

We claim that $QC = QA$. To see this, let us assume for the sake of a contradiction that $QC \neq QA$. Then arguing similarly as before but now with the triangles QA_1C and QC_1A we get

$$\angle A_1QC + \angle C_1QA = 180^\circ.$$

Adding this equation to (1), we get $\angle A_1QB_1 + \angle BQA = 360^\circ$, which is absurd as both summands lie in the interval $(0^\circ, 180^\circ)$.

This proves $QC = QA$; so the triangles QA_1C and QC_1A are congruent their sides being equal, which in turn yields

$$\angle A_1QC = \angle C_1QA. \tag{2}$$

Finally our Lemma (ii) is applicable to the triangles QA_1B and QB_1A. Indeed we have $QA_1 = QB_1$ and $A_1B = B_1A$ as usual, and the angle condition $\angle A_1BQ + \angle QAB_1 = 180^\circ$ holds as A and B lie on different sides of the chord QC in Ω. Consequently we have

$$\angle BQA_1 = \angle B_1QA. \tag{3}$$

From (1) and (3) we get

$$(\angle B_1QC + \angle B_1QA) + (\angle C_1QB - \angle BQA_1) = 180^\circ,$$

i.e. $\angle CQA + \angle A_1QC_1 = 180^\circ$. In light of (2) this may be rewritten as $2\angle CQA = 180^\circ$ and as Q lies on Ω this implies that the triangle ABC has a right angle at B.

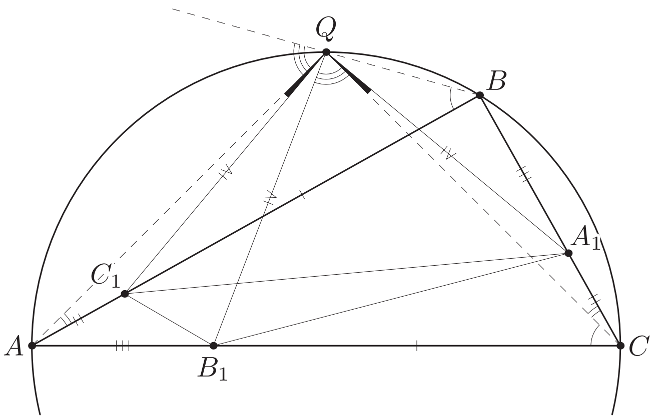


Figure 3

Comment 1. One may also check that Q is in the interior of Ω if and only if the triangle ABC is acute-angled.

Comment 2. The original proposal asked to prove the converse statement as well: if the triangle ABC is right-angled, then the point Q lies on its circumcircle. The Problem Selection Committee thinks that the above simplified version is more suitable for the competition.

Number Theory

N1. Let $\mathbb{Z}_{>0}$ be the set of positive integers. Find all functions $f\colon \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ such that

$$m^2 + f(n) \mid mf(m) + n$$

for all positive integers m and n.

(Malaysia)

Answer. $f(n) = n$.

Solution 1. Setting $m = n = 2$ tells us that $4 + f(2) \mid 2f(2) + 2$. Since $2f(2) + 2 < 2(4 + f(2))$, we must have $2f(2) + 2 = 4 + f(2)$, so $f(2) = 2$. Plugging in $m = 2$ then tells us that $4 + f(n) \mid 4 + n$, which implies that $f(n) \leqslant n$ for all n.

Setting $m = n$ gives $n^2 + f(n) \mid nf(n) + n$, so $nf(n) + n \geqslant n^2 + f(n)$ which we rewrite as $(n-1)(f(n) - n) \geqslant 0$. Therefore $f(n) \geqslant n$ for all $n \geqslant 2$. This is trivially true for $n = 1$ also.

It follows that $f(n) = n$ for all n. This function obviously satisfies the desired property.

Solution 2. Setting $m = f(n)$ we get $f(n)(f(n)+1) \mid f(n)f(f(n)) + n$. This implies that $f(n) \mid n$ for all n.

Now let m be any positive integer, and let $p > 2m^2$ be a prime number. Note that $p > mf(m)$ also. Plugging in $n = p - mf(m)$ we learn that $m^2 + f(n)$ divides p. Since $m^2 + f(n)$ cannot equal 1, it must equal p. Therefore $p - m^2 = f(n) \mid n = p - mf(m)$. But $p - mf(m) < p < 2(p - m^2)$, so we must have $p - mf(m) = p - m^2$, i.e., $f(m) = m$.

Solution 3. Plugging $m = 1$ we obtain $1 + f(n) \leqslant f(1) + n$, so $f(n) \leqslant n + c$ for the constant $c = f(1) - 1$. Assume that $f(n) \neq n$ for some fixed n. When m is large enough (e.g. $m \geqslant \max(n, c+1)$) we have

$$mf(m) + n \leqslant m(m + c) + n \leqslant 2m^2 < 2(m^2 + f(n)),$$

so we must have $mf(m) + n = m^2 + f(n)$. This implies that

$$0 \neq f(n) - n = m(f(m) - m),$$

which is impossible for $m > |f(n) - n|$. It follows that f is the identity function.

N2. Prove that for any pair of positive integers k and n there exist k positive integers $m_1, m_2, \ldots, m_k$ such that

$$1 + \frac{2^k - 1}{n} = \left(1 + \frac{1}{m_1}\right)\left(1 + \frac{1}{m_2}\right) \cdots \left(1 + \frac{1}{m_k}\right).$$

(Japan)

Solution 1. We proceed by induction on k. For $k = 1$ the statement is trivial. Assuming we have proved it for $k = j - 1$, we now prove it for $k = j$.

Case 1. $n = 2t - 1$ for some positive integer t.

Observe that

$$1 + \frac{2^j - 1}{2t - 1} = \frac{2(t + 2^{j-1} - 1)}{2t} \cdot \frac{2t}{2t - 1} = \left(1 + \frac{2^{j-1} - 1}{t}\right)\left(1 + \frac{1}{2t - 1}\right).$$

By the induction hypothesis we can find $m_1, \ldots, m_{j-1}$ such that

$$1 + \frac{2^{j-1} - 1}{t} = \left(1 + \frac{1}{m_1}\right)\left(1 + \frac{1}{m_2}\right) \cdots \left(1 + \frac{1}{m_{j-1}}\right),$$

so setting $m_j = 2t - 1$ gives the desired expression.

Case 2. $n = 2t$ for some positive integer t.

Now we have

$$1 + \frac{2^j - 1}{2t} = \frac{2t + 2^j - 1}{2t + 2^j - 2} \cdot \frac{2t + 2^j - 2}{2t} = \left(1 + \frac{1}{2t + 2^j - 2}\right)\left(1 + \frac{2^{j-1} - 1}{t}\right),$$

noting that $2t + 2^j - 2 > 0$. Again, we use that

$$1 + \frac{2^{j-1} - 1}{t} = \left(1 + \frac{1}{m_1}\right)\left(1 + \frac{1}{m_2}\right) \cdots \left(1 + \frac{1}{m_{j-1}}\right).$$

Setting $m_j = 2t + 2^j - 2$ then gives the desired expression.

Solution 2. Consider the base 2 expansions of the residues of $n - 1$ and $-n$ modulo 2^k:

$$\begin{aligned} n - 1 &\equiv 2^{a_1} + 2^{a_2} + \cdots + 2^{a_r} \pmod{2^k} \qquad &&\text{where } \ 0 \leqslant a_1 < a_2 < \ldots < a_r \leqslant k - 1, \\ -n &\equiv 2^{b_1} + 2^{b_2} + \cdots + 2^{b_s} \pmod{2^k} \qquad &&\text{where } \ 0 \leqslant b_1 < b_2 < \ldots < b_s \leqslant k - 1. \end{aligned}$$

Since $-1 \equiv 2^0 + 2^1 + \cdots + 2^{k-1} \pmod{2^k}$, we have $\{a_1, \ldots, a_r\} \cup \{b_1 \ldots, b_s\} = \{0, 1, \ldots, k - 1\}$ and $r + s = k$. Write

$$\begin{aligned} S_p &= 2^{a_p} + 2^{a_{p+1}} + \cdots + 2^{a_r} \qquad &&\text{for } \ 1 \leqslant p \leqslant r, \\ T_q &= 2^{b_1} + 2^{b_2} + \cdots + 2^{b_q} \qquad &&\text{for } \ 1 \leqslant q \leqslant s. \end{aligned}$$

Also set $S_{r+1} = T_0 = 0$. Notice that $S_1 + T_s = 2^k - 1$ and $n + T_s \equiv 0 \pmod{2^k}$. We have

$$\begin{aligned} 1 + \frac{2^k - 1}{n} &= \frac{n + S_1 + T_s}{n} = \frac{n + S_1 + T_s}{n + T_s} \cdot \frac{n + T_s}{n} \\ &= \prod_{p=1}^{r} \frac{n + S_p + T_s}{n + S_{p+1} + T_s} \cdot \prod_{q=1}^{s} \frac{n + T_q}{n + T_{q-1}} \\ &= \prod_{p=1}^{r} \left(1 + \frac{2^{a_p}}{n + S_{p+1} + T_s}\right) \cdot \prod_{q=1}^{s} \left(1 + \frac{2^{b_q}}{n + T_{q-1}}\right), \end{aligned}$$

so if we define

$$m_p = \frac{n + S_{p+1} + T_s}{2^{a_p}} \quad \text{for } 1 \leqslant p \leqslant r \quad \text{and} \quad m_{r+q} = \frac{n + T_{q-1}}{2^{b_q}} \quad \text{for } 1 \leqslant q \leqslant s,$$

the desired equality holds. It remains to check that every m_i is an integer. For $1 \leqslant p \leqslant r$ we have

$$n + S_{p+1} + T_s \equiv n + T_s \equiv 0 \pmod{2^{a_p}}$$

and for $1 \leqslant q \leqslant r$ we have

$$n + T_{q-1} \equiv n + T_s \equiv 0 \pmod{2^{b_q}}.$$

The desired result follows.

N3. Prove that there exist infinitely many positive integers n such that the largest prime divisor of $n^4 + n^2 + 1$ is equal to the largest prime divisor of $(n+1)^4 + (n+1)^2 + 1$.

(Belgium)

Solution. Let p_n be the largest prime divisor of $n^4 + n^2 + 1$ and let q_n be the largest prime divisor of $n^2 + n + 1$. Then $p_n = q_{n^2}$, and from

$$n^4 + n^2 + 1 = (n^2+1)^2 - n^2 = (n^2 - n + 1)(n^2 + n + 1) = ((n-1)^2 + (n-1) + 1)(n^2 + n + 1)$$

it follows that $p_n = \max\{q_n, q_{n-1}\}$ for $n \geqslant 2$. Keeping in mind that $n^2 - n + 1$ is odd, we have

$$\gcd(n^2 + n + 1, n^2 - n + 1) = \gcd(2n, n^2 - n + 1) = \gcd(n, n^2 - n + 1) = 1.$$

Therefore $q_n \neq q_{n-1}$.

To prove the result, it suffices to show that the set

$$S = \{n \in \mathbb{Z}_{\geqslant 2} \mid q_n > q_{n-1} \text{ and } q_n > q_{n+1}\}$$

is infinite, since for each $n \in S$ one has

$$p_n = \max\{q_n, q_{n-1}\} = q_n = \max\{q_n, q_{n+1}\} = p_{n+1}.$$

Suppose on the contrary that S is finite. Since $q_2 = 7 < 13 = q_3$ and $q_3 = 13 > 7 = q_4$, the set S is non-empty. Since it is finite, we can consider its largest element, say m.

Note that it is impossible that $q_m > q_{m+1} > q_{m+2} > \ldots$ because all these numbers are positive integers, so there exists a $k \geqslant m$ such that $q_k < q_{k+1}$ (recall that $q_k \neq q_{k+1}$). Next observe that it is impossible to have $q_k < q_{k+1} < q_{k+2} < \ldots$, because $q_{(k+1)^2} = p_{k+1} = \max\{q_k, q_{k+1}\} = q_{k+1}$, so let us take the smallest $\ell \geqslant k+1$ such that $q_\ell > q_{\ell+1}$. By the minimality of ℓ we have $q_{\ell-1} < q_\ell$, so $\ell \in S$. Since $\ell \geqslant k + 1 > k \geqslant m$, this contradicts the maximality of m, and hence S is indeed infinite.

Comment. Once the factorization of $n^4 + n^2 + 1$ is found and the set S is introduced, the problem is mainly about ruling out the case that

$$q_k < q_{k+1} < q_{k+2} < \ldots \tag{1}$$

might hold for some $k \in \mathbb{Z}_{>0}$. In the above solution, this is done by observing $q_{(k+1)^2} = \max(q_k, q_{k+1})$. Alternatively one may notice that (1) implies that $q_{j+2} - q_j \geqslant 6$ for $j \geqslant k+1$, since every prime greater than 3 is congruent to -1 or 1 modulo 6. Then there is some integer $C \geqslant 0$ such that $q_n \geqslant 3n - C$ for all $n \geqslant k$.

Now let the integer t be sufficiently large (e.g. $t = \max\{k+1, C+3\}$) and set $p = q_{t-1} \geqslant 2t$. Then $p \mid (t-1)^2 + (t-1) + 1$ implies that $p \mid (p-t)^2 + (p-t) + 1$, so p and q_{p-t} are prime divisors of $(p-t)^2 + (p-t) + 1$. But $p - t > t - 1 \geqslant k$, so $q_{p-t} > q_{t-1} = p$ and $p \cdot q_{p-t} > p^2 > (p-t)^2 + (p-t) + 1$, a contradiction.

N4. Determine whether there exists an infinite sequence of nonzero digits $a_1, a_2, a_3, \ldots$ and a positive integer N such that for every integer $k > N$, the number $\overline{a_k a_{k-1} \ldots a_1}$ is a perfect square.

(Iran)

Answer. No.

Solution. Assume that $a_1, a_2, a_3, \ldots$ is such a sequence. For each positive integer k, let $y_k = \overline{a_k a_{k-1} \ldots a_1}$. By the assumption, for each $k > N$ there exists a positive integer x_k such that $y_k = x_k^2$.

I. For every n, let 5^{γ_n} be the greatest power of 5 dividing x_n. Let us show first that $2\gamma_n \geqslant n$ for every positive integer $n > N$.

Assume, to the contrary, that there exists a positive integer $n > N$ such that $2\gamma_n < n$, which yields

$$y_{n+1} = \overline{a_{n+1} a_n \ldots a_1} = 10^n a_{n+1} + \overline{a_n a_{n-1} \ldots a_1} = 10^n a_{n+1} + y_n = 5^{2\gamma_n} \left(2^n 5^{n-2\gamma_n} a_{n+1} + \frac{y_n}{5^{2\gamma_n}} \right).$$

Since $5 \nmid y_n / 5^{2\gamma_n}$, we obtain $\gamma_{n+1} = \gamma_n < n < n+1$. By the same arguments we obtain that $\gamma_n = \gamma_{n+1} = \gamma_{n+2} = \ldots$. Denote this common value by γ.

Now, for each $k \geqslant n$ we have

$$(x_{k+1} - x_k)(x_{k+1} + x_k) = x_{k+1}^2 - x_k^2 = y_{k+1} - y_k = a_{k+1} \cdot 10^k.$$

One of the numbers $x_{k+1} - x_k$ and $x_{k+1} + x_k$ is not divisible by $5^{\gamma+1}$ since otherwise one would have $5^{\gamma+1} \mid \big((x_{k+1} - x_k) + (x_{k+1} + x_k)\big) = 2x_{k+1}$. On the other hand, we have $5^k \mid (x_{k+1} - x_k)(x_{k+1} + x_k)$, so $5^{k-\gamma}$ divides one of these two factors. Thus we get

$$5^{k-\gamma} \leqslant \max\{x_{k+1} - x_k, x_{k+1} + x_k\} < 2x_{k+1} = 2\sqrt{y_{k+1}} < 2 \cdot 10^{(k+1)/2},$$

which implies $5^{2k} < 4 \cdot 5^{2\gamma} \cdot 10^{k+1}$, or $(5/2)^k < 40 \cdot 5^{2\gamma}$. The last inequality is clearly false for sufficiently large values of k. This contradiction shows that $2\gamma_n \geqslant n$ for all $n > N$.

II. Consider now any integer $k > \max\{N/2, 2\}$. Since $2\gamma_{2k+1} \geqslant 2k+1$ and $2\gamma_{2k+2} \geqslant 2k+2$, we have $\gamma_{2k+1} \geqslant k+1$ and $\gamma_{2k+2} \geqslant k+1$. So, from $y_{2k+2} = a_{2k+2} \cdot 10^{2k+1} + y_{2k+1}$ we obtain $5^{2k+2} \mid y_{2k+2} - y_{2k+1} = a_{2k+2} \cdot 10^{2k+1}$ and thus $5 \mid a_{2k+2}$, which implies $a_{2k+2} = 5$. Therefore,

$$(x_{2k+2} - x_{2k+1})(x_{2k+2} + x_{2k+1}) = x_{2k+2}^2 - x_{2k+1}^2 = y_{2k+2} - y_{2k+1} = 5 \cdot 10^{2k+1} = 2^{2k+1} \cdot 5^{2k+2}.$$

Setting $A_k = x_{2k+2}/5^{k+1}$ and $B_k = x_{2k+1}/5^{k+1}$, which are integers, we obtain

$$(A_k - B_k)(A_k + B_k) = 2^{2k+1}. \tag{1}$$

Both A_k and B_k are odd, since otherwise y_{2k+2} or y_{2k+1} would be a multiple of 10 which is false by $a_1 \neq 0$; so one of the numbers $A_k - B_k$ and $A_k + B_k$ is not divisible by 4. Therefore (1) yields $A_k - B_k = 2$ and $A_k + B_k = 2^{2k}$, hence $A_k = 2^{2k-1} + 1$ and thus

$$x_{2k+2} = 5^{k+1} A_k = 10^{k+1} \cdot 2^{k-2} + 5^{k+1} > 10^{k+1},$$

since $k \geqslant 2$. This implies that $y_{2k+2} > 10^{2k+2}$ which contradicts the fact that y_{2k+2} contains $2k+2$ digits. The desired result follows.

Solution 2. Again, we assume that a sequence $a_1, a_2, a_3, \ldots$ satisfies the problem conditions, introduce the numbers x_k and y_k as in the previous solution, and notice that

$$y_{k+1} - y_k = (x_{k+1} - x_k)(x_{k+1} + x_k) = 10^k a_{k+1} \tag{2}$$

for all $k > N$. Consider any such k. Since $a_1 \neq 0$, the numbers x_k and x_{k+1} are not multiples of 10, and therefore the numbers $p_k = x_{k+1} - x_k$ and $q_k = x_{k+1} + x_k$ cannot be simultaneously multiples of 20, and hence one of them is not divisible either by 4 or by 5. In view of (2), this means that the other one is divisible by either 5^k or by 2^{k-1}. Notice also that p_k and q_k have the same parity, so both are even.

On the other hand, we have $x_{k+1}^2 = x_k^2 + 10^k a_{k+1} \geqslant x_k^2 + 10^k > 2x_k^2$, so $x_{k+1}/x_k > \sqrt{2}$, which implies that

$$1 < \frac{q_k}{p_k} = 1 + \frac{2}{x_{k+1}/x_k - 1} < 1 + \frac{2}{\sqrt{2} - 1} < 6. \tag{3}$$

Thus, if one of the numbers p_k and q_k is divisible by 5^k, then we have

$$10^{k+1} > 10^k a_{k+1} = p_k q_k \geqslant \frac{(5^k)^2}{6}$$

and hence $(5/2)^k < 60$ which is false for sufficiently large k. So, assuming that k is large, we get that 2^{k-1} divides one of the numbers p_k and q_k. Hence

$$\{p_k, q_k\} = \{2^{k-1} \cdot 5^{r_k} b_k, 2 \cdot 5^{k-r_k} c_k\} \quad \text{with nonnegative integers } b_k, c_k, r_k \text{ such that } b_k c_k = a_{k+1}.$$

Moreover, from (3) we get

$$6 > \frac{2^{k-1} \cdot 5^{r_k} b_k}{2 \cdot 5^{k-r_k} c_k} \geqslant \frac{1}{36} \cdot \left(\frac{2}{5}\right)^k \cdot 5^{2r_k} \quad \text{and} \quad 6 > \frac{2 \cdot 5^{k-r_k} c_k}{2^{k-1} \cdot 5^{r_k} b_k} \geqslant \frac{4}{9} \cdot \left(\frac{5}{2}\right)^k \cdot 5^{-2r_k},$$

so

$$\alpha k + c_1 < r_k < \alpha k + c_2 \qquad \text{for } \alpha = \tfrac{1}{2}\log_5\left(\tfrac{5}{2}\right) < 1 \text{ and some constants } c_2 > c_1. \tag{4}$$

Consequently, for $C = c_2 - c_1 + 1 - \alpha > 0$ we have

$$(k+1) - r_{k+1} \leqslant k - r_k + C. \tag{5}$$

Next, we will use the following easy lemma.

Lemma. Let s be a positive integer. Then $5^{s+2^s} \equiv 5^s \pmod{10^s}$.

Proof. Euler's theorem gives $5^{2^s} \equiv 1 \pmod{2^s}$, so $5^{s+2^s} - 5^s = 5^s(5^{2^s} - 1)$ is divisible by 2^s and 5^s.

Now, for every large k we have

$$x_{k+1} = \frac{p_k + q_k}{2} = 5^{r_k} \cdot 2^{k-2} b_k + 5^{k-r_k} c_k \equiv 5^{k-r_k} c_k \pmod{10^{r_k}} \tag{6}$$

since $r_k \leqslant k - 2$ by (4); hence $y_{k+1} \equiv 5^{2(k-r_k)} c_k^2 \pmod{10^{r_k}}$. Let us consider some large integer s, and choose the minimal k such that $2(k - r_k) \geqslant s + 2^s$; it exists by (4). Set $d = 2(k - r_k) - (s + 2^s)$. By (4) we have $2^s < 2(k - r_k) < \left(\frac{2}{\alpha} - 2\right) r_k - \frac{2c_1}{\alpha}$; if s is large this implies $r_k > s$, so (6) also holds modulo 10^s. Then (6) and the lemma give

$$y_{k+1} \equiv 5^{2(k-r_k)} c_k^2 = 5^{s+2^s} \cdot 5^d c_k^2 \equiv 5^s \cdot 5^d c_k^2 \pmod{10^s}. \tag{7}$$

By (5) and the minimality of k we have $d \leqslant 2C$, so $5^d c_k^2 \leqslant 5^{2C} \cdot 81 = D$. Using $5^4 < 10^3$ we obtain

$$5^s \cdot 5^d c_k^2 < 10^{3s/4} D < 10^{s-1}$$

for sufficiently large s. This, together with (7), shows that the sth digit from the right in y_{k+1}, which is a_s, is zero. This contradicts the problem condition.

N5. Fix an integer $k \geqslant 2$. Two players, called Ana and Banana, play the following *game of numbers*: Initially, some integer $n \geqslant k$ gets written on the blackboard. Then they take moves in turn, with Ana beginning. A player making a move erases the number m just written on the blackboard and replaces it by some number m' with $k \leqslant m' < m$ that is coprime to m. The first player who cannot move anymore loses.

An integer $n \geqslant k$ is called *good* if Banana has a winning strategy when the initial number is n, and *bad* otherwise.

Consider two integers $n, n' \geqslant k$ with the property that each prime number $p \leqslant k$ divides n if and only if it divides n'. Prove that either both n and n' are good or both are bad.

(Italy)

Solution 1. Let us first observe that the number appearing on the blackboard decreases after every move; so the game necessarily ends after at most n steps, and consequently there always has to be some player possessing a winning strategy. So if some $n \geqslant k$ is bad, then Ana has a winning strategy in the game with starting number n.

More precisely, if $n \geqslant k$ is such that there is a good integer m with $n > m \geqslant k$ and $\gcd(m, n) = 1$, then n itself is bad, for Ana has the following winning strategy in the game with initial number n: She proceeds by first playing m and then using Banana's strategy for the game with starting number m.

Otherwise, if some integer $n \geqslant k$ has the property that every integer m with $n > m \geqslant k$ and $\gcd(m, n) = 1$ is bad, then n is good. Indeed, if Ana can make a first move at all in the game with initial number n, then she leaves it in a position where the first player has a winning strategy, so that Banana can defeat her.

In particular, this implies that any two good numbers have a non–trivial common divisor. Also, k itself is good.

For brevity, we say that $n \longrightarrow x$ is a *move* if n and x are two coprime integers with $n > x \geqslant k$.

Claim 1. If n is good and n' is a multiple of n, then n' is also good.

Proof. If n' were bad, there would have to be some move $n' \longrightarrow x$, where x is good. As n' is a multiple of n this implies that the two good numbers n and x are coprime, which is absurd. □

Claim 2. If r and s denote two positive integers for which $rs \geqslant k$ is bad, then $r^2 s$ is also bad.

Proof. Since rs is bad, there is a move $rs \longrightarrow x$ for some good x. Evidently x is coprime to $r^2 s$ as well, and hence the move $r^2 s \longrightarrow x$ shows that $r^2 s$ is indeed bad. □

Claim 3. If $p > k$ is prime and $n \geqslant k$ is bad, then np is also bad.

Proof. Otherwise we choose a counterexample with n being as small as possible. In particular, np is good. Since n is bad, there is a move $n \longrightarrow x$ for some good x. Now $np \longrightarrow x$ cannot be a valid move, which tells us that x has to be divisible by p. So we can write $x = p^r y$, where r and y denote some positive integers, the latter of which is not divisible by p.

Note that $y = 1$ is impossible, for then we would have $x = p^r$ and the move $x \longrightarrow k$ would establish that x is bad. In view of this, there is a least power y^α of y that is at least as large as k. Since the numbers np and y^α are coprime and the former is good, the latter has to be bad. Moreover, the minimality of α implies $y^\alpha < ky < py = \frac{x}{p^{r-1}} < \frac{n}{p^{r-1}}$. So $p^{r-1} \cdot y^\alpha < n$ and consequently all the numbers $y^\alpha, py^\alpha, \ldots, p^r \cdot y^\alpha = p(p^{r-1} \cdot y^\alpha)$ are bad due to the minimal choice of n. But now by Claim 1 the divisor x of $p^r \cdot y^\alpha$ cannot be good, whereby we have reached a contradiction that proves Claim 3. □

We now deduce the statement of the problem from these three claims. To this end, we call two integers $a, b \geqslant k$ *similar* if they are divisible by the same prime numbers not exceeding k. We are to prove that if a and b are similar, then either both of them are good or both are bad. As in this case the product ab is similar to both a and b, it suffices to show the following: if $c \geqslant k$ is similar to some of its multiples d, then either both c and d are good or both are bad.

Assuming that this is not true in general, we choose a counterexample (c_0, d_0) with d_0 being as small as possible. By Claim 1, c_0 is bad whilst d_0 is good. Plainly d_0 is strictly greater than c_0 and hence the quotient $\frac{d_0}{c_0}$ has some prime factor p. Clearly p divides d_0. If $p \leqslant k$, then p divides c_0 as well due to the similarity, and hence d_0 is actually divisible by p^2. So $\frac{d_0}{p}$ is good by the contrapositive of Claim 2. Since $c_0 \mid \frac{d_0}{p}$, the pair $(c_0, \frac{d_0}{p})$ contradicts the supposed minimality of d_0. This proves $p > k$, but now we get the same contradiction using Claim 3 instead of Claim 2. Thereby the problem is solved.

Solution 2. We use the same analysis of the game of numbers as in the first five paragraphs of the first solution. Let us call a prime number p *small* in case $p \leqslant k$ and *big* otherwise. We again call two integers *similar* if their sets of small prime factors coincide.

Claim 4. For each integer $b \geqslant k$ having some small prime factor, there exists an integer x similar to it with $b \geqslant x \geqslant k$ and having no big prime factors.

Proof. Unless b has a big prime factor we may simply choose $x = b$. Now let p and q denote a small and a big prime factor of b, respectively. Let a be the product of all small prime factors of b. Further define n to be the least non–negative integer for which the number $x = p^n a$ is at least as large as k. It suffices to show that $b > x$. This is clear in case $n = 0$, so let us assume $n > 0$ from now on. Then we have $x < pk$ due to the minimality of n, $p \leqslant a$ because p divides a by construction, and $k < q$. Therefore $x < aq$ and, as the right hand side is a product of distinct prime factors of b, this implies indeed $x < b$. □

Let us now assume that there is a pair (a, b) of similar numbers such that a is bad and b is good. Take such a pair with $\max(a, b)$ being as small as possible. Since a is bad, there exists a move $a \longrightarrow r$ for some good r. Since the numbers k and r are both good, they have a common prime factor, which necessarily has to be small. Thus Claim 4 is applicable to r, which yields an integer r' similar to r containing small prime factors only and satisfying $r \geqslant r' \geqslant k$. Since $\max(r, r') = r < a \leqslant \max(a, b)$ the number r' is also good. Now let p denote a common prime factor of the good numbers r' and b. By our construction of r', this prime is small and due to the similarities it consequently divides a and r, contrary to $a \longrightarrow r$ being a move. Thereby the problem is solved.

Comment 1. Having reached Claim 4 of Solution 2, there are various other ways to proceed. For instance, one may directly obtain the following fact, which seems to be interesting in its own right:

Claim 5. Any two good numbers have a common small prime factor.

Proof. Otherwise there exists a pair (b, b') of good numbers with $b' \geqslant b \geqslant k$ all of whose common prime factors are big. Choose such a pair with b' being as small as possible. Since b and k are both good, there has to be a common prime factor p of b and k. Evidently p is small and thus it cannot divide b', which in turn tells us $b' > b$. Applying Claim 4 to b we get an integer x with $b \geqslant x \geqslant k$ that is similar to b and has no big prime divisors at all. By our assumption, b' and x are coprime, and as b' is good this implies that x is bad. Consequently there has to be some move $x \longrightarrow b^*$ such that b^* is good. But now all the small prime factors of b also appear in x and thus they cannot divide b^*. Therefore the pair (b^*, b) contradicts the supposed minimality of b'. □

From that point, it is easy to complete the solution: assume that there are two similar integers a and b such that a is bad and b is good. Since a is bad, there is a move $a \longrightarrow b'$ for some good b'. By Claim 5, there is a small prime p dividing b and b'. Due to the similarity of a and b, the prime p has to divide a as well, but this contradicts the fact that $a \longrightarrow b'$ is a valid move. Thereby the problem is solved.

Comment 2. There are infinitely many good numbers, e.g. all multiples of k. The increasing sequence $b_0, b_1, \ldots,$ of all good numbers may be constructed recursively as follows:

- Start with $b_0 = k$.
- If b_n has just been defined for some $n \geqslant 0$, then b_{n+1} is the smallest number $b > b_n$ that is coprime to none of $b_0, \ldots, b_n$.

This construction can be used to determine the set of good numbers for any specific k as explained in the next comment. It is already clear that if $k = p^\alpha$ is a prime power, then a number $b \geqslant k$ is good if and only if it is divisible by p.

Comment 3. Let $P > 1$ denote the product of all small prime numbers. Then any two integers $a, b \geqslant k$ that are congruent modulo P are similar. Thus the infinite word $W_k = (X_k, X_{k+1}, \ldots)$ defined by

$$X_i = \begin{cases} A & \text{if } i \text{ is bad} \\ B & \text{if } i \text{ is good} \end{cases}$$

for all $i \geqslant k$ is periodic and the length of its period divides P. As the prime power example shows, the true period can sometimes be much smaller than P. On the other hand, there are cases where the period is rather large; e.g., if $k = 15$, the sequence of good numbers begins with $15, 18, 20, 24, 30, 36, 40, 42, 45$ and the period of W_{15} is 30.

Comment 4. The original proposal contained two questions about the game of numbers, namely (a) to show that if two numbers have the same prime factors then either both are good or both are bad, and (b) to show that the word W_k introduced in the previous comment is indeed periodic. The Problem Selection Committee thinks that the above version of the problem is somewhat easier, even though it demands to prove a stronger result.

N6. Determine all functions $f\colon \mathbb{Q} \longrightarrow \mathbb{Z}$ satisfying

$$f\left(\frac{f(x)+a}{b}\right) = f\left(\frac{x+a}{b}\right) \tag{1}$$

for all $x \in \mathbb{Q}$, $a \in \mathbb{Z}$, and $b \in \mathbb{Z}_{>0}$. (Here, $\mathbb{Z}_{>0}$ denotes the set of positive integers.)

(Israel)

Answer. There are three kinds of such functions, which are: all constant functions, the floor function, and the ceiling function.

Solution 1. I. We start by verifying that these functions do indeed satisfy (1). This is clear for all constant functions. Now consider any triple $(x, a, b) \in \mathbb{Q} \times \mathbb{Z} \times \mathbb{Z}_{>0}$ and set

$$q = \left\lfloor \frac{x+a}{b} \right\rfloor.$$

This means that q is an integer and $bq \leqslant x + a < b(q + 1)$. It follows that $bq \leqslant \lfloor x \rfloor + a < b(q + 1)$ holds as well, and thus we have

$$\left\lfloor \frac{\lfloor x \rfloor + a}{b} \right\rfloor = \left\lfloor \frac{x+a}{b} \right\rfloor,$$

meaning that the floor function does indeed satisfy (1). One can check similarly that the ceiling function has the same property.

II. Let us now suppose conversely that the function $f\colon \mathbb{Q} \longrightarrow \mathbb{Z}$ satisfies (1) for all $(x, a, b) \in \mathbb{Q} \times \mathbb{Z} \times \mathbb{Z}_{>0}$. According to the behaviour of the restriction of f to the integers we distinguish two cases.

Case 1: There is some $m \in \mathbb{Z}$ such that $f(m) \neq m$.

Write $f(m) = C$ and let $\eta \in \{-1, +1\}$ and b denote the sign and absolute value of $f(m) - m$, respectively. Given any integer r, we may plug the triple $(m, rb - C, b)$ into (1), thus getting $f(r) = f(r - \eta)$. Starting with m and using induction in both directions, we deduce from this that the equation $f(r) = C$ holds for all integers r. Now any rational number y can be written in the form $y = \frac{p}{q}$ with $(p, q) \in \mathbb{Z} \times \mathbb{Z}_{>0}$, and substituting $(C - p, p - C, q)$ into (1) we get $f(y) = f(0) = C$. Thus f is the constant function whose value is always C.

Case 2: One has $f(m) = m$ for all integers m.

Note that now the special case $b = 1$ of (1) takes a particularly simple form, namely

$$f(x) + a = f(x + a) \qquad \text{for all } (x, a) \in \mathbb{Q} \times \mathbb{Z}. \tag{2}$$

Defining $f\left(\frac{1}{2}\right) = \omega$ we proceed in three steps.

Step A. We show that $\omega \in \{0, 1\}$.

If $\omega \leqslant 0$, we may plug $\left(\frac{1}{2}, -\omega, 1 - 2\omega\right)$ into (1), obtaining $0 = f(0) = f\left(\frac{1}{2}\right) = \omega$. In the contrary case $\omega \geqslant 1$ we argue similarly using the triple $\left(\frac{1}{2}, \omega - 1, 2\omega - 1\right)$.

Step B. We show that $f(x) = \omega$ for all rational numbers x with $0 < x < 1$.

Assume that this fails and pick some rational number $\frac{a}{b} \in (0, 1)$ with minimal b such that $f\left(\frac{a}{b}\right) \neq \omega$. Obviously, $\gcd(a, b) = 1$ and $b \geqslant 2$. If b is even, then a has to be odd and we can substitute $\left(\frac{1}{2}, \frac{a-1}{2}, \frac{b}{2}\right)$ into (1), which yields

$$f\left(\frac{\omega + (a-1)/2}{b/2}\right) = f\left(\frac{a}{b}\right) \neq \omega. \tag{3}$$

Recall that $0 \leqslant (a-1)/2 < b/2$. Thus, in both cases $\omega = 0$ and $\omega = 1$, the left-hand part of (3) equals ω either by the minimality of b, or by $f(\omega) = \omega$. A contradiction.

Thus b has to be odd, so $b = 2k+1$ for some $k \geqslant 1$. Applying (1) to $\left(\frac{1}{2}, k, b\right)$ we get

$$f\left(\frac{\omega + k}{b}\right) = f\left(\frac{1}{2}\right) = \omega. \tag{4}$$

Since a and b are coprime, there exist integers $r \in \{1, 2, \ldots, b\}$ and m such that $ra - mb = k + \omega$. Note that we actually have $1 \leqslant r < b$, since the right hand side is not a multiple of b. If m is negative, then we have $ra - mb > b \geqslant k + \omega$, which is absurd. Similarly, $m \geqslant r$ leads to $ra - mb < br - br = 0$, which is likewise impossible; so we must have $0 \leqslant m \leqslant r - 1$.

We finally substitute $\left(\frac{k+\omega}{b}, m, r\right)$ into (1) and use (4) to learn

$$f\left(\frac{\omega + m}{r}\right) = f\left(\frac{a}{b}\right) \neq \omega.$$

But as above one may see that the left hand side has to equal ω due to the minimality of b. This contradiction concludes our step B.

Step C. Now notice that if $\omega = 0$, then $f(x) = \lfloor x \rfloor$ holds for all rational x with $0 \leqslant x < 1$ and hence by (2) this even holds for all rational numbers x. Similarly, if $\omega = 1$, then $f(x) = \lceil x \rceil$ holds for all $x \in \mathbb{Q}$. Thereby the problem is solved.

Comment 1. An alternative treatment of Steps B and C from the second case, due to the proposer, proceeds as follows. Let square brackets indicate the floor function in case $\omega = 0$ and the ceiling function if $\omega = 1$. We are to prove that $f(x) = [x]$ holds for all $x \in \mathbb{Q}$, and because of Step A and (2) we already know this in case $2x \in \mathbb{Z}$. Applying (1) to $(2x, 0, 2)$ we get

$$f(x) = f\left(\frac{f(2x)}{2}\right),$$

and by the previous observation this yields

$$f(x) = \left[\frac{f(2x)}{2}\right] \qquad \text{for all } x \in \mathbb{Q}. \tag{5}$$

An easy induction now shows

$$f(x) = \left[\frac{f(2^n x)}{2^n}\right] \qquad \text{for all } (x, n) \in \mathbb{Q} \times \mathbb{Z}_{>0}. \tag{6}$$

Now suppose first that x is not an integer but can be written in the form $\frac{p}{q}$ with $p \in \mathbb{Z}$ and $q \in \mathbb{Z}_{>0}$ both being odd. Let d denote the multiplicative order of 2 modulo q and let m be any large integer. Plugging $n = dm$ into (6) and using (2) we get

$$f(x) = \left[\frac{f(2^{dm} x)}{2^{dm}}\right] = \left[\frac{f(x) + (2^{dm} - 1)x}{2^{dm}}\right] = \left[x + \frac{f(x) - x}{2^{dm}}\right].$$

Since x is not an integer, the square bracket function is continuous at x; hence as m tends to infinity the above fomula gives $f(x) = [x]$. To complete the argument we just need to observe that if some $y \in \mathbb{Q}$ satisfies $f(y) = [y]$, then (5) yields $f\left(\frac{y}{2}\right) = f\left(\frac{[y]}{2}\right) = \left[\frac{[y]}{2}\right] = \left[\frac{y}{2}\right]$.

Solution 2. Here we just give another argument for the second case of the above solution. Again we use equation (2). It follows that the set S of all zeros of f contains for each $x \in \mathbb{Q}$ exactly one term from the infinite sequence $\dots, x-2, x-1, x, x+1, x+2, \dots$.

Next we claim that

$$\text{if } (p,q) \in \mathbb{Z} \times \mathbb{Z}_{>0} \text{ and } \tfrac{p}{q} \in S, \text{ then } \tfrac{p}{q+1} \in S \text{ holds as well.} \tag{7}$$

To see this we just plug $\left(\frac{p}{q}, p, q+1\right)$ into (1), thus getting $f\left(\frac{p}{q+1}\right) = f\left(\frac{p}{q}\right) = 0$.

From this we get that

$$\text{if } x, y \in \mathbb{Q},\ x > y > 0, \text{ and } x \in S, \text{ then } y \in S. \tag{8}$$

Indeed, if we write $x = \frac{p}{q}$ and $y = \frac{r}{s}$ with $p, q, r, s \in \mathbb{Z}_{>0}$, then $ps > qr$ and (7) tells us

$$0 = f\left(\frac{p}{q}\right) = f\left(\frac{pr}{qr}\right) = f\left(\frac{pr}{qr+1}\right) = \dots = f\left(\frac{pr}{ps}\right) = f\left(\frac{r}{s}\right).$$

Essentially the same argument also establishes that

$$\text{if } x, y \in \mathbb{Q},\ x < y < 0, \text{ and } x \in S, \text{ then } y \in S. \tag{9}$$

From (8) and (9) we get $0 \in S \subseteq (-1, +1)$ and hence the real number $\alpha = \sup(S)$ exists and satisfies $0 \leqslant \alpha \leqslant 1$.

Let us assume that we actually had $0 < \alpha < 1$. Note that $f(x) = 0$ if $x \in (0, \alpha) \cap \mathbb{Q}$ by (8), and $f(x) = 1$ if $x \in (\alpha, 1) \cap \mathbb{Q}$ by (9) and (2). Let K denote the unique positive integer satisfying $K\alpha < 1 \leqslant (K+1)\alpha$. The first of these two inequalities entails $\alpha < \frac{1+\alpha}{K+1}$, and thus there is a rational number $x \in \left(\alpha, \frac{1+\alpha}{K+1}\right)$. Setting $y = (K+1)x - 1$ and substituting $(y, 1, K+1)$ into (1) we learn

$$f\left(\frac{f(y)+1}{K+1}\right) = f\left(\frac{y+1}{K+1}\right) = f(x).$$

Since $\alpha < x < 1$ and $0 < y < \alpha$, this simplifies to

$$f\left(\frac{1}{K+1}\right) = 1.$$

But, as $0 < \frac{1}{K+1} \leqslant \alpha$, this is only possible if $\alpha = \frac{1}{K+1}$ and $f(\alpha) = 1$. From this, however, we get the contradiction

$$0 = f\left(\frac{1}{(K+1)^2}\right) = f\left(\frac{\alpha+0}{K+1}\right) = f\left(\frac{f(\alpha)+0}{K+1}\right) = f(\alpha) = 1.$$

Thus our assumption $0 < \alpha < 1$ has turned out to be wrong and it follows that $\alpha \in \{0, 1\}$. If $\alpha = 0$, then we have $S \subseteq (-1, 0]$, whence $S = (-1, 0] \cap \mathbb{Q}$, which in turn yields $f(x) = \lceil x \rceil$ for all $x \in \mathbb{Q}$ due to (2). Similarly, $\alpha = 1$ entails $S = [0, 1) \cap \mathbb{Q}$ and $f(x) = \lfloor x \rfloor$ for all $x \in \mathbb{Q}$. Thereby the solution is complete.

Comment 2. It seems that all solutions to this problems involve some case distinction separating the constant solutions from the unbounded ones, though the "descriptions" of the cases may be different depending on the work that has been done at the beginning of the solution. For instance, these two cases can also be "f is periodic on the integers" and "f is not periodic on the integers". The case leading to the unbounded solutions appears to be the harder one.

In most approaches, the cases leading to the two functions $x \longmapsto \lfloor x \rfloor$ and $x \longmapsto \lceil x \rceil$ can easily be treated parallelly, but sometimes it may be useful to know that there is some symmetry in the problem interchanging these two functions. Namely, if a function $f\colon \mathbb{Q} \longrightarrow \mathbb{Z}$ satisfies (1), then so does the function $g\colon \mathbb{Q} \longrightarrow \mathbb{Z}$ defined by $g(x) = -f(-x)$ for all $x \in \mathbb{Q}$. For that reason, we could have restricted our attention to the case $\omega = 0$ in the first solution and, once $\alpha \in \{0, 1\}$ had been obtained, to the case $\alpha = 0$ in the second solution.

N7. Let ν be an irrational positive number, and let m be a positive integer. A pair (a, b) of positive integers is called *good* if

$$a\lceil b\nu\rceil - b\lfloor a\nu\rfloor = m. \qquad (*)$$

A good pair (a, b) is called *excellent* if neither of the pairs $(a-b, b)$ and $(a, b-a)$ is good. (As usual, by $\lfloor x\rfloor$ and $\lceil x\rceil$ we denote the integer numbers such that $x - 1 < \lfloor x\rfloor \leqslant x$ and $x \leqslant \lceil x\rceil < x + 1$.)

Prove that the number of excellent pairs is equal to the sum of the positive divisors of m.

(U.S.A.)

Solution. For positive integers a and b, let us denote

$$f(a, b) = a\lceil b\nu\rceil - b\lfloor a\nu\rfloor.$$

We will deal with various values of m; thus it is convenient to say that a pair (a, b) is m-*good* or m-*excellent* if the corresponding conditions are satisfied.

To start, let us investigate how the values $f(a + b, b)$ and $f(a, b + a)$ are related to $f(a, b)$. If $\{a\nu\} + \{b\nu\} < 1$, then we have $\lfloor(a + b)\nu\rfloor = \lfloor a\nu\rfloor + \lfloor b\nu\rfloor$ and $\lceil(a + b)\nu\rceil = \lceil a\nu\rceil + \lceil b\nu\rceil - 1$, so

$$f(a + b, b) = (a + b)\lceil b\nu\rceil - b(\lfloor a\nu\rfloor + \lfloor b\nu\rfloor) = f(a, b) + b(\lceil b\nu\rceil - \lfloor b\nu\rfloor) = f(a, b) + b$$

and

$$f(a, b + a) = a(\lceil b\nu\rceil + \lceil a\nu\rceil - 1) - (b + a)\lfloor a\nu\rfloor = f(a, b) + a(\lceil a\nu\rceil - 1 - \lfloor a\nu\rfloor) = f(a, b).$$

Similarly, if $\{a\nu\} + \{b\nu\} \geqslant 1$ then one obtains

$$f(a + b, b) = f(a, b) \quad \text{and} \quad f(a, b + a) = f(a, b) + a.$$

So, in both cases one of the numbers $f(a + b, a)$ and $f(a, b + a)$ is equal to $f(a, b)$ while the other is greater than $f(a, b)$ by one of a and b. Thus, exactly one of the pairs $(a + b, b)$ and $(a, b + a)$ is excellent (for an appropriate value of m).

Now let us say that the pairs $(a + b, b)$ and $(a, b + a)$ are the *children* of the pair (a, b), while this pair is their *parent*. Next, if a pair (c, d) can be obtained from (a, b) by several passings from a parent to a child, we will say that (c, d) is a *descendant* of (a, b), while (a, b) is an *ancestor* of (c, d) (a pair is neither an ancestor nor a descendant of itself). Thus each pair (a, b) has two children, it has a unique parent if $a \neq b$, and no parents otherwise. Therefore, each pair of distinct positive integers has a unique ancestor of the form (a, a); our aim is now to find how many m-excellent descendants each such pair has.

Notice now that if a pair (a, b) is m-excellent then $\min\{a, b\} \leqslant m$. Indeed, if $a = b$ then $f(a, a) = a = m$, so the statement is valid. Otherwise, the pair (a, b) is a child of some pair (a', b'). If $b = b'$ and $a = a' + b'$, then we should have $m = f(a, b) = f(a', b') + b'$, so $b = b' = m - f(a', b') < m$. Similarly, if $a = a'$ and $b = b' + a'$ then $a < m$.

Let us consider the set S_m of all pairs (a, b) such that $f(a, b) \leqslant m$ and $\min\{a, b\} \leqslant m$. Then all the ancestors of the elements in S_m are again in S_m, and each element in S_m either is of the form (a, a) with $a \leqslant m$, or has a unique ancestor of this form. From the arguments above we see that all m-excellent pairs lie in S_m.

We claim now that the set S_m is finite. Indeed, assume, for instance, that it contains infinitely many pairs (c, d) with $d > 2m$. Such a pair is necessarily a child of $(c, d-c)$, and thus a descendant of some pair (c, d') with $m < d' \leqslant 2m$. Therefore, one of the pairs $(a, b) \in S_m$ with $m < b \leqslant 2m$

has infinitely many descendants in S_m, and all these descendants have the form $(a, b + ka)$ with k a positive integer. Since $f(a, b + ka)$ does not decrease as k grows, it becomes constant for $k \geqslant k_0$, where k_0 is some positive integer. This means that $\{a\nu\} + \{(b + ka)\nu\} < 1$ for all $k \geqslant k_0$. But this yields $1 > \{(b + ka)\nu\} = \{(b + k_0a)\nu\} + (k - k_0)\{a\nu\}$ for all $k > k_0$, which is absurd.

Similarly, one can prove that S_m contains finitely many pairs (c, d) with $c > 2m$, thus finitely many elements at all.

We are now prepared for proving the following crucial lemma.

Lemma. Consider any pair (a, b) with $f(a, b) \neq m$. Then the number $g(a, b)$ of its m-excellent descendants is equal to the number $h(a, b)$ of ways to represent the number $t = m - f(a, b)$ as $t = ka + \ell b$ with k and ℓ being some nonnegative integers.

Proof. We proceed by induction on the number N of descendants of (a, b) in S_m. If $N = 0$ then clearly $g(a, b) = 0$. Assume that $h(a, b) > 0$; without loss of generality, we have $a \leqslant b$. Then, clearly, $m - f(a, b) \geqslant a$, so $f(a, b + a) \leqslant f(a, b) + a \leqslant m$ and $a \leqslant m$, hence $(a, b + a) \in S_m$ which is impossible. Thus in the base case we have $g(a, b) = h(a, b) = 0$, as desired.

Now let $N > 0$. Assume that $f(a + b, b) = f(a, b) + b$ and $f(a, b + a) = f(a, b)$ (the other case is similar). If $f(a, b) + b \neq m$, then by the induction hypothesis we have

$$g(a, b) = g(a + b, b) + g(a, b + a) = h(a + b, b) + h(a, b + a).$$

Notice that both pairs $(a + b, b)$ and $(a, b + a)$ are descendants of (a, b) and thus each of them has strictly less descendants in S_m than (a, b) does.

Next, each one of the $h(a + b, b)$ representations of $m - f(a + b, b) = m - b - f(a, b)$ as the sum $k'(a + b) + \ell' b$ provides the representation $m - f(a, b) = ka + \ell b$ with $k = k' < k' + \ell' + 1 = \ell$. Similarly, each one of the $h(a, b + a)$ representations of $m - f(a, b + a) = m - f(a, b)$ as the sum $k'a + \ell'(b + a)$ provides the representation $m - f(a, b) = ka + \ell b$ with $k = k' + \ell' \geqslant \ell' = \ell$. This correspondence is obviously bijective, so

$$h(a, b) = h(a + b, b) + h(a, b + a) = g(a, b),$$

as required.

Finally, if $f(a, b) + b = m$ then $(a + b, b)$ is m-excellent, so $g(a, b) = 1 + g(a, b + a) = 1 + h(a, b + a)$ by the induction hypothesis. On the other hand, the number $m - f(a, b) = b$ has a representation $0 \cdot a + 1 \cdot b$ and sometimes one more representation as $ka + 0 \cdot b$; this last representation exists simultaneously with the representation $m - f(a, b + a) = ka + 0 \cdot (b + a)$, so $h(a, b) = 1 + h(a, b + a)$ as well. Thus in this case the step is also proved. $\square$

Now it is easy to finish the solution. There exists a unique m-excellent pair of the form (a, a), and each other m-excellent pair (a, b) has a unique ancestor of the form (x, x) with $x < m$. By the lemma, for every $x < m$ the number of its m-excellent descendants is $h(x, x)$, which is the number of ways to represent $m - f(x, x) = m - x$ as $kx + \ell x$ (with nonnegative integer k and ℓ). This number is 0 if $x \nmid m$, and m/x otherwise. So the total number of excellent pairs is

$$1 + \sum_{x \mid m,\, x < m} \frac{m}{x} = 1 + \sum_{d \mid m,\, d > 1} d = \sum_{d \mid m} d,$$

as required.

Comment. Let us present a sketch of an outline of a different solution. The plan is to check that the number of excellent pairs does not depend on the (irrational) number ν, and to find this number for some appropriate value of ν. For that, we first introduce some geometrical language. We deal only with the excellent pairs (a, b) with $a \neq b$.

Part I. Given an irrational positive ν, for every positive integer n we introduce two integral points $F_\nu(n) = (n, \lfloor n\nu \rfloor)$ and $C_\nu(n) = (n, \lceil n\nu \rceil)$ on the coordinate plane Oxy. Then $(*)$ reads as $[OF_\nu(a)C_\nu(b)] = m/2$; here $[\cdot]$ stands for the signed area. Next, we rewrite in these terms the condition on a pair (a, b) to be excellent. Let ℓ_ν, ℓ_ν^+, and ℓ_ν^- be the lines determined by the equations $y = \nu x$, $y = \nu x + 1$, and $y = \nu x - 1$, respectively.

a). Firstly, we deal with all excellent pairs (a, b) with $a < b$. Given some value of a, all the points C such that $[OF_\nu(a)C] = m/2$ lie on some line $f_\nu(a)$; if there exist any good pairs (a, b) at all, this line has to contain at least one integral point, which happens exactly when $\gcd(a, \lfloor a\nu \rfloor) \mid m$.

Let $P_\nu(a)$ be the point of intersection of ℓ_ν^+ and $f_\nu(a)$, and let $p_\nu(a)$ be its abscissa; notice that $p_\nu(a)$ is irrational if it is nonzero. Now, if (a, b) is good, then the point $C_\nu(b)$ lies on $f_\nu(a)$, which means that the point of $f_\nu(a)$ with abscissa b lies between ℓ_ν and ℓ_ν^+ and is integral. If in addition the pair $(a, b - a)$ is not good, then the point of $f_\nu(a)$ with abscissa $b - a$ lies above ℓ_ν^+ (see Fig. 1). Thus, the pair (a, b) with $b > a$ is excellent exactly when $p_\nu(a)$ lies between $b - a$ and b, and the point of $f_\nu(a)$ with abscissa b is integral (which means that this point is $C_\nu(b)$).

Notice now that, if $p_\nu(a) > a$, then the number of excellent pairs of the form (a, b) (with $b > a$) is $\gcd(a, \lfloor a\nu \rfloor)$.

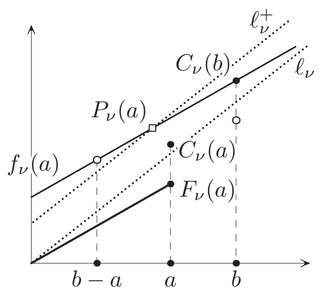


Figure 1

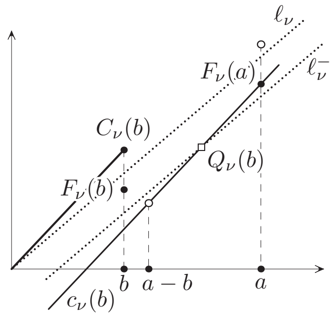


Figure 2

b). Analogously, considering the pairs (a, b) with $a > b$, we fix the value of b, introduce the line $c_\nu(b)$ containing all the points F with $[OFC_\nu(b)] = m/2$, assume that this line contains an integral point (which means $\gcd(b, \lceil b\nu \rceil) \mid m$), and denote the common point of $c_\nu(b)$ and ℓ_ν^- by $Q_\nu(b)$, its abscissa being $q_\nu(b)$. Similarly to the previous case, we obtain that the pair (a, b) is excellent exactly when $q_\nu(a)$ lies between $a - b$ and a, and the point of $c_\nu(b)$ with abscissa a is integral (see Fig. 2). Again, if $q_\nu(b) > b$, then the number of excellent pairs of the form (a, b) (with $a > b$) is $\gcd(b, \lceil b\nu \rceil)$.

Part II, sketchy. Having obtained such a description, one may check how the number of excellent pairs changes as ν grows. (Having done that, one may find this number for one appropriate value of ν; for instance, it is relatively easy to make this calculation for $\nu \in \left(1, 1 + \frac{1}{m}\right)$.)

Consider, for the initial value of ν, some excellent pair (a, t) with $a > t$. As ν grows, this pair eventually stops being excellent; this happens when the point $Q_\nu(t)$ passes through $F_\nu(a)$. At the same moment, the pair $(a + t, t)$ becomes excellent instead.

This process halts when the point $Q_\nu(t)$ eventually disappears, i.e. when ν passes through the ratio of the coordinates of the point $T = C_\nu(t)$. Hence, the point T afterwards is regarded as $F_\nu(t)$. Thus, all the old excellent pairs of the form (a, t) with $a > t$ disappear; on the other hand, the same number of excellent pairs with the first element being t just appear.

Similarly, if some pair (t, b) with $t < b$ is initially ν-excellent, then at some moment it stops being excellent when $P_\nu(t)$ passes through $C_\nu(b)$; at the same moment, the pair $(t, b-t)$ becomes excellent. This process eventually stops when $b - t < t$. At this moment, again the second element of the pair becomes fixed, and the first one starts to increase.

These ideas can be made precise enough to show that the number of excellent pairs remains unchanged, as required.

We should warn the reader that the rigorous elaboration of Part II is technically quite involved, mostly by the reason that the set of moments when the collection of excellent pairs changes is infinite. Especially much care should be applied to the limit points of this set, which are exactly the points when the line ℓ_ν passes through some point of the form $C_\nu(b)$.

The same ideas may be explained in an algebraic language instead of a geometrical one; the same technicalities remain in this way as well.